

A NOTE ON THE HARNACK INEQUALITY FOR ELLIPTIC EQUATIONS IN DIVERGENCE FORM

DONGSHENG LI AND KAI ZHANG

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ABSTRACT. In 1957, De Giorgi proved the Hölder continuity for elliptic equations in divergence form and Moser gave a new proof in 1960. In the next year, Moser obtained the Harnack inequality. In this note, we point out that the Harnack inequality was hidden in De Giorgi's work.

1. THE HARNACK INEQUALITY

Consider the following elliptic equation:

$$(1.1) \quad (a^{ij}u_i)_j = 0 \quad \text{in } Q_6,$$

where $(a^{ij})_{n \times n}$ is uniformly elliptic with ellipticity constants λ and Λ . In this note, $Q_r(x_0)$ denotes the cube with center x_0 and side-length r , and $Q_r := Q_r(0)$.

In 1961, Moser [8] obtained the following Harnack inequality:

Theorem 1.1. *Let $u \geq 0$ be a weak solution of (1.1). Then*

$$(1.2) \quad \sup_{Q_1} u \leq C \inf_{Q_1} u,$$

where C depends only on n , λ and Λ .

The method used in [8] is that first one estimates the upper and lower bound of u in terms of $\|u\|_{L^{p_0}}$ and $\|u^{-1}\|_{L^{p_0}}$ respectively by an iteration for some $p_0 > 0$ and then joins these two estimates together to obtain (1.2) by the John-Nirenberg inequality.

In 1957, De Giorgi [3] proved the Hölder continuity for weak solutions of (1.1) and Moser [7] gave a new proof later. The following are two of the main results in [3] and [7] (see also [1]):

Theorem 1.2. *Let $u \geq 0$ be a weak subsolution of (1.1). Then*

$$(1.3) \quad \|u\|_{L^\infty(Q_1)} \leq C \|u\|_{L^2(Q_3)},$$

where C depends only on n , λ and Λ .

Theorem 1.3. *Let $u \geq 0$ be a weak subsolution of (1.1). Then for any $c_0 > 0$, there exists a constant c depending only on n , λ , Λ and c_0 such that*

$$(1.4) \quad m\{x \in Q_1 : u(x) > c\} > c_0 \Rightarrow u > 1 \text{ in } Q_3,$$

where m denotes the Lebesgue measure.

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In this note, we will prove Theorem 1.1 by the above two theorems directly. That is, De Giorgi's proof implies the Harnack inequality. This was first noticed by DiBenedetto [4]. Some other new approaches to Harnack inequality can be found in [5] and [6], where U. Gianazza and V. Vespri [6] required only a qualitative boundedness of solutions, which is different from what is shown here.

2. PROOF OF THEOREM 1.1

In the following, we present the key points for obtaining Theorem 1.1 by Theorems 1.2 and 1.3. First, we show that Theorem 1.2 implies the following local maximum principle:

Lemma 2.1. *Let $u \geq 0$ be a weak subsolution of (1.1) and $p_0 > 0$. Then*

$$(2.1) \quad \|u\|_{L^\infty(Q_1)} \leq C \|u\|_{L^{p_0}(Q_3)},$$

where C depends only on n , λ , Λ and p_0 .

Proof. By the interpolation for L^p functions and (1.3), for any $\varepsilon > 0$, we have

$$\|u\|_{L^\infty(Q_1)} \leq \varepsilon \|u\|_{L^\infty(Q_3)} + C(\varepsilon) \|u\|_{L^{p_0}(Q_3)}$$

whose scaling version is

$$(2.2) \quad r^{\frac{n}{p_0}} \|u\|_{L^\infty(Q_r(x_0))} \leq \varepsilon r^{\frac{n}{p_0}} \|u\|_{L^\infty(Q_{3r}(x_0))} + C(\varepsilon) \|u\|_{L^{p_0}(Q_{3r}(x_0))}$$

for any $Q_{6r}(x_0) \subset Q_6$.

Given $x_0 \in Q_3$, denote by d_{x_0} the distance between x_0 and ∂Q_3 . Then the cube $Q_{6r}(x_0) \subset Q_3$ where $r = d_{x_0}/3\sqrt{n}$, and we have

$$\begin{aligned} d_{x_0}^{\frac{n}{p_0}} |u(x_0)| &\leq C(n) r^{\frac{n}{p_0}} \|u\|_{L^\infty(Q_r(x_0))} \\ &\leq \varepsilon C(n) r^{\frac{n}{p_0}} \|u\|_{L^\infty(Q_{3r}(x_0))} + C(\varepsilon) \|u\|_{L^{p_0}(Q_{3r}(x_0))} \text{ (by (2.2))} \\ &\leq \varepsilon C(n) \sup_{x \in Q_3} d_x^{\frac{n}{p_0}} |u(x)| + C(\varepsilon) \|u\|_{L^{p_0}(Q_3)}. \end{aligned}$$

Take the supremum over Q_3 and ε small enough such that $\varepsilon C(n) < 1/2$. Then,

$$\sup_{x \in Q_3} d_x^{\frac{n}{p_0}} |u(x)| \leq C \|u\|_{L^{p_0}(Q_3)},$$

which implies (2.1). □

Next, we show that Theorem 1.3 implies the weak Harnack inequality:

Lemma 2.2. *Let $u \geq 0$ be a weak subsolution of (1.1). Then*

$$(2.3) \quad \|u\|_{L^p(Q_1)} \leq C \inf_{Q_3} u,$$

where $p > 0$ and C depend only on n , λ and Λ .

Proof. Without loss of generality, we assume that $\inf_{Q_3} u = 1$ and we only need to prove that there exists a constant c depending only on n , λ and Λ such that

$$(2.4) \quad m\{x \in Q_1 : u(x) > c^k\} \leq \frac{1}{2^k},$$

for $k = 1, 2, \dots$

We prove (2.4) by induction. Take $c_0 = 1/2$ in Theorem 1.3. Then there exists a constant c depending only on n , λ and Λ such that (1.4) holds. Hence, (2.4) holds

for $k = 1$ since we assume that $\inf_{Q_3} u = 1$. Suppose that (2.4) holds for $k \leq k_0 - 1$. Let

$$A := \{x \in Q_1 : u(x) > c^{k_0}\} \quad \text{and} \quad B := \{x \in Q_1 : u(x) > c^{k_0-1}\}.$$

We need to prove $m(A) \leq m(B)/2$. By the Calderón-Zygmund cube decomposition (see [2, Lemma 4.2]), we only need to prove that for any $Q_r(x_0) \subset Q_1$,

$$m(A \cap Q_r(x_0)) > \frac{1}{2}m(Q_r(x_0)) \Rightarrow Q_{3r}(x_0) \cap Q_1 \subset B,$$

which is exactly the scaling version of (1.4) for $v = u/c^{k_0-1}$. \square

Now, Theorem 1.1 follows clearly by combining (2.1) and (2.3).

Remark 2.3. “ $u \geq 0$ ” can be removed in Lemma 2.1 and a corresponding estimate for u^+ holds. As for elliptic equations in non-divergence form, we also have the local maximum principle (Lemma 2.1) and the weak Harnack inequality (Lemma 2.2) respectively (see [2, Theorem 4.8]). In fact, this note is inspired by [2].

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SCHOOL OF MATHEMATICS AND STATISTICS, XI'AN JIAOTONG UNIVERSITY, XI'AN 710049, PEOPLE'S REPUBLIC OF CHINA

E-mail address: lidsh@mail.xjtu.edu.cn

SCHOOL OF MATHEMATICS AND STATISTICS, XI'AN JIAOTONG UNIVERSITY, XI'AN 710049, PEOPLE'S REPUBLIC OF CHINA

E-mail address: zkkzk@stu.xjtu.edu.cn