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# REES ALGEBRAS AND $p_g$-IDEALS IN A TWO-DIMENSIONAL NORMAL LOCAL DOMAIN

TOMOHIRO OKUMA, KEI-ICHI WATANABE, AND KEN-ICHI YOSHIDA

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Abstract. The authors previously introduced the notion of $p_g$-ideals for two-dimensional excellent normal local domain over an algebraically closed field in terms of resolution of singularities. In this note, we give several ring-theoretic characterizations of $p_g$-ideals. For instance, an $\mathfrak{m}$-primary ideal $I \subset A$ is a $p_g$-ideal if and only if the Rees algebra $\mathcal{R}(I)$ is a Cohen-Macaulay normal domain.

## 1. Introduction

In [9, Sect. 7], Lipman proved that for any integrally closed $\mathfrak{m}$-primary ideal $I$ in a rational singularity of dimension 2, $I^2 = QI$ holds for every minimal reduction $Q$ of $I$ and that all powers of $I$ are integrally closed. This implies that the Rees algebra $\mathcal{R}(I) = \bigoplus_{n \geq 0} I^n$ is a Cohen-Macaulay normal domain. Moreover, for any two integrally closed $\mathfrak{m}$-primary ideals $I$, $J$ in a two-dimensional rational singularity, one can choose general elements $a \in I$ and $b \in J$ so that $IJ = aJ + bI$. This fact implies that the bigraded Rees algebra $\mathcal{R}(I, J)$ is a Cohen-Macaulay ring. An ideal theory in a two-dimensional rational singularity is established based upon these facts.

In [12], the authors introduced the notion of $p_g$-ideals for two-dimensional normal local domains using a resolution of singularities; see Section 2 for the definition and basic properties. Notice that the notion of $p_g$-ideals is a natural generalization of an integrally closed $\mathfrak{m}$-primary ideal in a two-dimensional rational singularity; see [12].

The main purpose of this note is to give several ring-theoretic characterizations of $p_g$-ideals. Namely, we prove the following theorem.

**Theorem 1.1** (see Corollary 3.3 and Theorem 4.1)**.** *Let $(A, \mathfrak{m})$ be a two-dimensional excellent normal local domain over an algebraically closed field. Let $I \subset A$ be an $\mathfrak{m}$-primary ideal, and let $Q$ be a minimal reduction of $I$. Then the following conditions are equivalent*:

(1) *$I$ is a $p_g$-ideal.*
(2) *$I^2 = QI$ and $\overline{I^n} = I^n$ for every $n \geq 1$, where $\overline{J}$ denotes the integral closure of an ideal $J$.*

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(3) *The Rees algebra* $\mathcal{R}(I)$ *is a Cohen-Macaulay normal domain.*
(4) $\overline{e}_2(I) = 0$.

Let us explain the organization of the paper. In Section 2, we recall the definition and several basic properties for $p_g$-ideals. For instance, $IJ = aJ + bI$ holds true for any two $p_g$-ideals $I$, $J$ and general elements $a \in I$, $b \in J$. In Section 3, we give a characterization of $p_g$-ideals in terms of normal Hilbert polynomials. Namely, the vanishing of the second normal Hilbert coefficient of $I$ yields that the ideal is a $p_g$-ideal (see Theorem 3.2). In Section 4, we give a characterization of $p_g$-ideals in terms of Rees algebras. Namely, an ideal $I$ is a $p_g$-ideal if and only if the Rees algebra $\mathcal{R}(I)$ is a Cohen-Macaulay normal domain. Applying these results, one can find some examples of $p_g$-ideals.

## 2. Basic results

Throughout this paper, let $(A, \mathfrak{m})$ be a two-dimensional excellent normal local domain containing an algebraically closed field $k$ and $f\colon X \to \operatorname{Spec} A$ a resolution of singularities with exceptional divisor $E := f^{-1}(\mathfrak{m})$ unless otherwise specified. Let $E = \bigcup_{i=1}^{r} E_i$ be the decomposition into irreducible components of $E$.

First, we recall the definition of $p_g$-ideals. For the definition of the integral closure and the reduction of ideals, refer to the textbook [14]. An $\mathfrak{m}$-primary ideal $I$ is said to be *represented on* $X$ if the ideal sheaf $I\mathcal{O}_X$ is invertible and $I = H^0(X, I\mathcal{O}_X)$. If $I$ is represented on $X$, then there exists an anti-nef cycle $Z$ such that $I\mathcal{O}_X = \mathcal{O}_X(-Z)$; $I$ is also said to be *represented by* $Z$ and written as $I = I_Z$. Note that such an ideal $I$ is integrally closed in $A$. See [9, Sect. 18]: note that an ideal $I$ is integrally closed if and only if it is complete ([9, Sect. 5]).

We say that $\mathcal{O}_X(-Z)$ *has no fixed component* if

$$H^0(\mathcal{O}_X(-Z)) \neq H^0(\mathcal{O}_X(-Z - E_i))$$

for every $E_i \subset E$, i.e., the base locus of the linear system $H^0(\mathcal{O}_X(-Z))$ does not contain any component of $E$.

We denote by $h^1(\mathcal{O}_X(-Z))$ the length $\ell_A(H^1(\mathcal{O}_X(-Z)))$. It is known that $h^1(\mathcal{O}_X)$ is independent of the choice of the resolution of singularities. The invariant $p_g(A) := h^1(\mathcal{O}_X)$ is called the *geometric genus* of $A$.

In [12, Theorem 3.1], the authors proved $h^1(\mathcal{O}_X(-Z)) \leq p_g(A)$ if $\mathcal{O}_X(-Z)$ has no fixed component. Based upon this result, they introduced the notion of $p_g$-ideals. The definition of $p_g$-ideal is independent of the choice of the resolution of singularities ([12, Lemma 3.4]).

**Definition 2.1** (**$p_g$-ideals, $p_g$-cycles**)**.** A cycle $Z > 0$ is called a *$p_g$-cycle* if $\mathcal{O}_X(-Z)$ is generated and $h^1(\mathcal{O}_X(-Z)) = p_g(A)$. An $\mathfrak{m}$-primary ideal $I$ is called a *$p_g$-ideal* if $I$ is represented by a $p_g$-cycle on some resolution.

Assume that $p_g(A) = 0$. Such a ring $A$ is called a rational singularity. Then every anti-nef cycle is a $p_g$-cycle (Lipman [9, Theorem 12.1]). See [12, Proposition 3.10] for another characterization of $p_g$-ideals in the case of $p_g(A) > 0$.

In what follows, let us discuss whether $Z + Z'$ is a $p_g$-cycle.

**Proposition 2.2** (see [12, Theorem 3.5]). *Let $Z$, $Z'$ be anti-nef cycles on the resolution $X \to \operatorname{Spec} A$ such that $\mathcal{O}_X(-Z)$ and $\mathcal{O}_X(-Z')$ are generated. Take general elements $a \in I_Z$, $b \in I_{Z'}$, so that the natural homomorphism $b\mathcal{O}_X(-Z) \oplus a\mathcal{O}_X(-Z') \to \mathcal{O}_X(-Z-Z')$ is surjective, and put*

$$\begin{aligned} \varepsilon(Z, Z') &:= \ell_A(I_{Z+Z'}/aI_{Z'} + bI_Z) \\ &= p_g(A) - h^1(\mathcal{O}_X(-Z)) - h^1(\mathcal{O}_X(-Z')) + h^1(\mathcal{O}_X(-Z-Z')). \end{aligned}$$

*Then:*

(1) *If $Z$ is a $p_g$-cycle on $X$, then $\varepsilon(Z, Z') = 0$ for any $Z'$. In particular, if $a \in I_Z$ and $b \in I_{Z'}$ are general elements, then*

$$I_{Z+Z'} = aI_{Z'} + bI_Z.$$

(2) *Assume that $Z$ is a $p_g$-cycle. Then $Z'$ is a $p_g$-cycle if and only if so is $Z + Z'$.*

(3) *If $Z + Z'$ is a $p_g$-cycle for some cycle $Z'$, then so is $Z$.*

*Proof.* (1), (2) It follows from [12, Theorem 3.5].

(3) Let $\alpha \in H^0(\mathcal{O}_X(-Z'))$ be a general element. Then $\operatorname{div}_X(\alpha) = Z' + H$, where $H$ is the proper transform of $\operatorname{div}_{\operatorname{Spec} A}(\alpha)$. From the exact sequence

$$0 \to \mathcal{O}_X(-Z) \xrightarrow{\times\alpha} \mathcal{O}_X(-Z-Z') \to \mathcal{C} \to 0 \tag{2.1}$$

we obtain $h^1(\mathcal{O}_X(-Z)) \geq h^1(\mathcal{O}_X(-Z-Z')) = p_g(A)$. Hence $h^1(\mathcal{O}_X(-Z)) = p_g(A)$ by [12, Theorem 3.10]. □

The following corollary immediately follows from Proposition 2.2.

**Corollary 2.3** ([12, Corollary 3.6]). *Let $I$, $J$ be $\mathfrak{m}$-primary integrally closed ideals.*

(1) *Assume that $I$ is a $p_g$-ideal. For general elements $a \in I$, $b \in J$, we have $IJ = aJ + bI$.*

(2) *If $I$ and $J$ are $p_g$-ideals, then $IJ$ is also a $p_g$-ideal.*

(3) *If $IJ$ is a $p_g$-ideal, then so are $I$ and $J$.*

## 3. The normal Hilbert polynomials

For an $\mathfrak{m}$-primary ideal $I \subset A$, there exist integers $\bar{e}_0(I)$, $\bar{e}_1(I)$, $\bar{e}_2(I)$ such that

$$\ell_A(A/\overline{I^{n+1}}) = \bar{e}_0(I)\binom{n+2}{2} - \bar{e}_1(I)\binom{n+1}{1} + \bar{e}_2(I) \quad \text{for large enough } n \gg 0.$$

Then

$$P_I(n) = \bar{e}_0(I)\binom{n+2}{2} - \bar{e}_1(I)\binom{n+1}{1} + \bar{e}_2(I)$$

is called the *normal Hilbert polynomial* of $I$. See e.g. [7].

**Lemma 3.1.** *Let $Z > 0$ be a cycle such that $\mathcal{O}_X(-Z)$ has no fixed component. Then:*

(1) *$h^1(\mathcal{O}_X(-nZ)) \geq h^1(\mathcal{O}_X(-(n+1)Z))$ for $n \geq 0$.*

(2) *If we put $n_0 = \min\left\{n \in \mathbb{Z}_{\geq 0} \;\middle|\; h^1(\mathcal{O}_X(-nZ)) = h^1(\mathcal{O}_X(-(n+1)Z))\right\}$, then $n_0 \leq p_g(A)$ and $h^1(\mathcal{O}_X(-nZ)) = h^1(\mathcal{O}_X(-n_0Z))$ for all $n \geq n_0$.*

*Proof.* (1) follows from the argument of Proposition 2.2.

(2) From the exact sequence

$$0 \to \mathcal{O}_X(-nZ) \to \mathcal{O}_X(-(n+1)Z)^{\oplus 2} \to \mathcal{O}_X(-(n+2)Z) \to 0,$$

we obtain that $h^1(\mathcal{O}_X(-nZ)) \geq 2 \cdot h^1(\mathcal{O}_X(-(n+1)Z)) - h^1(\mathcal{O}_X(-(n+2)Z))$. Thus if $h^1(\mathcal{O}_X(-nZ)) = h^1(\mathcal{O}_X(-(n+1)Z))$ is satisfied, then $h^1(\mathcal{O}_X(-(n+1)Z)) = h^1(\mathcal{O}_X(-(n+2)Z))$ holds true. □

The following result, the so-called *Kato's Riemann-Roch formula* ([8]), plays an important role in the next theorem. For an anti-nef cycle on $Z$ on $X$ and $I_Z = H^0(\mathcal{O}_X(-Z))$, we have

$$\ell_A(A/I_Z) + h^1(\mathcal{O}_X(-Z)) = -\frac{Z^2 + K_X Z}{2} + p_g(A), \tag{3.1}$$

where $K_X$ denotes the canonical divisor of $X$.

**Theorem 3.2.** *Assume that $I$ is represented by a cycle $Z > 0$. Let $P_I(n)$ be a normal Hilbert polynomial of $I$. Then*

(1) $P_I(n) = \ell_A(A/\overline{I^{n+1}})$ *for all* $n \geq p_g(A) - 1$.
(2) $\bar{e}_0(I) = e_0(I) = -Z^2$.
(3) $\bar{e}_1(I) = e_0(I) - \ell_A(A/I) + \big(p_g(A) - h^1(\mathcal{O}_X(-Z))\big) = \dfrac{-Z^2 + ZK_X}{2}$.
(4) $\bar{e}_2(I) = p_g(A) - h^1(\mathcal{O}_X(-nZ))$ *for all* $n \geq p_g(A)$.

*Proof.* It follows from the Riemann-Roch formula (3.1) that

$$\begin{aligned}\ell_A(A/\overline{I^{n+1}}) &= -\frac{(n+1)^2 Z^2 + (n+1)ZK_X}{2} + p_g(A) - h^1(\mathcal{O}_X(-(n+1)Z)) \\ &= -Z^2\binom{n+2}{2} - \frac{-Z^2 + ZK_X}{2}\binom{n+1}{1} \\ &\qquad + p_g(A) - h^1(\mathcal{O}_X(-(n+1)Z)).\end{aligned}$$

Since $h^1(\mathcal{O}_X(-nZ))$ is stable for $n \geq p_g(A)$ by Lemma 3.1, we obtain the required assertions. □

As a corollary, we obtain a simple characterization of $p_g$-ideals in terms of normal Hilbert coefficients.

**Corollary 3.3.** *The following conditions are equivalent*:

(1) *$I$ is a $p_g$-ideal.*
(2) $\bar{e}_1(I) = e_0(I) - \ell_A(A/I)$.
(3) $\bar{e}_2(I) = 0$.

*Proof.* (1) $\Longrightarrow$ (2) follows from Theorem 3.2.

(2) $\Longrightarrow$ (3): By assumption, $I = I_Z$ is a $p_g$-ideal. Hence $I_{nZ} = I^n$ is a $p_g$-ideal by Corollary 2.3(2), and thus $\bar{e}_2(I) = 0$ by Theorem 3.2.

(3) $\Longrightarrow$ (1): Theorem 3.2 yields that $h^1(\mathcal{O}_X(-(n+1)Z)) = p_g(A)$ for $n \gg 0$ and thus $I_{nZ} = I^n$ is a $p_g$-ideal. By Corollary 2.3(3), we obtain that $I_Z$ is also a $p_g$-ideal. □

For any cycle $Z$ on $X$, we put $Z^{\perp} = \sum_{ZE_i=0} E_i$.

**Proposition 3.4.** *Let $Z > 0$ be a cycle such that $\mathcal{O}_X(-Z)$ has no fixed component. If $C$ is the cohomological cycle on $Z^\perp$, i.e., the smallest cycle with*

$$h^1(\mathcal{O}_C) = \max_{D>0, D_{red} \le Z^\perp} h^1(\mathcal{O}_D),$$

*then $\mathcal{O}_C \cong \mathcal{O}_C(-n_0 Z)$ and $h^1(\mathcal{O}_C) = h^1(\mathcal{O}_X(-n_0 Z)) = p_g(A) - \bar{e}_2(I_Z)$, where $n_0$ is an integer given by Lemma 3.1.*

*Proof.* Let $D > 0$ satisfy that $\mathrm{Supp}(D) = Z^\perp$ and $DE_i < 0$ for all $E_i \le Z^\perp$. There exist $m, n \in \mathbb{N}$ such that $H^1(\mathcal{O}_X(-nD - mZ)) = 0$ (cf. the proof of [12, Proposition 3.10]). Then $H^1(\mathcal{O}_X(-mZ)) = H^1(\mathcal{O}_{nD}(-mZ))$. Since $\mathcal{O}_{nD}(-mZ) \cong \mathcal{O}_{nD}$, $h^1(\mathcal{O}_X(-mZ)) = h^1(\mathcal{O}_C)$ for sufficiently large $m$. □

*Remark* 3.5. Assume that $\mathcal{O}_X(-Z)$ is generated. Let $E^{(1)}, \ldots, E^{(k)}$ be the connected components of $Z^\perp$ and assume that each $E^{(i)}$ contracts to a normal surface singularity isomorphic to $(A_i, \mathfrak{m}_i)$. Then we have $p_g(A) = \bar{e}_2(I_Z) + \sum_{i=1}^k p_g(A_i)$ (cf. [11, Corollary 4.5]).

**Example 3.6.** Let $e \ge 2$ be an integer, and let $A = k[[x, y, z]]/(x^e + y^e + z^e)$. Then the Poincaré series of $k[x, y, z]/(x^e + y^e + z^e)$ is equal to

$$\sum_{k \ge 0} \ell_A(\mathfrak{m}^k/\mathfrak{m}^{k+1}) t^k = \frac{1 - t^e}{(1-t)^3} = \frac{1 + t + t^2 + \cdots + t^{e-1}}{(1-t)^2}.$$

It follows that

$$\ell_A(A/\mathfrak{m}^{n+1}) = \begin{cases} e\binom{n+2}{2} - \dfrac{e(e-1)}{2}\binom{n+1}{1} + \dfrac{e(e-1)(e-2)}{6} & (n \ge e), \\ \dfrac{(n+1)(n+2)(n+3)}{6} & (n \le e-1). \end{cases}$$

Hence

$$\begin{cases} \bar{e}_0(\mathfrak{m}) &=& e_0(\mathfrak{m}) = e, \\ \bar{e}_1(\mathfrak{m}) &=& e_1(\mathfrak{m}) = \dfrac{e(e-1)}{2}, \\ \bar{e}_2(\mathfrak{m}) &=& e_2(\mathfrak{m}) = \dfrac{e(e-1)(e-2)}{6} = p_g(A); \quad \text{see [15, (4.11)]}. \end{cases}$$

Write $\mathfrak{m} = I_Z$ for some anti-nef cycle $Z$ on some resolution $X \to \operatorname{Spec} A$. Then $h^1(\mathcal{O}_X(-kZ)) = 0$ for every $k \ge e$. On the other hand,

$$\frac{e(e-1)}{2} = \bar{e}_1(\mathfrak{m}) = e_0(\mathfrak{m}) - \ell_A(A/\mathfrak{m}) + p_g(A) - h^1(\mathcal{O}_X(-Z))$$

yields $h^1(\mathcal{O}_X(-Z)) = \dfrac{(e-1)(e-2)(e-3)}{6}$.

Furthermore, since $ZK = 2 \cdot \overline{e}_1(\mathfrak{m}) - (-Z^2) = e(e-2)$ and $e_0(\mathfrak{m}^k) = k^2 e$, we have

$$\begin{aligned} h^1(\mathcal{O}_X(-kZ)) &= e_0(\mathfrak{m}^k) - \ell_A(A/\mathfrak{m}^k) + p_g(A) - \frac{-(kZ)^2 + (kZ)K}{2} \\ &= k^2 e - \binom{k+2}{3} + \binom{e}{3} - \frac{k^2 e + ke(e-2)}{2} \\ &= \frac{(e-k)(e-k-1)(e-k-2)}{6} = \binom{e-k}{3} \end{aligned}$$

for each $k = 1, 2, \ldots, e-1$. In particular, we get

$$h^1(\mathcal{O}_X(-(e-3)Z)) = 1 \quad \text{and} \quad h^1(\mathcal{O}_X(-(e-2)Z)) = 0,$$

and thus $n_0 = e-2$ in Lemma 3.1.

## 4. The Rees algebra

Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d$, and let $I$ be an ideal of $A$. Now consider three $A$-algebras, which are called blow-up algebras,

$$\begin{aligned} \mathcal{R}(I) &:= A[It] = \bigoplus_{n \geq 0} I^n t^n \subset A[t], \\ \mathcal{R}'(I) &:= A[It, t^{-1}] = \bigoplus_{n \in \mathbb{Z}} I^n t^n \subset A[t, t^{-1}], \\ G(I) &:= \mathcal{R}(I)/I\mathcal{R}(I) \cong \mathcal{R}'(I)/t^{-1}\mathcal{R}'(I). \end{aligned}$$

The algebra $\mathcal{R}(I)$ (resp. $\mathcal{R}'(I)$, $G(I)$) is called the *Rees algebra* (resp. *the extended Rees algebra*, *the associated graded ring*) of $I$.

The main purpose of this section is to characterize $p_g$-ideals in terms of blow-up algebras.

**Theorem 4.1.** *Let $(A, \mathfrak{m})$ be a two-dimensional excellent normal local domain over an algebraically closed field, and let $I \subset A$ be an $\mathfrak{m}$-primary ideal. Then the following conditions are equivalent*:

1. *$I$ is a $p_g$-ideal in the sense of Definition* 2.1.
2. *$I^2 = QI$ for some minimal reduction $Q$ of $I$, and $\overline{I^n} = I^n$ holds true for every $n \geq 1$.*
3. *$\mathcal{R}(I)$ is a Cohen-Macaulay normal domain.*
4. *$\mathcal{R}'(I)$ is a Cohen-Macaulay normal domain with $a(G(I)) < 0$, where $a(G(I))$ denotes the $a$-invariant of the graded ring $G(I)$; see* [3, Definition 3.14].

*Proof.* (1) $\Longrightarrow$ (2): It follows from Corollary 2.3.

(2) $\Longrightarrow$ (3): Since $I^2 = QI$ for some minimal reduction $Q$ of $I$, $\mathcal{R}(I)$ is Cohen-Macaulay by Valabrega–Valla [16] and Goto–Shimoda [1]. Moreover, since $A$ is normal and $\overline{I^n} = I^n$ for every $n \geq 1$, $\mathcal{R}(I)$ is a normal domain.

(4) $\Longleftrightarrow$ (3) $\Longrightarrow$ (2) follows from Goto–Shimoda [1] and Herzog et al. [5, Proposition 2.1.2].

(2) $\Longrightarrow$ (1): Assume that $I^n$ is integrally closed for $n \geq 1$ and that $I^2 = QI$ for a minimal reduction $Q$ of $I$. Suppose that $I$ is represented by a cycle $Z$ on $X$.

Consider the following exact sequence given by general elements of $I = I_Z$ and $I_{nZ}$ (see [12, (2.3)]):

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(-Z) \oplus \mathcal{O}_X(-nZ) \to \mathcal{O}_X(-(n+1)Z) \to 0.$$

Since $QI^n = I^{n+1} = \overline{I^{n+1}}$, we obtain that $\varepsilon(Z, nZ) = 0$ for $n \geq 1$. Therefore, $p_g(A) = h^1(\mathcal{O}_X(-Z))$ because $h^1(\mathcal{O}_X(-nZ))$ is stable for $n \gg 0$. □

The following two examples are known.

**Example 4.2** (cf. Lipman [10, Example 3])**.** Let $A$ be a two-dimensional rational singularity. Then any integrally closed $\mathfrak{m}$-primary ideal $I$ is a $p_g$-ideal and $\mathcal{R}(I)$ is a Cohen-Macaulay normal domain.

**Example 4.3.** Let $A$ be a complete Gorenstein local ring with $p_g(A) > 0$. If $\mathfrak{m}$ is a $p_g$-ideal of $A$, then $\mathfrak{m}$ is stable, that is, $\mathfrak{m}^2 = Q\mathfrak{m}$ for some minimal reduction $Q$ of $\mathfrak{m}$. Since $A$ is Gorenstein, we obtain that $A$ is a hypersurface of degree 2. So we may assume that $A = K[[x, y, z]]/(f)$, where $f = x^2 + g(y, z)$. As $A$ is not rational, $g(y, z) \in (y, z)^3$. Moreover, since $R(\mathfrak{m})$ is normal, we have $g(y, z) \notin (y, z)^4$.

Conversely, if $A = K[[x, y, z]]/(x^2 + g(y, z))$, where $g(y, z) \in (y, z)^3 \setminus (y, z)^4$, then for every $n$, $\mathfrak{m}^n = (y, z)^n + x(y, z)^{n-1}$ and is integrally closed. Then, since $\mathfrak{m}$ is stable and $\mathfrak{m}^n$ is integrally closed for every $n \geq 1$, $\mathfrak{m}$ is a $p_g$-ideal.

The next example gives a hypersurface local ring $A$ whose maximal ideal is a $p_g$-ideal and $p_g(A) = p$ for a given integer $p \geq 1$.

**Example 4.4.** Let $p \geq 1$ be an integer, and let $k$ be an algebraically closed field. Let $B = k[x, y, z]/(x^2 + y^3 + z^{6p+1})$. If we put $\deg x = 3(6p + 1)$, $\deg y = 2(6p + 1)$ and $\deg z = 6$, then $A$ can be regarded as a quasi-homogeneous $k$-algebra with $a(A) = 6p - 5$. In particular,

$$p_g(B) = \sum_{i=0}^{6p-5} \dim_k B_i = p; \quad \text{(cf. [13, 19])}.$$

Moreover, if we put $X = xt$, $Y = yt$, $Z = zt$ and $U = t^{-1}$, then the extended Rees algebra of $\mathfrak{m} = (x, y, z)$ is

$$\mathcal{R}'(\mathfrak{m}) \cong k[X, Y, Z, U]/(F),$$

where $F = X^2 + Y^3U + Z^{6p+1}U^{6p-1}$. Since the Jacobian ideal is

$$\left(\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}, \frac{\partial F}{\partial Z}, \frac{\partial F}{\partial U}, F\right) = (X, Y^2U, Z^{6p}U^{6p-1}, Y^3 + (6p-1)Z^{6p+1}U^{6p-2}),$$

one can check the $(R_1)$-condition of $\mathcal{R}'(\mathfrak{m})$. Thus $\mathcal{R}'(\mathfrak{m})$ is a normal domain because it is Cohen-Macaulay.

Now let us put $A = B_{(x,y,z)}$ and $\mathfrak{m} = (x, y, z)A$. Then we can conclude that $A$ is a two-dimensional normal hypersurface with $p_g(A) = p$ and that $\mathfrak{m}$ is a $p_g$-ideal by applying the theorem above.

Similarly, if we consider $I_k = (x, y, z^k)A$ and $Q_k = (y, z^k)$ for $k = 2, 3, \ldots, 3p$, then $I_k^2 = Q_kI_k$ and $\mathcal{R}'(I_k)$ is a normal domain. Hence $I_k$ is a $p_g$-ideal.

The next example gives a hypersurface local ring $A$ whose maximal ideal is not a $p_g$-ideal and $p_g(A) = p$ for a given integer $p \geq 1$.

**Example 4.5.** Let $p \geq 1$ be an integer. Let $A = k[x,y,z]_{(x,y,z)}/(x^2+y^4+z^{4p+1})$. Then $A$ is a two-dimensional normal hypersurface with $p_g(A) = p$. Then $\mathfrak{m} = (x,y,z)$ is *not* a $p_g$-ideal and $I_k = (x,y,z^k)$ is a $p_g$-ideal for every $k = 2,3,\ldots,2p$ because $\mathcal{R}'(I_k)$ is normal but $\mathcal{R}'(\mathfrak{m})$ is not.

Furthermore, $\mathfrak{m}^k$ is not a $p_g$-ideal for every $k \geq 1$ by Corollary 2.3.

It is not so difficult to extend our result to the case of bigraded Rees algebras. Let $I, J \subset A$ be ideals. Then

$$\mathcal{R}(I,J) := A[It_1, Jt_2] = \bigoplus_{n=1}^{\infty}\bigoplus_{m=1}^{\infty} I^m J^n t_1^m t_2^n \subset A[t_1,t_2]$$

is called the *multi-Rees algebra* of $I$ and $J$.

**Corollary 4.6.** *Let $(A,\mathfrak{m})$ be a two-dimensional excellent normal local domain over an algebraically closed field, and let $I$, $J$ be $\mathfrak{m}$-primary ideals. Then the following conditions are equivalent*:

(1) *$I$ and $J$ are $p_g$-ideals.*
(2) *$\mathcal{R}(I,J)$ is a Cohen-Macaulay normal domain.*
(3) *$I$, $J$ are integrally closed and $\mathcal{R}(IJ)$ is a Cohen-Macaulay normal domain.*

*Proof.* (1) $\Longrightarrow$ (2): Since $I$ and $J$ are $p_g$-ideals, $\mathcal{R}(I)$ and $\mathcal{R}(J)$ are Cohen-Macaulay and $IJ = aJ + bI$ for some joint reduction $(a,b)$ of $(I,J)$; see [14, Sect. 17]. Hence $\mathcal{R}(I,J)$ is Cohen-Macaulay by [6, Corollary 3.5] (see also e.g. [4, 17, 18]). Since $S = \mathcal{R}(I)$ is a normal domain and $JI^k$ is integrally closed for every $k \geq 1$, $\mathcal{R}(I,J)$ is normal.

(2) $\Longrightarrow$ (1): Since $\mathcal{R}(I,J)$ is Cohen-Macaulay, $\mathcal{R}(I)$ and $\mathcal{R}(J)$ are Cohen-Macaulay by [6, Corollary 3.5]. Since $\mathcal{R}(I)$ and $\mathcal{R}(J)$ are pure subrings of $\mathcal{R}(I,J)$, they are normal domains. Hence $I$ and $J$ are $p_g$-ideals by Theorem 4.1.

(1) $\Longleftrightarrow$ (3): It follows from Theorem 4.1 and Corollary 2.3. □

*Remark* 4.7. By a similar argument as in the proof of (1) $\Longrightarrow$ (2), we can obtain that the multi-Rees algebra $\mathcal{R}(I_1,\ldots,I_r)$ is a Cohen-Macaulay normal domain for every $p_g$-ideal $I_1,\ldots,I_r$.

*Remark* 4.8. Assume that $A$ is a rational singularity. Let $I$ and $J$ be $\mathfrak{m}$-primary integrally closed ideals of $A$. Then $I$ and $J$ are $p_g$-ideals and thus $\mathcal{R}(I)$, $\mathcal{R}(J)$ and $\mathcal{R}(I,J)$ are Cohen-Macaulay normal domains. In fact, S. Goto, N. Matsuoka, N. Taniguchi and the third author [2] prove that $\mathcal{R}(I)$ and $\mathcal{R}(J)$ are almost Gorenstein. Moreover, Verma [18] proved that they admit minimal multiplicities.

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Department of Mathematical Sciences, Faculty of Science, Yamagata University, Yamagata, 990-8560, Japan

*E-mail address*: okuma@sci.kj.yamagata-u.ac.jp

Department of Mathematics, College of Humanities and Sciences, Nihon University, Setagaya-ku, Tokyo, 156-8550, Japan

*E-mail address*: watanabe@math.chs.nihon-u.ac.jp

Department of Mathematics, College of Humanities and Sciences, Nihon University, Setagaya-ku, Tokyo, 156-8550, Japan

*E-mail address*: yoshida@math.chs.nihon-u.ac.jp