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LOWER ORDER PERTURBATION AND GLOBAL ANALYTIC VECTORS FOR A CLASS OF GLOBALLY ANALYTIC HYPOELLIPTIC OPERATORS

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Abstract. In this work we return to the class of globally analytic hypoelliptic Hörmander's operators defined on the N-dimensional torus introduced by Cordaro and Himonas and prove that if P is any operator in this class, then a perturbation of P by an analytic pseudodifferential operator with degree smaller than the subelliptic index of P remains globally analytic hypoelliptic. We also study the Gevrey regularity of the Gevrey vectors for such a class and at the end we also show that Cordaro and Himonas's result can be extended to a similar class of operators now defined in a product of compact Lie group by a compact manifold.

Introduction

A natural question in the theory of analytic linear partial differential operators is when does an operator which is analytic hypoelliptic remain so after a perturbation by an operator of smaller order? This is not a simple question to answer: already in 1982 E. Stein ([19], see also [14] for a more general result) proved that if $\square_b$ denotes the Kohn Laplacian on the Heisenberg group and if $\lambda \in \mathbb{C}$ is not zero, then $\square_b + \lambda$ is analytic hypoelliptic. Noticing that $\square_b$ is not analytic hypoelliptic, the analytic hypoellipticity of $\square_b + \lambda$ is not preserved by perturbations of order zero. On the other hand if P is an analytic hypoelliptic sum of squares operator satisfying Hörmander's condition [11], A. Bove has suggested to us that it should be true that $P + A$ is still analytic hypoelliptic if A is an analytic pseudodifferential operator with order smaller than the subelliptic index of P.

This is one of the questions we address in this note. We adopt the global viewpoint and return to the class of Hörmander's operators, here denoted by $\mathfrak{O}(\mathbb{T}^N)$, on the N-dimensional torus $\mathbb{T}^N$ introduced in [5], [6]. We prove (Theorem 1.1) that given an operator $P \in \mathfrak{O}(\mathbb{T}^N)$, which as proved in [6] is globally analytic hypoelliptic, then $P + A$ is also globally analytic hypoelliptic if A is any analytic pseudodifferential operator on the torus whose order is smaller than the subelliptic index of P. Other interesting results related to such a class of operators can be found in Himonas-Petronilho [9], [10], Petronilho [17] and Albanese-Jornet [1], and an analogous study in the smooth category can be found in Parmeggiani [15].

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It is important to point out that although operators $P \in \mathfrak{O}(\mathbb{T}^N)$ belong to the more general model described in [4], this is no longer true for their perturbations $P + A$.

In Section 3 we study the regularity of the global Gevrey vectors for sum of squares of operators belonging to $\mathfrak{O}(\mathbb{T}^N)$ and prove (Theorem 2.1) that every global Gevrey vector of order $s \geq 1$ belongs to $G^{sr}(\mathbb{T}^N)$ (here r is the number of brackets needed to span the tangent space and $2/r$ is then the subelliptic index of P according to [18]). We also show that this is the best possible result in this generality, after applying an important criterion due to Métivier [16]. Thus in this (very) particular case we obtain a complete answer for this regularity question, one that is still missing in the local and more general case (see for instance the survey article [3]).

In Section 3 we depart from the torus and extend the result in [6] to a similar class of operators P now defined in a product $G \times M$, where G is a compact Lie group and M a compact manifold (which we assume is endowed with a Riemannian metric, although this is not essential). Such an extension is not automatic, and relies on the crucial results stated in Lemmas 4.1 and 4.3. Furthermore it strengthens the importance of the method used throughout this note: the constant use of the microlocal regularity of the Gevrey vectors for certain partially elliptic operators that commute with P.

1. The class of operators under study

A. First we establish the notation. Let $\mathbb{T}^N = \mathbb{R}^N/(2\pi\mathbb{Z}^N)$ be the N-dimensional torus, where the coordinates are written as $y = (y_1, \dots, y_N)$. As usual we denote by $G^s(\mathbb{T}^N)$, $s \geq 1$, the class of Gevrey functions of order s on $\mathbb{T}^N$. Notice that $G^1(\mathbb{T}^N) = C^\omega(\mathbb{T}^N)$ is the space of real-analytic functions on $\mathbb{T}^N$. We recall that $u \in G^s(\mathbb{T}^N)$ if and only if $u \in C^\infty(\mathbb{T}^N)$ and there is a constant $C > 0$ such that

$$\|D^\alpha u\|_{L^\infty(\mathbb{T}^N)} \leq C^{|\alpha|+1}(\alpha!)^s, \quad \forall \alpha \in \mathbb{Z}_+^N,$$

where $D^\alpha = D_1^{\alpha} \cdots D_N^{\alpha_N}$, $D_j = -\sqrt{-1}\, \partial/\partial y_j$. Notice that an equivalent definition is obtained after replacing the L^∞ norm by the L^p norm for any $p \geq 1$. In terms of the Fourier coefficients, for a function $u \in C^\infty(\mathbb{T}^N)$ to belong to $G^s(\mathbb{T}^N)$ it is necessary and sufficient that there is a constant $\delta > 0$ such that

$$\|\{e^{\delta|\eta|^{1/s}} \hat{u}(\eta)\}_{\eta \in \mathbb{Z}^N}\|_{\ell_\infty(\mathbb{Z}^N)} < \infty.$$

Here we write the Fourier expansion of u as

$$u(y) = \sum_{\eta \in \mathbb{Z}^N} e^{i\eta \cdot y}\ \hat{u}(\eta), \text{ where } \hat{u}(\eta) = \frac{1}{(2\pi)^N} \int_{\mathbb{T}^N} e^{-i\eta \cdot y} u(y) dy.$$

Of course equivalent characterizations can be obtained after replacing the norm in $\ell_\infty(\mathbb{T}^N)$ by the norm in $\ell_p(\mathbb{T}^N)$ for any $p \geq 1$.

Likewise we recall the scale of Sobolev spaces in $\mathbb{T}^N$. If $s \in \mathbb{R}$, we denote by $H^s(\mathbb{T}^N)$ the space of all $u \in \mathcal{D}'(\mathbb{T}^N)$ such that

$$\left\{(1 + |\eta|^2)^{s/2} \hat{u}(\eta)\right\}_{\eta \in \mathbb{Z}^N} \in \ell_2(\mathbb{Z}^N).$$

The spaces $H^s(\mathbb{T}^N)$ are Hilbert spaces with norm

$$\|u\|_s = \left\{ \sum_{\eta \in \mathbb{Z}^N} (1 + |\eta|^2)^s |\hat{u}(\eta)|^2 \right\}^{1/2}.$$

When there is no risk of confusion we shall sometimes write $\|\cdot\|$ for the L^2-norm $\|\cdot\|_0$.

B. Let $P = P(y, D)$ be a real-analytic, linear partial differential operator on $\mathbb{T}^N$ of order $m \geq 1$. We say that P is *globally G^s-hypoelliptic* ($s \geq 1$) if given $u \in \mathcal{D}'(\mathbb{T}^N)$ such that $Pu \in G^s(\mathbb{T}^N)$, then $u \in G^s(\mathbb{T}^N)$. When $s = 1$ we also say that P is *globally analytic hypoelliptic*. Also, $u \in \mathcal{D}'(\mathbb{T}^N)$ is called a *global Gevrey vector of order $s \geq 1$ for P* if $P^j u \in L^2(\mathbb{T}^N)$ for every $j \in \mathbb{Z}_+$ and, for some constant $C > 0$,

$$\|P^j u\|_0 \leq C^{j+1} j!^{sm}, \quad j \in \mathbb{Z}_+.$$

When $s = 1$ we simply say that u is a *global analytic vector for P*. Needless to say that in this definition the space $L^2(\mathbb{T}^N)$ can be replaced by any space $L^p(\mathbb{T}^N)$ or even $H^s(\mathbb{T}^N)$.

The space of all global Gevrey vectors of order $s \geq 1$ for P will be denoted by $G^s(\mathbb{T}^N; P)$. It is not difficult to see that $G^s(\mathbb{T}^N) \subset G^s(\mathbb{T}^N; P)$ for every P and every $s \geq 1$.

The following result will often be used in our work (cf. [2]).

Proposition 1.1. *Let $u \in G^s(\mathbb{T}^N; P)$. If $(y, \eta) \in T^*\mathbb{T}^N \setminus 0$ does not belong to the characteristic set of $P = P(y, D)$, then $(y, \eta) \notin WF_s(u)$, the G^s-wave-front set of u. In particular, if P is elliptic, then $G^s(\mathbb{T}^N) = G^s(\mathbb{T}^N; P)$.*

C. We write $\mathbb{T}^N$ as the product of two tori $\mathbb{T}^N = \mathbb{T}^m \times \mathbb{T}^n$ with variables written as $y = (x, t)$ where $x = (x_1, \ldots, x_m) \in \mathbb{T}^m$ and $t = (t_1, \ldots, t_n) \in \mathbb{T}^n$. Under this decomposition the Fourier series will be written as

$$u(x,t) = \sum_{(\xi,\tau)\in\mathbb{Z}^N} e^{i\xi\cdot x + i\tau\cdot t}\ \hat{u}(\xi,\tau), \text{ where}$$

$$\hat{u}(\xi,\tau) = \frac{1}{(2\pi)^N} \int_{\mathbb{T}^N} e^{-i\xi\cdot x - i\tau\cdot t} u(x,t)\,dx\,dt.$$

We shall consider on $\mathbb{T}^N$ vector fields of the form

$$L_j = \sum_{i=1}^{n} a_{ji}(t)\frac{\partial}{\partial t_i} + \sum_{\ell=1}^{m} b_{j\ell}(t)\frac{\partial}{\partial x_\ell}, \quad j = 0, 1, \ldots, \nu, \tag{1.1}$$

where $a_{ji}, b_{j\ell} \in C^\omega(\mathbb{T}^N)$ are real valued. We assume the following conditions:

(A-1) $L_0, L_1, \ldots, L_\nu$ and their brackets of length at most r span the tangent space at every point of $\mathbb{T}^N$ (Hörmander's condition).

(A-2) The vector fields $\sum_{i=1}^n a_{ji}(t)\partial_{t_i}$, $j = 1, \ldots, \nu$, span $T_t(\mathbb{T}^n)$ for every $t \in \mathbb{T}^N$.

An operator $P = P(t, D_x, D_t)$ is said to belong to the class $\mathfrak{O}(\mathbb{T}^N)$ if it is of the form

$$P = -\sum_{j=1}^{\nu} L_j^2 + L_0, \tag{1.2}$$

where L_j are of the form (1.1) and satisfy (A-1) and (A-2). We recall the following result proved in [6].

Theorem 1.1. *If $P \in \mathfrak{O}(\mathbb{T}^N)$, then for all $a \in C^\omega(\mathbb{T}^N)$ the operator $P + a$ is globally analytic hypoelliptic on $\mathbb{T}^N$.*

2. Lower order perturbations for operators in the class $\mathfrak{O}(\mathbb{T}^N)$

A. Let $L^\sigma_\omega(\mathbb{T}^N)$ denote the space of all (classical) analytic pseudodifferential operators on $\mathbb{T}^N$ of order $\le \sigma$. We write

$$L^\infty_\omega(\mathbb{T}^N) = \bigcup_{\sigma\in\mathbb{R}} L^\sigma_\omega(\mathbb{T}^N).$$

We shall also denote by $SL^\sigma_\omega(\mathbb{T}^N)$ the space of all functions $a(y,\eta) \in C^\infty(\mathbb{T}^N \times \mathbb{Z}^N)$ such that for every $\alpha \in \mathbb{Z}^N_+$, there exists a constant $C > 0$ such that

$$|D^\alpha_y a(y,\eta)| \le C^{|\alpha|+1}\alpha!(1+|\eta|)^\sigma. \tag{2.1}$$

We stress that condition (2.1) is equivalent, in terms of the Fourier transform, to

$$|\hat{a}(\zeta,\eta)| \le Ce^{-\delta|\zeta|}(1+|\eta|)^\sigma, \tag{2.2}$$

for some constants C and δ greater than zero.

Write $e_\eta(x) = \exp\{ix\cdot\eta\}$. If $A \in L^\sigma_\omega(\mathbb{T}^N)$, then it can be shown that

$$a(y,\eta) = A[e_\eta](y)e_{-\eta}(y), \quad \eta \in \mathbb{Z}^N,$$

defines an element in $SL^\sigma_\omega(\mathbb{T}^N)$; we shall refer to $a(y,\eta)$ as the *discrete symbol of* A. Notice that we can write

$$A[u](y) = \sum_{\eta\in\mathbb{Z}^N} e_\eta(y)a(y,\eta)\hat{u}(\eta). \tag{2.3}$$

B. If $P \in \mathfrak{O}(\mathbb{T}^N)$, then for some $\varepsilon > 0$ precisely defined in terms of Hörmander's condition (A-1) (cf. [18]), it follows that P is *ε-subelliptic*, that is, the following holds:

Given $s \in \mathbb{R}$ and $u \in \mathcal{D}'(\mathbb{T}^N)$ such that $Pu \in H^s(\mathbb{T}^N)$, then $u \in H^{s+\varepsilon}(\mathbb{T}^N)$.

Remark 2.1. In the important case when $L_0 = 0$, that is, when $P \in \mathfrak{O}(\mathbb{T}^N)$ is a sum of squares, we have $\varepsilon = 2/r$ (cf. [18, p. 310, Theorem 16]).

A standard application of the closed graph theorem and an interpolation argument gives

Lemma 2.1. *Let P belong to $\mathfrak{O}(\mathbb{T}^N)$ and let $\varepsilon > 0$ be as above. Then there is $C > 0$ such that*

$$\|u\|_\varepsilon \le C\left(\|Pu\| + C^k\|u\|_{-k}\right), \ \forall k \in \mathbb{Z}_+ \text{ and } u \in C^\infty(\mathbb{T}^N). \tag{2.4}$$

C. We are now ready to prove our first result.

Theorem 2.1. *Let P belong to $\mathfrak{O}(\mathbb{T}^N)$ and let $\varepsilon > 0$ be as above. If $A \in L^\sigma_\omega(\mathbb{T}^N)$ with $\sigma < \varepsilon$, then $P + A$ is globally analytic hypoelliptic.*

Proof. Let $u \in \mathcal{D}'(\mathbb{T}^N)$ be such that

$$(P+A)u = f, \ \ f \in C^\omega(\mathbb{T}^N). \tag{2.5}$$

Because $\sigma < \varepsilon$, it follows easily that $P + A$ is also ε-subelliptic and consequently $u \in C^\infty(\mathbb{T}^N)$. Replacing u by $\Delta_x^k u$ in (2.4) gives[1]

$$\|\Delta_x^k u\|_\varepsilon \le C\left(\|\Delta_x^k (P+A)u\| + \|\Delta_x^k Au\| + C^{2k}\|u\|\right).$$

In terms of the Fourier transform we have

$$\|\Delta_x^k Au\| = \left(\sum_{\xi,\tau} ||\xi|^{2k}\widehat{Au}(\xi,\tau)|^2\right)^{1/2}. \tag{2.6}$$

Since

$$|\xi|^{2k} \le \sum_{j=0}^{2k}\binom{2k}{j}|\xi-\xi'|^j|\xi'|^{2k-j},$$

and denoting by $a(x,t,\xi,\tau) \in SL^\sigma_\omega(\mathbb{T}^N)$ the discrete symbol of A, we obtain

$$||\xi|^{2k}\widehat{Au}(\xi,\tau)| \le \sum_{j=0}^{2k}\binom{2k}{j}\left(\sum_{\xi',\tau'}|\xi-\xi'|^j|\widehat{a}(\xi-\xi',\tau-\tau',\xi',\tau')||\xi'|^{2k-j}|\hat{u}(\xi',\tau')|\right).$$

Applying Minkowsky's inequality we can estimate the right hand side of (2.6) by

$$\sum_{j=0}^{2k}\binom{2k}{j}\left[\sum_{\xi,\tau}\underbrace{\left(\sum_{\xi',\tau'}|\xi-\xi'|^j|\widehat{a}(\xi-\xi',\tau-\tau',\xi',\tau')||\xi'|^{2k-j}|\hat{u}(\xi',\tau')|\right)^2}_{(\bigstar)}\right]^{1/2}.$$

Now since $a \in SL^\sigma_\omega(\mathbb{T}^N)$, there are constants $C_1 > 0$ and $\delta > 0$ such that

$$|\widehat{a}(\xi-\xi',\tau-\tau',\xi',\tau')| \le C_1 e^{-\delta(|\xi-\xi'|+|\tau-\tau'|)}(1+|\xi'|+|\tau'|)^\sigma.$$

Hence,

$$\begin{aligned}(\bigstar) \le &\left(\sum_{\xi',\tau'}|\xi-\xi'|^{2j}|\widehat{a}(\xi-\xi',\tau-\tau',\xi',\tau')|\right)\\ &\times\left(\sum_{\xi',\tau'}|\widehat{a}(\xi-\xi',\tau-\tau',\xi',\tau')||\xi'|^{2(2k-j)}|\hat{u}(\xi',\tau')|^2\right)\\ \le C_1&\left(\sum_{\xi',\tau'}|\xi-\xi'|^{2j}e^{-\delta(|\xi-\xi'|+|\tau-\tau'|)}(1+|\xi'|+|\tau'|)^\sigma\right)\\ &\times\left(\sum_{\xi',\tau'}|\widehat{a}(\xi-\xi',\tau-\tau',\xi',\tau')||\xi'|^{2(2k-j)}|\hat{u}(\xi',\tau')|^2\right).\end{aligned}$$

If we recall the elementary inequality

$$(1+|\xi'|+|\tau'|) \le (1+|\xi-\xi'|+|\tau-\tau'|)(1+|\xi|+|\tau|),$$

[1]Here the Laplace operator in the x-variables, Δ_x^k, could be replaced by ∂_x^α without any additional technical difficulties. Our choice in this case was motivated by an analogous study that will be presented in the last section when we deal with the case of compact Lie groups.

we thus obtain

$$(\bigstar) \leq C_1 C_2^{2j} j!^2 (1+|\xi|+|\tau|)^\sigma \sum_{\xi',\tau'} |\widehat{a}(\xi-\xi',\tau-\tau',\xi',\tau')||\xi'|^{2(2k-j)}|\hat{u}(\xi',\tau')|^2,$$

where C_2 depends on δ, σ and N. Therefore the right hand side of (2.6) is estimated by

(2.7)

$$C_1 \sum_{j=0}^{2k} \frac{C_2^j (2k)!}{(2k-j)!} \times \Bigg(\underbrace{\sum_{\xi,\tau} (1+|\xi|+|\tau|)^\sigma \sum_{\xi',\tau'} |\widehat{a}(\xi-\xi',\tau-\tau',\xi',\tau')||\xi'|^{2(2k-j)}|\hat{u}(\xi',\tau')|^2}_{(\bigstar\bigstar)} \Bigg)^{1/2}.$$

Again, since $a \in SL_\omega^\sigma(\mathbb{T}^N)$, we can estimate

$$\begin{aligned}(\bigstar\bigstar) &\leq C_1 \sum_{\xi,\tau}(1+|\xi|+|\tau|)^\sigma \\ &\qquad \times \sum_{\xi',\tau'}(1+|\xi'|+|\tau'|)^\sigma e^{-\delta(|\xi-\xi'|+|\tau-\tau'|)}|\xi'|^{2(2k-j)}|\hat{u}(\xi',\tau')|^2 \\ &\leq C_1 \sum_{\xi',\tau'}(1+|\xi'|+|\tau'|)^{2\sigma}|\xi'|^{2(2k-j)}|\hat{u}(\xi',\tau')|^2 \\ &\qquad \times \sum_{\xi,\tau}(1+|\xi-\xi'|+|\tau-\tau'|)^\sigma e^{-\delta(|\xi-\xi'|+|\tau-\tau'|)} \\ &\leq C_3^2 \|\Delta_x^{2k-j} u\|_\sigma^2.\end{aligned}$$

Replacing this in (2.7), we conclude that

$$\begin{aligned}\|\Delta_x^k A u\| &\leq C_4 \|\Delta_x^k u\|_\sigma + C_4 C_2^{2k}(2k)!\|u\|_\sigma \\ &\quad + C_4 \sum_{j=1}^{2k-1} \frac{C_2^j (2k)!}{(2k-j)!} \sum_{\xi',\tau'} (1+|\xi'|+|\tau'|)^{2\sigma}|\xi'|^{2(2k-j)}|\hat{u}(\xi',\tau')|^2.\end{aligned}$$

Since

$$|\xi'|^{2k-j} \leq \frac{2k-j}{2k}\rho_j |\xi'|^{2k} + \frac{j}{2k}\rho_j^{-\frac{2k-j}{j}},$$

where ρ_j are positive arbitrary constants, we can estimate the sum in the preceding inequality by

$$C_4 \sum_{j=0}^{2k-1} \frac{C_2^j (2k-1)!}{(2k-j-1)!}\, \rho_j \|\Delta_x^k u\|_\sigma + C_4 \sum_{j=1}^{2k} \frac{C_2^j (2k-1)!}{(2k-j)!}\, j\rho_j^{-\frac{2k-j}{j}} \|u\|_\sigma.$$

Choosing $\rho_j = (2C_2)^{-j}(2k-j-1)!((2k-1)!)^{-1}$, we conclude that

$$\|\Delta_x^k A u\| \leq C_4 \|\Delta_x^k u\|_\sigma + 2C_4 C_5^{2k} k!^2 \|u\|_\sigma.$$

Summing up, we have obtained

$$\|\Delta_x^k u\|_\varepsilon \leq C\|\Delta_x^k f\|_0 + C^{2k+1}k!^2(\|f\|+\|u\|_\sigma) + C\|\Delta_x^k u\|_\sigma.$$

Since by hypothesis $\sigma < \varepsilon$, again an interpolation argument gives

$$\|\Delta_x^k u\|_\varepsilon \leq C\|\Delta_x^k f\|_0 + C^{2k+1}k!^2(\|f\| + \|u\|_\sigma),$$

which shows that $u \in G^1(\mathbb{T}^N; \Delta_x)$. By Proposition 1.1 we conclude that $(x, t, \xi, \tau) \notin WF_a(u)$, the analytic wave front set of u, if $\xi \neq 0$. On the other hand $(x, t, 0, \tau) \notin WF_a(u)$ if $\tau \neq 0$ since $P+A$ is an analytic pseudodifferential operator on $\mathbb{T}^N$ which is elliptic at this point. Consequently u is an analytic function, and our proof is now complete. □

3. Analytic and Gevrey vectors for operators in the class $\mathfrak{O}(\mathbb{T}^N)$

A. Our main theorem in this section is the following.

Theorem 3.1. *Let $P \in \mathfrak{O}(\mathbb{T}^N)$ be a sum of squares, that is, P is as in section* 1.C *with $L_0 = 0$. Also let r be as in* (A-1). *Then $G^s(\mathbb{T}^N; P) \subset G^{rs}(\mathbb{T}^N)$, for every $s \geq 1$. Moreover this result is sharp.*

Notice that $r = 1$ corresponds to the case when P is elliptic and this statement is thus in accordance with the well-known local results (cf. [13], [16]).

Proof. Recall from the discussion in Section 2 that P is $2/r$-subelliptic. In particular if $u \in G^s(\mathbb{T}^N; P)$, then u belongs to $C^\infty(\mathbb{T}^N)$ and for some constant $C > 0$ it holds that

$$\|v\|_{2/r} \leq C\left(\|Pv\| + C^k\|v\|_{-2k}\right), \quad k \in \mathbb{Z}_+, \quad v \in C^\infty(\mathbb{T}^N).$$

Replacing v by $\Delta_x^k u$ gives

$$\begin{aligned}\|\Delta_x^k u\|_{2/r} &\leq C\left(\|\Delta_x^k Pu\| + C^k\|u\|\right)\\ &\leq C\left(\|\Delta_x^{k-1/r} Pu\|_{2/r} + C^k\|u\|\right).\end{aligned}$$

If we iterate the process we obtain

$$\|\Delta_x^k u\|_{2/r} \leq C^q\|\Delta_x^{k-q/r}P^q u\|_{2/r} + C^{k+q}\sum_{i=0}^{q-1}\|P^i u\|.$$

Choosing $q = kr$ implies

$$\|\Delta_x^k u\| \leq C^{kr}\|P^{kr}u\|_{2/r} + C^{kr+r}\sum_{i=0}^{kr-1}\|P^i u\| \leq A^{kr+1}(kr)!^{2s}$$

for some constant $A > 0$. Then $u \in G^{sr}(\mathbb{T}^N; \Delta_x)$. By Proposition 1.1 we conclude that $(x, t, \xi, \tau) \notin WF_{sr}(u)$ if $\xi \neq 0$. On the other hand $(x, t, 0, \tau) \notin WF_s(u)$ if $\tau \neq 0$ since P is elliptic at this point (cf. Proposition 1.1). Consequently $u \in G^{rs}(\mathbb{T}^N)$.

To prove the sharpness of our result consider an operator $Q = \sum_{j=1}^{\nu} L_j^2 \in \mathfrak{O}(\mathbb{T}^2)$ which is formally self-adjoint and for which $r = 2$. One example of such an operator is the periodic version of the Baouendi-Grushin operator in $\mathbb{T}^2$,

$$Q_0 = \partial_t^2 + (\sin t)^2\partial_x^2.$$

We will show that $G^s(\mathbb{T}^2; Q) \not\subset G^\sigma(\mathbb{T}^N)$ for every $s \geq 1$ and every $s < \sigma < 2s$.

Indeed suppose that for a pair of real numbers $s_* \geq 1$ and $\sigma_* \in]s_*, 2s_*[$ we have $G^{s_*}(\mathbb{T}^2, ; Q) \subset G^{\sigma_*}(\mathbb{T}^N)$. From [16, Theorem 3.5], or rather its natural extension when the open set Ω is replaced by $\mathbb{T}^N$, we conclude the validity of the estimate

$$\|v\|_{2s_*/\sigma_*} \leq C\left\{\|Qv\| + \|v\|\right\}, \quad v \in C^\infty(\mathbb{T}^N).$$

Since $2s_*/\sigma_* > 1$ and since Q has order 2, this inequality implies that Q is subelliptic with loss of $\delta < 1$ derivatives. On the other hand if $q(x,t,\xi,\tau)$ denotes the principal symbol of Q, then $q \leq 0$ and hence $dq = 0$ when $q = 0$. By [12, Proposition 27.1.7] this gives a contradiction, which proves our claim. □

4. Global analytic hypoellipticity for a class of second order operators on compact Lie groups

A. Let G be a compact Lie group of dimension m and let M be a compact real-analytic Riemannian manifold of dimension n. We set $\Omega = G \times M$. Let dx be the Haar measure on G and dt the volume density on M induced by its Riemannian metric. We have on Ω a structure of a Riemannian manifold, whose associated Laplace-Beltrami operator will be denoted by Δ_Ω.

Let $0 = \lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_j \leq \ldots$ be the sequence of eigenvalues of Δ_Ω and let φ_j, $j = 0, 1, \ldots$, be the correponding sequence of eigenfunctions. We can suppose that $\{\varphi_j\}$ is an orthonormal basis for $L^2(\Omega)$, where the latter is defined by the product measure $dxdt$. For $s \in \mathbb{R}$ the Sobolev space $H^s(\Omega)$ can be identified to the set of all $u \in \mathcal{D}'(\Omega)$ such that

$$\|f\|_s^2 = \sum_{j=0}^{\infty} (1+\lambda_j)^s \, |\hat{u}_j|^2 < \infty,$$

where $\hat{u}_j = u(\overline{\varphi_j})$. Notice the following interpolation inequality, which is valid for $\delta > 0$ and r, s and $t \in \mathbb{R}$ such that $r < s < t$:

$$\|f\|_s \leq \delta \|f\|_t + \delta^{-\frac{s-r}{t-s}} \|f\|_r, \quad f \in H^t(\Omega).$$

Following the convention adopted in the preceding sections we shall write $\|\cdot\|_0 = \|\cdot\|$.

B. As before a real-analytic partial differential operator P defined on Ω is said to be *globally analytic hypoelliptic* if $u \in \mathcal{D}'(\Omega)$ and $Pu \in C^\omega(\Omega)$ imply $u \in C^\omega(\Omega)$.

We now state and prove our final result. Let $Y_1, \ldots, Y_m$ be a basis for the Lie algebra of G.

Theorem 4.1. *Let $Z_0, Z_1, \ldots, Z_\nu$ be real-analytic vector fields in M such that $Z_1, \ldots, Z_\nu$ span TM at each point. Define*

$$L_j = Z_j + \sum_{i=1}^{m} b_{ji} Y_i, \quad j = 0, 1, \ldots, \nu,$$

where $b_{ji} \in C^\omega(M)$ and assume that $\{L_0, L_1, \ldots, L_\nu\}$ satisfies Hörmander's condition at every point of Ω. Then for every $a \in C^\omega(\Omega)$ the operator

$$P = \sum_{j=1}^{\nu} L_j^2 + L_0 + a$$

is globally analytic hypoelliptic in Ω.

The proof of Theorem 4.1 will be carried out following the same lines of the proof of [6, Theorem 1.1] although it is technically a bit more involved. The key point is the following result.

Lemma 4.1. *There is a basis $X_1, \dots, X_m$ for the Lie algebra of G such that*

$$\Delta_G \doteq -\sum_{j=1}^{m} X_j^2$$

is an elliptic operator on G which commutes with all elements of the Lie algebra of G. In particular, considering Δ_G as an operator in Ω, Δ_G commutes with all vector fields L_j and then

$$[P - a, \Delta_G] = 0. \tag{4.1}$$

Proof. By [8, Theorem 3.6.2] we can decompose the Lie algebra $\mathfrak{g}$ of G as $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{z}$, where $\mathfrak{g}'$ is semisimple, that is, the Killing form on $\mathfrak{g}$ is definite negative, and $\mathfrak{z}$ is the center of $\mathfrak{g}$. As in [7] we can find a basis $X_1, \dots, X_p$ of $\mathfrak{g}'$ such that $-X_1^2 - \dots - X_p^2$ commutes with every element of $\mathfrak{g}'$ and hence with every element of $\mathfrak{g}$. To complete the argument it suffices to select any basis $X_{p+1}, \dots, X_m$ of $\mathfrak{z}$; the basis $X_1 \dots, X_m$ of $\mathfrak{g}$ satisfies the required properties. ☐

Proof of Theorem 4.1. Let $u \in \mathcal{D}'(\Omega)$ be such that $Pu = f \in C^\omega(\Omega)$. By [11] P is hypoelliptic and as in Section 2.B we can find constants $C > 0$ and $\varepsilon > 0$ such that

$$\|v\|_\varepsilon \leq C\left(\|Pv\| + \|v\|\right), \quad v \in C^\infty(\Omega). \tag{4.2}$$

In particular $u \in C^\infty(\mathbb{T}^N)$.

Next we observe the decomposition $T^*\Omega = T^*G \times T^*M$ and that P is elliptic at any point $\theta = (\theta_1, \theta_2)$, where θ_1 projects to the origin at the fiber and $\theta_2 \in T^*M \setminus 0$. Hence the theorem will be proved (cf. the argument at the end of the proof of Theorem 2.1) if we can show that u is a global analytic vector for Δ_G, that is, that there exists a constant $B > 0$ such that

$$\|\Delta_G^j u\| \leq B^{j+1} j!^2 \text{ if } j \leq k \tag{4.3}$$

for every $k \geq 1$. The proof of (4.3) will be carried out by induction on k, its validity for $k = 1$ granted if we choose B sufficiently large.

Before we continue we pause for two auxiliary results.

Lemma 4.2. *If* (4.3) *holds for fixed B and k, then*

$$\|X_I \Delta_G^{j-|I|} u\| \leq B^{j-|I|/2+1} j!(j - |I|)! \tag{4.4}$$

is valid for $j \leq k$ and for all ordered sequences $I = (i_1, \dots, i_\ell)$ of elements in $\{1, \dots, m\}$, with length $|I| \doteq \ell \leq j$.

Proof. From the inequality

$$\|X_I v\|^2 \leq \langle \Delta_G^{|I|} v, v\rangle \leq \|\Delta_G^{|I|} v\|\|v\|, \quad v \in C^\infty(\Omega),$$

we obtain, setting $v = \Delta_G^{j-|I|} u$ (here j and I are as in the statement),

$$\|X_I \Delta_G^{j-|I|} u\| \leq \|\Delta_G^{j} u\|^{1/2} \|\Delta_G^{j-|I|} u\|^{1/2} \leq \left\{B^{j+1} j!^2\right\}^{1/2} \left\{B^{j-|I|+1} (j-|I|)!^2\right\}^{1/2},$$

which gives the result. □

For the next result we omit the proof, which follows by an induction argument. The notation is as in Lemma 4.2.

Lemma 4.3. *If $f, g \in C^\infty(G)$ and $k \in \mathbb{Z}_+$, then*

$$\Delta_G^k(fg) = \sum_{\ell=0}^{k} 2^\ell \binom{k}{\ell} \sum_{j=0}^{k-\ell} \binom{k-\ell}{j} \sum_{|I|=\ell} (X_I \Delta_G^j f)(X_I \Delta_G^{k-\ell-j} g).$$

We now return to the proof of Theorem 4.1. We first select a constant $A > 0$ such that

$$\|X_I \Delta_G^k f\| \leq A^{k+|I|/2+1} k!^2 |I|!, \qquad \|X_I \Delta_G^k a\|_\infty \leq A^{k+|I|/2+1} k!^2 |I|!$$

for all $k \in \mathbb{Z}_+$ and $I = (i_1, \ldots, i_{|I|})$.

We assume the validity of (4.3) for k replaced by $k-1$ and take $v = \Delta_G^k u$ in (4.2). Making use of (4.1) we have, for some fixed $C > 0$,

$$\|\Delta_G^k u\|_\varepsilon \leq C \left(\|\Delta_G^k f\| + \|[\Delta_G^k, a]u\| + \|\Delta_G^k u\|_{-2}\right). \tag{4.5}$$

Now Lemma 4.3 gives the estimate

$$\begin{aligned} \|[\Delta_G^k, a]u\| &= \|\Delta_G^k(au) - a\Delta_G^k u\| \\ &\leq \sum_{\ell=1}^{k} 2^\ell \binom{k}{\ell} \sum_{j=0}^{k-\ell} \binom{k-\ell}{j} \sum_{|I|=\ell} \|X_I \Delta_G^j a\|_\infty \|X_I \Delta_G^{k-\ell-j} u\| \\ &\quad + \sum_{j=1}^{k} \binom{k}{j} \|\Delta_G^j a\|_\infty \|\Delta_G^{k-j} u\| \doteq S_1 + S_2. \end{aligned}$$

Firstly we observe that by the induction hypothesis we can estimate

$$S_2 \leq \sum_{j=1}^{k} \binom{k}{j} A^{j+1} j!^2 B^{k-j+1} (k-j)!^2 \leq AB^{k+1} k!^2 \sum_{j=1}^{k} \left(\frac{A}{B}\right)^j.$$

For the other term we use again the induction hypothesis in conjunction with Lemma 4.2. We obtain

$$\begin{aligned} S_1 &\leq \sum_{\ell=1}^{k} 2^\ell \binom{k}{\ell} \sum_{j=0}^{k-\ell} \binom{k-\ell}{j} \sum_{|I|=\ell} A^{j+\frac{\ell}{2}+1} j!^2 \ell! B^{k-\frac{\ell}{2}-j+1} (k-j)!(k-j-\ell)! \\ &\leq AB^{k+1} k!^2 \sum_{\ell=1}^{k} 2^\ell m^\ell \sum_{j=0}^{k-\ell} \left(\frac{A}{B}\right)^{j+\ell/2}. \end{aligned}$$

Finally, if $\gamma > 0$ is such that $\|\Delta_G v\|_{-2} \leq \gamma\|v\|$, $v \in C^\infty(\Omega)$, gathering together in (4.5) all the information obtained, and assuming $A \ll B$, we get

$$\|\Delta_{_G}^k u\| \leq CB^{k+1}k!^2 \left(\frac{A^{k+1}}{B^{k+1}} + A\sum_{j=1}^{k}\left(\frac{A}{B}\right)^j + A\sum_{\ell=1}^{k} 2^\ell m^\ell \sum_{j=0}^{k-\ell}\left(\frac{A}{B}\right)^{j+\ell/2} + \frac{\gamma}{Bk^2}\right)$$
$$\leq CB^{k+1}k!^2\left(\frac{A}{B} + \frac{A^2}{B-A} + \frac{8mA}{B^{1/2}} + \frac{\gamma}{B}\right).$$

If we choose $B > 0$ so large such that the term between parentheses is $\leq 1/C$, our induction argument proves that (4.3) indeed holds for every k. The proof of Theorem 4.1 is now complete. □

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