

## FINITE GROUPS WITH ODD SYLOW NORMALIZERS

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ABSTRACT. We determine the non-abelian composition factors of the finite groups with Sylow normalizers of odd order. As a consequence, among others, we prove the McKay conjecture and the Alperin weight conjecture for these groups at these primes.

### 1. INTRODUCTION

Suppose that  $G$  is a finite group,  $p$  is an odd prime number, and  $P$  is a Sylow  $p$ -subgroup of  $G$ . In [GMN], the first two authors and G. Malle determined the non-abelian composition factors of  $G$  if  $P = \mathbf{N}_G(P)$ . This led to proving a strong form of the McKay conjecture in [N] and [NTV] for these groups at the prime  $p$ . Now, instead of assuming that  $\mathbf{N}_G(P)/P$  is trivial, we assume that it has odd order. Although in this case the structure of  $G$  can be fairly complicated, we nevertheless are able to have control on the non-abelian composition factors of  $G$ .

**Theorem A.** *Let  $G$  be a finite group, let  $p$  be a prime, and  $P \in \text{Syl}_p(G)$ . Assume that  $|\mathbf{N}_G(P)|$  is odd. If  $S$  is a non-abelian composition factor of  $G$ , then  $|S|$  is divisible by  $p$ , and either  $S$  has cyclic Sylow  $p$ -subgroups or  $S = \text{PSL}_2(q)$ , for some power  $q = p^f \equiv 3 \pmod{4}$ .*

There are too many almost simple groups with odd Sylow normalizers to be listed. However, once Theorem A is proved, we can apply the results in [IMN2], [KS], [S1] and [S2] to establish the main counting conjectures for groups with odd Sylow normalizers. (Recall that  $\text{Irr}_{p'}(G)$  is the set of the irreducible complex characters of  $G$  with degree not divisible by  $p$ .)

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**Corollary B.** *Let  $G$  be a finite group, let  $p$  be a prime and  $P \in \text{Syl}_p(G)$ . If  $|\mathbf{N}_G(P)|$  is odd, then*

$$|\text{Irr}_{p'}(G)| = |\text{Irr}_{p'}(\mathbf{N}_G(P))|.$$

*Furthermore, the Alperin-McKay conjecture and the blockwise Alperin weight conjecture hold true for  $G$ , for the prime  $p$ .*

In view of the recent proof by G. Malle and B. Späth [MS] of the McKay conjecture for the prime 2, we can now state that the McKay conjecture holds for the prime  $p$  and for all finite groups  $G$  whenever  $|\mathbf{N}_G(P)/P|$  is odd for  $P \in \text{Syl}_p(G)$ .

Contrary to the case where  $\mathbf{N}_G(P) = P$  (see [N] and [NTV]), there does not seem to exist a canonical choice-free bijection  $\text{Irr}_{p'}(G) \rightarrow \text{Irr}_{p'}(\mathbf{N}_G(P))$  if  $|\mathbf{N}_G(P)|$  is odd. (If  $G$  is solvable, A. Turull did find a canonical bijection in [T].) At least, we know for a fact that there cannot exist a bijection that commutes with complex conjugation. In this case, the trivial character would be the only real-valued irreducible character of  $G$  of  $p'$ -degree, and this is plainly false: it was proved in [IMN1] that for  $p > 3$  every non-solvable group has a non-trivial real-valued irreducible character of  $p'$ -degree. On the other hand, we prove the following.

**Theorem C.** *Let  $G$  be a finite group and let  $p$  be a prime, let  $P \in \text{Syl}_p(G)$ , and assume that  $|\mathbf{N}_G(P)|$  is odd. If  $\chi \in \text{Irr}_{p'}(G)$  is real-valued, then  $N \leq \ker(\chi)$  for every solvable  $N \triangleleft G$ .*

Finally, we remark that it is a basic problem in character theory to investigate how the character table of a finite group reflects (and is reflected by) its group theoretical properties. We wished to be able to detect from the character table of  $G$  whether or not  $|\mathbf{N}_G(P)|$  is odd, but this seems to be an elusive problem (at least, if  $P$  is not abelian). When finding properties which are detectable in the character table (as Theorem C), it is easy to prove that if  $|\mathbf{N}_G(P)|$  has odd order, then the only real class of  $\bar{G}$  with  $p'$ -size is the identity, for every factor group  $\bar{G}$  of  $G$ . This condition characterizes that  $|\mathbf{N}_G(P)|$  is odd in many groups, such as solvable groups, or groups with an abelian Sylow  $p$ -subgroup. (We write down proofs of these elementary results in Section 4 below.) A general if and only if condition, however, remains to be discovered.

## 2. PROOFS OF THEOREM A AND COROLLARY B

Our results rely on the following result which will be proved in the next section:

**Theorem 2.1.** *Suppose that  $G$  is an almost simple group with socle  $S$ . Let  $P \in \text{Syl}_p(G)$  such that  $G = SP$ ,  $p$  divides  $|S|$ , and  $|\mathbf{N}_G(P)|$  is odd. Then either  $S$  has cyclic Sylow  $p$ -subgroups or  $S = \text{PSL}_2(q)$  for some power  $q = p^f \equiv 3 \pmod{4}$ .*

First we prove Theorem A, which we restate:

**Theorem 2.2.** *Let  $G$  be a finite group, let  $p$  be a prime, and  $P \in \text{Syl}_p(G)$ . Assume that  $|\mathbf{N}_G(P)|$  is odd. If  $S$  is a non-abelian composition factor of  $G$ , then  $|S|$  is divisible by  $p$ , and either  $S$  has cyclic Sylow  $p$ -subgroups or  $S = \text{PSL}_2(q)$ , for some power  $q = p^f \equiv 3 \pmod{4}$ .*

*Proof.* We argue by induction on  $|G|$ . Since the hypotheses are inherited by factor groups, it is enough to show that if  $N$  is a minimal non-abelian normal subgroup of  $G$ , and  $S \triangleleft N$  is simple, then  $|S|$  is divisible by  $p$  and either  $S$  has cyclic Sylow  $p$ -subgroups or  $S = \text{PSL}_2(q)$ , for some prime power  $q = p^f \equiv 3 \pmod{4}$ . First of all,

if  $|N|$  is not divisible by  $p$ , then  $\mathbf{C}_N(P) = \mathbf{N}_N(P)$  has odd order, and therefore  $N$  is solvable by a standard application of the Classification of Finite Simple Groups to coprime action (see Theorem 3.4 of [IMN1]). Hence, we may assume that  $|S|$  has order divisible by  $p$ . Now,  $NP$  has an odd order Sylow normalizer, so we may assume that  $NP = G$ . Let  $Q = P \cap N$ . We have that  $\mathbf{N}_G(P)/P$  is isomorphic to  $\mathbf{C}_{\mathbf{N}_N(Q)/Q}(P)$ .

We can write  $N = S^{x_1} \times \dots \times S^{x_t}$ , where  $U = \{x_1, \dots, x_t\}$  is a complete set of representatives of right cosets of  $\mathbf{N}_P(S)$  in  $P$ . Set  $R = P \cap S = Q \cap S$ ,  $H = \mathbf{S}\mathbf{N}_P(S)$  and  $P_1 = \mathbf{N}_P(S) \in \text{Syl}_p(H)$ . We have that  $\mathbf{N}_H(P_1)/P_1$  is isomorphic to  $\mathbf{C}_{\mathbf{N}_S(R)/R}(P_1)$ . If  $xR \in \mathbf{C}_{\mathbf{N}_S(R)/R}(P_1)$  is an involution, then it is straightforward to check that

$$z := \prod_{u \in U} x^u \in \mathbf{N}_N(Q)$$

and that  $zQ$  is an involution centralized by  $P$ . But this is impossible since  $\mathbf{C}_{\mathbf{N}_N(Q)/Q}(P)$  has odd order by hypothesis. Hence, we conclude that  $H$  is an almost simple group with socle  $S$  and  $H/S$  a  $p$ -group having a Sylow  $p$ -subgroup  $P_1$  with normalizer of odd order. Now we apply Theorem 2.1.  $\square$

Once Theorem A is completed, Corollary B, which we restate, easily follows.

**Corollary 2.3.** *Let  $G$  be a finite group, let  $p$  be a prime and  $P \in \text{Syl}_p(G)$ . If  $|\mathbf{N}_G(P)|$  is odd, then*

$$|\text{Irr}_{p'}(G)| = |\text{Irr}_{p'}(\mathbf{N}_G(P))|.$$

*Moreover, the Alperin-McKay conjecture and the blockwise Alperin weight conjecture hold true for  $G$ , for the prime  $p$ .*

*Proof.* It is well known (see e.g. [Is, Corollary (11.21)]) that if a prime  $r$  divides the order of the Schur multiplier of a non-abelian simple group  $S$ , then the Sylow  $r$ -subgroups of  $S$  are non-cyclic. In particular, if Sylow  $p$ -subgroups of  $S$  are cyclic, then the same holds for its universal cover. Hence, it follows from [KS, Theorem 1.1, Corollary 6.9, Corollary 7.1] that simple groups with cyclic Sylow  $p$ -subgroups satisfy the inductive Alperin-McKay condition and the inductive condition for the blockwise Alperin weight conjecture (for the prime  $p$ ). Next, it is proved in [S1] and [S2] that  $PSL_2(q)$  with  $q = p^f \equiv 3 \pmod{4}$  also satisfies both inductive conditions (although the case  $q = 3^f > 9$  is not explicitly stated there, but follows with the same argument). Now, if  $G$  is a finite group with odd Sylow normalizers, it follows from Theorem A, elementary group theory, and the classification of the subgroups of  $PSL_2(q)$  with  $q \equiv 3 \pmod{4}$ , that every non-abelian composition factor of order divisible by  $p$  of every subgroup of  $G$  either has cyclic Sylow  $p$ -subgroups, or is isomorphic to some  $PSL_2(q')$  with  $q' \equiv 3 \pmod{4}$ . Hence the main results of [IMN2], [S1] and [S2] apply.  $\square$

### 3. PROOF OF THEOREM 2.1

We begin with a simple observation:

**Lemma 3.1.** *Keep the notation and hypothesis of Theorem 2.1. Then  $S \cap P$  contains a non-real  $p$ -central element  $z$  of order  $p$ . In fact, every  $p$ -element  $1 \neq y \in S$  that is  $p$ -central in  $G$  must be non-real.*

*Proof.* By assumption,  $1 \neq Q := P \cap S \triangleleft P$ , and so we can choose  $1 \neq z \in \Omega_1(\mathbf{Z}(P)) \cap Q$ . Since  $p > 2$  and  $2 \nmid |\mathbf{N}_G(P)|$ ,  $z$  is not real in  $\mathbf{N}_G(P)$ , whence  $z$  is not real in  $G$  (and so in  $S$ ) by Burnside’s fusion control lemma. The second conclusion also follows by applying the same argument to  $y$ .  $\square$

**Corollary 3.2.** *Suppose that the simple group  $S$  in Theorem 2.1 is of Lie type defined over  $\mathbb{F}_q$ .*

- (i) *If  $p \nmid q$ , then  $S$  can only be possibly of types  $A_n$  or  ${}^2A_n$  with  $n \geq 2$ ,  $D_n$  or  ${}^2D_n$  with  $2 \nmid n \geq 5$ ,  $E_6$  or  ${}^2E_6$ .*
- (ii) *Suppose  $p|q$ . Then either  $q \equiv 3 \pmod{4}$ , or  $p \in \{3, 5\}$  and  $S$  is an exceptional group of Lie type. Furthermore,  $S$  is not of types  $D_4, {}^2D_4, {}^3D_4$ .*

*Proof.* Recall that  $p$  is odd.

(i) Assume the contrary. By [TZ, Proposition 3.1] all semisimple elements in  $G$  are real. In particular, the element  $z$  described in Lemma 3.1 is real, a contradiction. Note that our argument also applies to  ${}^2F_4(2)'$ .

(ii) Assume the contrary. By Lemma 2.2 and Theorem 1.4 of [TZ], all unipotent elements in  $G$  are real. Hence the element  $z$  described in Lemma 3.1 is real, again a contradiction.  $\square$

**Lemma 3.3.** *Let  $S = A_n$  be an alternating group.*

- (i) *If  $n \geq p + 2$ , then any element  $t$  of order  $p$  in  $S$  is real. In particular,  $|\mathbf{N}_S(Q)|$  is even for  $Q \in \text{Syl}_2(S)$ .*
- (ii) *Similarly, if  $n \geq 2$  and  $P \in \text{Syl}_p(S_n)$ , then  $|\mathbf{N}_{S_n}(P)|$  is even.*
- (iii) *Theorem 2.1 holds if  $S = A_n$  is an alternating group.*

*Proof.* (i) Write  $t$  as a product of  $b$  disjoint cycles. Then  $t$  has  $a := n - bp$  fixed points. As  $n \geq p + 2$ , we have that  $a \geq 2$  or  $b \geq 2$ . Hence  $t$  is centralized by an odd permutation, and so it is real in  $S$ . Arguing as in the proof of Lemma 3.1, we see that  $|\mathbf{N}_S(Q)|$  is even.

(ii) The statement is obvious if  $n < p$ . If  $n \geq p$ , then any element (of order  $p$ ) in  $S_n$  is real, so we can argue as in the proof of Lemma 3.1.

(iii) Applying (i) to the element  $z$  obtained in Lemma 3.1 we see that  $n = p$  or  $p + 1$ . In particular, the Sylow  $p$ -subgroups of  $S$  are cyclic. (In fact, one can also show that  $p \equiv 3 \pmod{4}$  in this case, but we do not need this stronger conclusion.)  $\square$

**Lemma 3.4.** *Theorem 2.1 holds true if  $S$  is any of the 26 sporadic simple groups.*

*Proof.* Since  $p > 2$ ,  $G = S$  in this case. Now if  $P$  is non-cyclic, then one can check using [W] that  $|\mathbf{N}_G(P)|$  is even. Otherwise  $P$  is cyclic as desired.  $\square$

In what follows, we use the notation  $SL^\epsilon$  to denote  $SL$  when  $\epsilon = +$  and  $SU$  when  $\epsilon = -$ , and similarly for  $GL^\epsilon, PSL^\epsilon$ , etc. We also use  $E_6^\epsilon$  to denote  $E_6$  when  $\epsilon = +$  and  ${}^2E_6$  when  $\epsilon = -$ .

**Proposition 3.5.** *Theorem 2.1 holds if  $S = PSL_n^\epsilon(q)$  with  $n \geq 3$ ,  $\epsilon = \pm$ , and  $p \nmid q$ .*

*Proof.* (a) We view  $S = L/\mathbf{Z}(L)$ , where  $L := SL_n^\epsilon(q) \triangleleft H := GL(V) \cong GL_n^\epsilon(q)$ , and  $V = \mathbb{F}_q^n$  for  $\epsilon = +$ ,  $V = \mathbb{F}_{q^2}^n$  for  $\epsilon = -$ . Fix a basis  $(e_1, \dots, e_n)$  of  $V$ , which is orthonormal if  $\epsilon = -$ . If  $q = r^f$  for a prime  $r$ , then we can use this basis to define

a field automorphism  $\sigma$  of  $H$ ,  $L$ , and  $S$ , which sends a matrix  $(x_{ij}) \in H$  (written in the chosen basis) to  $(x_{ij}^r)$ , and let  $\sigma_0$  denote the  $p$ -part of  $\sigma$ .

We will use the description of a Sylow  $p$ -subgroup  $R$  of  $H$  as given in [Hall, §2] using the results of [GL, Chap. 4, §10] and [GLS, Chap. 4, §4.10]. In particular,  $R = R_T \rtimes R_W$ , with  $R_T$  (the ‘‘toral part’’ of  $R$ ) being homocyclic abelian. Furthermore, there is a direct sum (orthogonal if  $\epsilon = -$ ) decomposition of  $V$  (compatible with the chosen basis of  $V$ ) as  $R_T$ -module:

$$V = V_0 \perp V_1 \perp \dots \perp V_m,$$

with  $V_i \cong V_1$  for  $1 \leq i \leq m$  of dimension  $e := \text{ord}_p(\epsilon q)$ , and  $0 \leq \dim V_0 < e$ . Next,  $R_T = R_1 \times \dots \times R_m$  with  $R_i$  a cyclic subgroup of a cyclic maximal torus  $T_i$  of  $GL^\epsilon(V_i)$  and  $|T_i| = q^e - \epsilon^e$ . Furthermore, there is a subgroup  $\Sigma$  of  $\mathbf{N}_H(R_T)$  with  $\Sigma$  isomorphic to the symmetric group  $\mathbf{S}_m$  acting naturally on the sets  $\{V_1, \dots, V_m\}$ , and  $\{R_1, \dots, R_m\}$ , and with  $R_W$  (the ‘‘Weyl part’’ of  $R$ ) being a Sylow  $p$ -subgroup of  $\Sigma$ .

Since  $\sigma_0$  normalizes  $GL^\epsilon(V_1)$ , we can embed  $\sigma_0$  in a Sylow  $p$ -subgroup  $\tilde{R}_1$  of  $GL^\epsilon(V_1) \rtimes \langle \sigma_0 \rangle$ , and then take  $R_1 = \tilde{R}_1 \cap GL^\epsilon(V_1)$  to ensure  $R_1$  to be  $\sigma_0$ -stable. The subgroups  $R_i$  are then constructed using the (fixed) isomorphism  $V_i \cong V_1$  for  $1 \leq i \leq m$ . By its construction,  $R$  is  $\sigma_0$ -stable. Then  $LR$  is  $\sigma_0$ -stable and  $p \nmid |H/LR|$ . Since  $p > 2$  and  $G = SP \leq \text{Aut}(S)$ , by a suitable conjugation in  $\text{Aut}(S)$  we may assume that  $G \leq H^*/\mathbf{Z}(H^*)$ , where  $H^* := LR \rtimes \langle \sigma_0 \rangle$  and  $P \leq P^*/(P^* \cap \mathbf{Z}(H^*))$ , where  $P^* = R \rtimes \langle \sigma_0 \rangle$ . Set  $Q^* := L \cap P^*$ .

(b) Here we consider the case  $p|(q - \epsilon)$ , whence  $e = 1$  and  $m = n$ . Let  $V_i$ ,  $1 \leq i \leq m$ , be spanned by a vector  $e_i$  in such a way that  $\Sigma$  acts on  $\{e_1, e_2, \dots, e_m\}$ .

(b1) Consider the case  $n \geq p + 2$ . Then, by Lemma 3.3(i), we can find an involution  $t \in \mathbf{N}_{A_n}(R_W)$ . By the choice, we have that  $t \in L \setminus \mathbf{Z}(H)$ ; furthermore,  $t$  normalizes  $R_W$ ,  $R_T$ ,  $R$ , and commutes with  $\sigma_0$ . Next, if  $x \in R_1 < R_T$ , then  $txt^{-1}x^{-1} \in L \cap R_T \leq Q^*$ . Since  $P^* = Q^*R_1\langle \sigma_0 \rangle$ , it follows that  $t$  centralizes  $P^*/Q^*$ . We have shown that

$$tQ^* \in \mathbf{C}_{\mathbf{N}_L(Q^*)/Q^*}(P^*/Q^*).$$

Since  $Q = Q^*/(Q^* \cap \mathbf{Z}(H^*)) \in \text{Syl}_p(S)$ , we get

$$tQ \in \mathbf{C}_{\mathbf{N}_S(Q)/Q}(P/Q) = \mathbf{N}_S(P)/Q.$$

Thus we have shown that  $|\mathbf{N}_S(P)|$  is even, a contradiction.

(b2) Next assume that  $n = p + 1$ . Then we can choose  $R_W$  to be generated by the  $p$ -cycle permuting  $e_1, \dots, e_p$  cyclically and fixing  $e_{p+1}$ . By choosing  $t(e_{p+1}) = \pm e_{p+1}$  suitably, we again get a  $\sigma_0$ -invariant monomial matrix  $t \in \mathbf{N}_L(R_W) \setminus \mathbf{Z}(H)$  of order 2, and then repeat the arguments in (b1) to see that  $|\mathbf{N}_S(P)|$  is even.

If  $3 \leq n < p$ , then  $R_W = 1$ . Choosing  $t = \text{diag}(-1, -1, 1, \dots, 1)$  if  $2 \nmid q$  and

$$t : e_1 \leftrightarrow e_2, e_i \mapsto e_i, 3 \leq i \leq n,$$

if  $2|q$ , we then get a  $\sigma_0$ -invariant monomial involution  $t \in \mathbf{N}_L(R_W) \setminus \mathbf{Z}(H)$  and finish as above.

(b3) It remains to consider the case  $n = p$ . Fix  $\omega \in \mathbb{F}_{q^2}^\times$  of order  $p$  and consider  $y := \text{diag}(\omega, \omega^2, \dots, \omega^{p-1}, 1)$ . We can also choose  $R_W$  to be generated by the  $p$ -cycle permuting  $e_1, \dots, e_p$  cyclically. Then one can check that  $y \in \mathbf{Z}(Q)$ . Note that  $p|(q - \epsilon)$  implies that  $\omega^{\sigma_0} = \omega$  and so  $[y, \sigma_0] = 1$ . Also,  $y$  commutes with  $T_1$ .

Since  $P^* = Q^*R_1\langle\sigma_0\rangle$ , we see that  $y \in \mathbf{Z}(P) \cap Q$ . On the other hand,  $y$  is inverted by the element

$$v : e_i \leftrightarrow e_{p-i}, \quad 1 \leq i \leq p-1, \quad e_p \mapsto (-1)^{(p-1)/2}e_p,$$

of  $L$ , contradicting Lemma 3.1.

(c) From now on we may assume that  $p \nmid q(q - \epsilon)$ ; in particular,  $R < L$  and  $e > 1$ . Arguing as in (b), we see that it suffices to produce a  $\sigma_0$ -invariant involution  $t \in L \setminus \mathbf{Z}(L)$  that induces an element in  $\mathbf{N}_{S_m}(R_W)$  and normalizes  $R_T$ . Also note that Sylow  $p$ -subgroups of  $S$  are cyclic if  $m = 1$ , so we may assume that  $m \geq 2$ . Now, if  $2|q$ , then  $\mathbf{N}_\Sigma(R_W)$  is contained in  $L$  (and  $\sigma_0$ -fixed pointwise), and so we are done by Lemma 3.3(ii). Similarly, if  $2|e$ , then any transposition in  $\Sigma$  “flips” two  $e$ -dimensional subspaces  $V_i$  and  $V_j$  and so has determinant 1, whence we are again done by Lemma 3.3(ii).

So we may assume  $2 \nmid qe$ . If  $m \geq p + 2$ , then we can apply Lemma 3.3(i). If  $4 \leq m < p$ , then we can take  $t \in \Sigma$  that represents (12)(34) under  $\Sigma \cong S_m$ . If  $p > m = 2, 3$ , then take  $s \in \Sigma$  that represents (12) under  $\Sigma \cong S_m$  and set  $t := (-1_{V_1})s$ . Finally, assume that  $m = p$  or  $p + 1$ . In this case, we may assume that  $R_W$  is generated by the element represented by the  $p$ -cycle  $(1, 2, \dots, p)$  in  $S_m$ , and choose an involution  $s \in \mathbf{N}_\Sigma(R_W)$ ; in particular,  $s$  acts trivially on  $V_{p+1}$  if  $m > p$ . Then we can take  $t = su^j$  with  $u = -1_{V_1 \oplus V_2 \oplus \dots \oplus V_p}$  and  $j \in \{0, 1\}$  chosen suitably.  $\square$

**Proposition 3.6.** *Theorem 2.1 holds if  $S = P\Omega_{2n}^\epsilon(q)$  with  $2 \nmid n \geq 5$ ,  $\epsilon = \pm$ , and  $p \nmid q$ .*

*Proof.* (a) We view  $S = L/\mathbf{Z}(L)$ , where  $L = \Omega(V) \cong \Omega_{2n}^\epsilon(q)$  and  $V = \mathbb{F}_q^{2n}$  is a quadratic space of type  $\epsilon$ , and let  $H = GO(V)$ . Also define  $\epsilon_p = +$  if  $e = \text{ord}_p(q)$  is odd and  $\epsilon_p = -$  if  $2|e$ , and let  $d := \text{lcm}(2, e)$ . We again use the description of  $R = R_T \rtimes R_W \in \text{Syl}_p(H)$  as in part (a) of the proof of Proposition 3.5. The only differences are that  $V_i \cong V_1$  for  $1 \leq i \leq m$  is a quadratic space of dimension  $d$  and type  $\epsilon_p$ , either  $\dim V_0 < d$  or  $\dim V_0 = d$  but  $V_0$  has type  $-\epsilon_p$ , and  $|T_1| = q^{d/2} - \epsilon_p$ .

We will choose a basis compatible with the decomposition  $V = V_0 \perp V_1 \perp \dots \perp V_m$  and the isomorphisms  $V_i \cong V_1$  for  $1 \leq i \leq m$ , and use this basis to define the field automorphism  $\sigma : (x_{ij}) \mapsto (x_{ij}^r)$  if  $r$  is the prime dividing  $q$ . Let  $\sigma_0$  denote the  $p$ -part of  $\sigma$ . As in the proof of Proposition 3.5, we can construct  $R$  in such a way that all  $R_i$  and  $R$  are  $\sigma_0$ -stable. As  $p > 2$ , we may identify  $Q \in \text{Syl}_p(S)$  with  $R$ . Furthermore, since  $n \geq 5$  and  $G = SP$ , conjugating suitably in  $\text{Aut}(S)$ , we may assume that  $P \leq P^* := R \rtimes \langle\sigma_0\rangle$ .

(b) First we consider the case  $V_0 \neq 0$ . Since  $2|d = \dim V_1$ , we then have that  $\dim V_0 \geq 2$ . It follows that  $R < \Omega_{2n-2}^\pm(q) < L$ . Since  $2|(n-1)$ , the element  $z$  obtained in Lemma 3.1 is real in  $\Omega_{2n-2}^\pm(q)$  by [TZ, Proposition 3.1] and so in  $L$  as well, a contradiction.

Next suppose that  $p \nmid m$  but  $m > 1$ . Then we can choose  $R_W \in \text{Syl}_p(\Sigma)$  to be contained in  $S_{m-1}$  under  $\Sigma \cong S_m$ . As the  $p$ -group  $\langle\sigma_0\rangle$  acts on  $\Omega_1(R_m) \neq 1$ , we can take  $1 \neq z \in \Omega_1(R_m)$  to be  $\sigma_0$ -fixed, and then note that  $z \in \mathbf{Z}(P) \cap R$ . On the other hand, [TZ, Proposition 3.1] and the oddness of  $n$  again imply that  $z$  is real in  $\Omega_{2n-2}^\pm(q) < L$ , contradicting Lemma 3.1.

(c) If  $m = 1$ , then Sylow  $p$ -subgroups of  $S$  are cyclic. From now on we may therefore assume that  $V_0 = 0$ ,  $m \geq 2$ , and  $p|m$ . It suffices to produce a  $\sigma_0$ -invariant

involution  $t \in L \setminus \mathbf{Z}(L)$  that induces an element in  $\mathbf{N}_{S_m}(R_W)$  and normalizes  $R_T$ . Indeed, such an element would belong to  $\mathbf{N}_L(P)$  and so  $|\mathbf{N}_S(P)|$  would be even.

If  $m \geq 2p$ , then by Lemma 3.3(i), we can choose an involution  $t \in \mathbf{N}_{A_m}(R_W)$  which is then  $\sigma_0$ -fixed and belongs to  $L \setminus \mathbf{Z}(L)$ , and so we are done.

So we may assume that  $m = p$  and so  $R_W \cong C_p$ . We can find an involution  $s \in \mathbf{N}_\Sigma(R_W)$  which is represented by a disjoint product of  $(p-1)/2$  transpositions  $s_i \in \mathfrak{S}_p$ ,  $1 \leq i \leq (p-1)/2$ . Note that each  $s_i$  flips two  $d$ -dimensional subspaces  $V_k$  and  $V_{k'}$  (and fixes all remaining  $V_i$  pointwise), so  $\det(s_i) = 1$  and  $s \in SO(V)$ .

(d) Here we assume that  $2 \nmid q$ . Since all  $V_i$  have type  $\epsilon_p$  and  $V$  has type  $\epsilon$ , we get  $\epsilon = \epsilon_p$ . Next,  $n = dm/2$  is odd, so  $d \equiv 2n \pmod{4}$ . If  $-1_V \notin \Omega(V)$ , then we can take  $t = (-1_V)^j s$  for a suitable  $j \in \{0, 1\}$  to get  $t \in L \setminus \mathbf{Z}(L)$  as desired.

Assume  $-1_V \in \Omega(V)$ . Then  $\epsilon = (-1)^{n(q-1)/2}$  by [KL, Proposition 2.5.13(ii)], and so  $-1_{V_i} \in \Omega(V_i)$  since  $d \equiv 2n \pmod{4}$  and  $\epsilon = \epsilon_p$ . Consider for instance a flip  $\tau : V_1 \leftrightarrow V_2$  in  $\Sigma$  that sends an orthogonal basis  $(e_1, \dots, e_d)$  of  $V_1$  to a basis  $(f_1, \dots, f_d)$  of  $V_2$ . Let  $Q$  denote the quadratic form on  $V$  and let  $\rho_v$  denote the reflection corresponding to any non-singular  $v \in V$ . As  $-1_{V_i} = \prod_{i=1}^d \rho_{e_i} \in \Omega(V_i)$ , we have that  $\prod_{i=1}^d Q(e_i) \in \mathbb{F}_q^{\times 2}$ . Now  $\tau = \prod_{i=1}^d \rho_{e_i - f_i}$  and

$$\prod_{i=1}^d Q(e_i - f_i) = \prod_{i=1}^d (Q(e_i) + Q(f_i)) = 2^d \prod_{i=1}^d Q(e_i) \in \mathbb{F}_q^{\times 2}.$$

We have shown that  $\tau \in \Omega(V)$ . As a consequence,  $s_i \in \Omega(V) = L$  for all  $i$ , and so we can just set  $t := s = \prod_i s_i$ .

(e) Finally, let  $2|q$ . Again, we consider a flip  $\tau : V_1 \leftrightarrow V_2$  in  $\Sigma$  that sends a basis  $(e_1, \dots, e_d)$  of  $V_1$  to a basis  $(f_1, \dots, f_d)$  of  $V_2$ . Then  $\tau$  fixes the maximal totally singular subspace  $M := \langle e_1 + f_1, \dots, e_d + f_d \rangle_{\mathbb{F}_q}$  of  $V_1 \oplus V_2$ . It follows by [KL, Lemma 2.5.8] that  $\tau \in \Omega(V_1 \oplus V_2) \leq L$ . Hence,  $s_i \in L$  for all  $i$ , and we are done by taking  $t := s = \prod_i s_i$ .  $\square$

The following simple argument is useful in various situations:

**Lemma 3.7.** *Let  $M = NP$  be a finite group with a normal subgroup  $N$ ,  $p$  be a prime, and  $P \in \text{Syl}_p(M)$ . Let  $A$  be a subgroup of  $N$  that contains  $Q := P \cap N$  as a normal subgroup. Suppose that  $M$  acts transitively on the set of  $N$ -conjugates of  $A$ . Then, for any prime  $r$ , some Sylow  $r$ -subgroup  $X$  of  $\mathbf{N}_N(A)/Q$  is fixed by a Sylow  $p$ -subgroup of  $M$  containing  $Q$ .*

*Proof.* Note that  $Q \in \text{Syl}_p(A)$  is normal in  $A$  and so  $Q \triangleleft \mathbf{N}_N(A)$ . By assumption,  $NP = M = \mathbf{N}_M(A)N$ . Hence, without any loss, we may replace  $(N, M)$  with  $(\mathbf{N}_M(A), \mathbf{N}_N(A))$  and assume that  $A \triangleleft M$ . Now the  $p$ -group  $P$  acts on the set of Sylow  $r$ -subgroups of  $N/Q$  which has  $p'$ -size, and so it must have a fixed point.  $\square$

**Proposition 3.8.** *Theorem 2.1 holds if  $S$  is a simple group of type  $E_6^{\epsilon}(q)$  with  $\epsilon = \pm$  and  $p \nmid q$ .*

*Proof.* (i) We can view  $S$  as  $[H, H]$  where  $H := E_6^{\epsilon}(q)_{\text{ad}}$ . If  $p \nmid (q^5 - \epsilon)(q^9 - \epsilon)$ , then we can embed  $Q := P \cap S$  in a subgroup  $X \cong F_4(q)$  of  $S$  and conclude that the element  $z$  obtained in Lemma 3.1 is real in  $X$  by [TZ, Proposition 3.1], a contradiction. On the other hand, if  $p|(q^5 - \epsilon)(q^9 - \epsilon)$  but  $p \nmid (q^3 - \epsilon)$ , then Sylow  $p$ -subgroups of  $S$  are cyclic. So we may assume that  $p|(q^3 - \epsilon)$ .

Suppose that  $p|(q^3 - \epsilon)$  but  $p \nmid (q - \epsilon)$ ; in particular,  $(q, \epsilon) \neq (2, -)$ . By the main result of [LSS] (see Table 5.2 therein),  $H$  has a unique conjugacy class of maximal subgroups  $A$  (of maximal rank) of type  $T \cdot 3^{1+2} \cdot SL_2(3)$ , where  $T$  is a maximal torus of order  $(q^2 + \epsilon q + 1)^3$ . Now we can view  $Q$  as  $\mathbf{O}_p(T)$  and apply Lemma 3.7 to  $(M, N, A) = (GH, H, A)$  to conclude that  $P$  fixes a Sylow 2-subgroup  $B \cong Q_8$  of  $A/Q$  (note that  $GH$  is a group as  $G \leq \text{Aut}(S)$  normalizes  $H$  and that  $\mathbf{N}_H(A) = A = \mathbf{N}_H(Q)$  by maximality of  $A$ ). It then follows that  $P$  fixes the subgroup  $[B, B] \cong C_2$  which is contained in  $\mathbf{N}_S(Q)/Q$ . Thus  $P$  fixes an involution in  $\mathbf{N}_S(Q)/Q$  and so  $|\mathbf{N}_G(P)|$  is even, a contradiction.

Assume now that  $p|(q - \epsilon)$  but  $p \neq 3$  (whence  $p \geq 5$  and  $q \geq 4$ ). By the main result of [LSS] (see Table 5.1 therein),  $H$  has a unique conjugacy class of maximal subgroups  $A$  (of maximal rank) of type  $C_d \cdot (PSL_2(q) \times PSL_6^\epsilon(q)) \cdot C_{de}$ , where  $d = \gcd(2, q - 1)$  and  $e = \gcd(3, q - \epsilon)$ . By the Frattini argument, we may assume that  $P$  normalizes  $A$ . Next, the assumption on  $p$  implies that we can view  $Q$  as a Sylow  $p$ -subgroup of  $A$  and so contained in  $C_d \cdot (PSL_2(q) \times PSL_6^\epsilon(q))$ . As  $P$  normalizes the component  $A_1 := C_d \cdot PSL_2(q)$  of  $[A, A]$ , we may choose the element  $z$  in Lemma 3.1 to be contained in  $A_1$ . But then  $z$  is real in  $A_1$ , again a contradiction.

(ii) Finally, we consider the case  $p = 3|(q - \epsilon)$ . View  $S = L/Z$  for  $L = E_6^\epsilon(q)_{\text{sc}} = \mathcal{G}^F$  and  $Z = \mathbf{Z}(L)$ , where  $\mathcal{G}$  is a simple, simply connected algebraic group of type  $E_6$  over  $\overline{\mathbb{F}}_q$  and  $F : \mathcal{G} \rightarrow \mathcal{G}$  a Frobenius endomorphism. According to [MT, Theorem 25.11], there is a unique  $L$ -conjugacy class of maximal tori  $\mathcal{T}$  in  $\mathcal{G}$  such that  $T := \mathcal{T}^F$  has order  $(q - \epsilon)^6$ , and  $\mathbf{N}_L(\mathcal{T})/T = W(E_6)$  by [MT, Proposition 25.3]. Next, we can view  $Q = \hat{Q}/Z$  for some  $\hat{Q} \in \text{Syl}_3(L)$ . By the Frattini argument, we may assume that  $P$  normalizes  $A := \mathbf{N}_L(\mathcal{T})$  and  $T = \text{sol}(A)$ ; moreover,  $\hat{Q} > \mathbf{O}_3(T)$  and  $\bar{Q} := \hat{Q}T/T \in \text{Syl}_3(W(E_6))$ . Recall that  $W(E_6) = SU_4(2) \rtimes C_2$ . Since  $P$  normalizes  $[A/T, A/T] \cong SU_4(2)$  and  $\bar{Q}$ , we see that  $P$  normalizes the subgroups  $C > B > T$  of  $A$ , where

$$B/T = \mathbf{N}_{SU_4(2)}(\bar{Q}) = \bar{Q} \rtimes C_2, \quad C/T = \mathbf{N}_{W(E_6)}(\bar{Q}) = \bar{Q} \rtimes C_2^2.$$

Setting  $O := \mathbf{O}_3(T)$ , observe that  $\mathbf{N}_B(\hat{Q})O/O = \mathbf{N}_{B/O}(\hat{Q}O/O)$  and so the previous equalities imply that

$$2|\hat{Q}| \text{ divides } |\mathbf{N}_B(\hat{Q})| = |\mathbf{N}_C(\hat{Q})|/2.$$

According to [MN, Table 1],  $\mathbf{N}_L(\hat{Q}) = \hat{Q} \rtimes C_2^2$ . As  $\mathbf{N}_C(\hat{Q}) \leq \mathbf{N}_L(\hat{Q})$ , we conclude that  $\mathbf{N}_B(\hat{Q}) = \hat{Q} \rtimes C_2$ , with the quotient  $\mathbf{N}_B(\hat{Q})/\hat{Q} \cong C_2$  fixed by  $P$ . Since  $\mathbf{N}_S(Q) = \mathbf{N}_L(\hat{Q})/Z$ , we have therefore shown that  $\mathbf{N}_S(Q)/Q$  contains a  $P$ -fixed involution. Thus  $|\mathbf{N}_G(P)|$  is even, again a contradiction.  $\square$

**Proposition 3.9.** *Theorem 2.1 holds if  $S$  is a simple group of Lie type in characteristic  $p$ .*

*Proof.* As  $p > 2$ , we can view  $S = L/Z$  for  $L = \mathcal{G}^F$  and  $Z = \mathbf{Z}(L)$ , where  $\mathcal{G}$  is a simple, simply connected algebraic group over  $\overline{\mathbb{F}}_q$  and  $F : \mathcal{G} \rightarrow \mathcal{G}$  a Steinberg endomorphism. Since  $p \nmid |Z|$ , we can identify  $Q := P \cap S$  with a Sylow  $p$ -subgroup of  $L$ . It is well known that  $\mathbf{N}_L(Q) = QT = \mathcal{B}^F$ , where  $T = \mathcal{T}^F$ ,  $\mathcal{T}$  is an  $F$ -stable maximal torus of  $\mathcal{G}$  contained in an  $F$ -stable Borel subgroup  $\mathcal{B}$  of  $\mathcal{G}$ , see e.g., [GLS, §2.3]. As  $\mathcal{G}$  is not of type  $D_4$  by Corollary 3.2 and  $p > 2$ , it follows that the subgroup of field automorphisms in  $\text{Out}(S)$  contains a Sylow  $p$ -subgroup of  $\text{Out}(S)$ .

Hence, we may assume (using the uniqueness of  $(\mathcal{B}, \mathcal{T})$  up to conjugacy) that  $G \leq H := L \rtimes \langle \sigma \rangle$  for a suitable field automorphism  $\sigma$  and  $\sigma$  acts on  $T$ . Any 2-element in  $T^\sigma$  will then belong to  $\mathbf{C}_{\mathbf{N}_L(Q)/Q}(P)$ ; also,  $\mathbf{N}_S(Q)/Q = (\mathbf{N}_L(Q)/Q)/(QZ/Q)$  and  $QZ/Q \cong Z$ . Hence,  $|\mathbf{N}_G(P)|$  is even whenever the 2-rank  $a$  of  $T^\sigma$  is larger than the 2-rank  $b$  of  $Z$ .

Note that  $T^\sigma = \mathcal{T}^{F'}$  for a suitable Frobenius endomorphism  $F'$  of  $\mathcal{G}$ . Now, if  $\mathcal{G}$  is of type  $D_n$  with  $n \geq 4$ , then  $a \geq 3$  and  $b \leq 2$ . In all other cases  $Z$  is cyclic (see e.g. [GLS, Theorem 2.5.12]) and so  $b \leq 1$ . So we only need to consider the cases where  $a \leq b \leq 1$ , whence  $L$  is of type  $SL_2$  (note that  $(a, b) = (1, 0)$  for  $L$  of type  $SU_3$  or  ${}^2G_2$ ). Finally, when  $L = SL_2(q)$  we must have  $q \equiv 3 \pmod{4}$  by Corollary 3.2(ii).  $\square$

Theorem 2.1 now follows from Corollary 3.2, Lemmas 3.3, 3.4, and Propositions 3.5, 3.6, 3.8, and 3.9.  $\square$

#### 4. ODD SYLOW NORMALIZERS AND CHARACTER TABLES

In this section we prove several reflections (of elementary nature) in the character table of the existence of odd Sylow normalizers. The first of these is our Theorem C.

**Theorem 4.1.** *Let  $G$  be a finite group and let  $p$  be a prime, let  $P \in \text{Syl}_p(G)$ , and assume that  $|\mathbf{N}_G(P)|$  is odd. If  $\chi \in \text{Irr}_{p'}(G)$  is real-valued, then  $N \leq \ker(\chi)$  for every solvable  $N \triangleleft G$ .*

*Proof.* Let  $\chi \in \text{Irr}_{p'}(G)$ . Since  $\chi$  has  $p'$ -degree, it follows that  $\chi_N$  has a  $P$ -invariant constituent  $\theta$ . By the Frattini argument, two such  $P$ -invariant constituents are  $\mathbf{N}_G(P)$ -conjugate. Now, since  $\chi$  is real-valued, it follows that  $\bar{\theta}$  also lies under  $\chi$ . Hence, there exists  $g \in \mathbf{N}_G(P)$  such that  $\theta^g = \bar{\theta}$ . Then  $g^2$  fixes  $\theta$ . Since  $\mathbf{N}_G(P)$  has odd order, it follows that  $g$  fixes  $\theta$ , and therefore, we have that  $\theta$  is a real-valued  $P$ -invariant character. Now, by Lemma 2.1 of [NS], we have that  $\theta$  has a real extension  $\eta$  to  $NP$ . Now,  $NP$  is a solvable group with a Sylow  $p$ -subgroup  $P$  and having odd Sylow normalizer. By Theorem A of [IMN1], we have that  $\eta$  is principal, and therefore  $\theta$  is principal.  $\square$

If  $G$  has no non-trivial real conjugacy classes of  $p'$ -size, then it is not true that this condition is inherited by normal subgroups. Take  $G$  the semidirect product of the extra-special 3-group  $F$  of exponent 3 acted on by an involution that inverts  $F/\mathbf{Z}(F)$  and centralizes  $\mathbf{Z}(F)$ . Set  $p = 3$ . Then  $G/Z$  possesses a real class of size 3.

**Theorem 4.2.** *Suppose that  $G$  is a finite group, let  $p$  be an odd prime, and let  $P \in \text{Syl}_p(G)$ . If  $|\mathbf{N}_G(P)|$  is odd, then for every  $N \triangleleft G$  the only real class of  $G/N$  with  $p'$ -size is the trivial class.*

*Proof.* Suppose that  $|\mathbf{N}_G(P)|$  is odd. If  $N \triangleleft G$ , then  $\mathbf{N}_{G/N}(PN/N) = \mathbf{N}_G(P)N/N \cong \mathbf{N}_G(P)/\mathbf{N}_N(P)$  also has odd order. So assume that  $x^G$  is a real class of  $p'$ -size. Then we may assume that there is some  $t \in G$  such that  $x^t = x^{-1}$  and  $P \leq \mathbf{C}_G(x)$ . Then  $P^t \leq \mathbf{C}_G(x)$  and  $P^t = P^v$  for some  $v \in \mathbf{C}_G(x)$ . Then  $n = tv^{-1} \in \mathbf{N}_G(P)$  and  $x^n = x^{-1}$ . Thus  $x^{n^2} = x$ . Since  $\langle n \rangle = \langle n^2 \rangle$ , we conclude that  $x = x^{-1}$  and so  $x^2 = 1$ . Since  $P \leq \mathbf{C}_G(x)$  we have that  $x \in \mathbf{C}_G(P) \leq \mathbf{N}_G(P)$ . Thus  $x = 1$ , because  $\mathbf{N}_G(P)$  has odd order.  $\square$

If  $\mathbf{N}_G(P)$  has odd order, there is a deeper necessary condition that can be read off the character table (assuming the Galois version of the McKay conjecture [N]): *If  $\sigma$  is the Galois automorphism that complex-conjugates  $p$ -power roots of unity and fixes  $p'$ -roots of unity, then every  $\sigma$ -fixed  $p'$ -degree irreducible character of  $G$  is  $p$ -rational.*

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