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CONTINUITY OF THE SOLUTION TO THE L_p MINKOWSKI PROBLEM

GUANGXIAN ZHU

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Abstract. For $p > 1$ with $p \neq n$, it is proved that the weak convergence of L_p surface area measures implies the convergence of the corresponding convex bodies in the Hausdorff metric and that the solution to the L_p Minkowski problem is continuous with respect to p.

1. Introduction

A convex body in n-dimensional Euclidean space, $\mathbb{R}^n$, is a compact convex set with non-empty interior. One of the fundamental concepts in convex geometry is the L_p surface area measure introduced by Lutwak [25]. Let $p \in \mathbb{R}$ and K be a convex body that contains the origin in its interior; then the L_p surface area measure, $S_p(K, \cdot)$, of K is the Borel measure on S^{n-1} defined for each Borel $\omega \subset S^{n-1}$ by

$$S_p(K, \omega) = \int_{x \in \nu_K^{-1}(\omega)} (x \cdot \nu_K(x))^{1-p} d\mathcal{H}^{n-1}(x),$$

where $\nu_K : \partial' K \to S^{n-1}$ is the Gauss map of K, defined on $\partial' K$, the set of points of ∂K that have a unique outer unit normal, and $\mathcal{H}^{n-1}$ is an $(n-1)$-dimensional Hausdorff measure.

The L_p surface area measure of a convex body K is an important generalization of the classical surface area measure, $S_1(K, \cdot)$ (also denoted by S_K), of K. In recent years, the L_p surface area measure was studied in, e.g., [3–5, 9, 11–15, 17–19, 22–28, 30].

Let K, K_i be convex bodies that contain the origin in their interiors. It is well known that if K_i converges to K in the Hausdorff metric, then $S_p(K_i, \cdot)$ converges weakly to $S_p(K, \cdot)$. The opposite question is of great interest: Assume that $S_p(K_i, \cdot)$ converges weakly to $S_p(K, \cdot)$. Does K_i converge to K in Hausdorff metric? While it is well known that the answer to this question is affirmative when $p = 1$ (up to translation), it is not necessarily positive for all $p \in \mathbb{R}$. For example, let $p = n$, K_1 be an origin-symmetric convex body and $K_i = \frac{1}{i} K_1$; then $S_n(K_i, \cdot) = S_n(K, \cdot)$ for all i but K_i converges to the origin.

It is the first aim of this paper to give an affirmative answer to this question for all $p > 1$ with $p \neq n$. To this end, let $\mathcal{K}_o^n$ denote the set of convex bodies in $\mathbb{R}^n$ that contain the origin in their interiors.

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Theorem 1.1. *Let $p \in (1,n) \cup (n,\infty)$ and $K, K_i \in \mathcal{K}_o^n$. If $S_p(K_i,\cdot)$ converges weakly to $S_p(K,\cdot)$, then K_i converges to K in the Hausdorff metric.*

Remark. The statement of Theorem 1.1 is well known for the case $p = 1$, which is closely related to the classical Minkowski problem and has many important applications. For references, see the book of Schneider [29].

In [25], Lutwak posed the following L_p version of the classical Minkowski problem.

L_p **Minkowski problem:** *For fixed p, what are necessary and sufficient conditions for a finite Borel measure μ on S^{n-1} so that μ is the L_p surface area measure of a convex body in $\mathbb{R}^n$?*

Today, the L_p Minkowski problem is a core problem in convex geometric analysis and was studied by, e.g., Lutwak [25], Lutwak and Oliker [26], Lutwak, Yang and Zhang [28], Chou and Wang [6], Guan and Lin [10], Haberl, Lutwak, Yang and Zhang [11], Böröczky, Lutwak, Yang and Zhang [4, 5], Stancu [31, 32], Huang, Liu and Xu [18], Lu and Wang [21], Böröczky, Hegedűs and Zhu [3], Jian, Lu and Zhu [20], and Zhu [34–37]. The solutions of the L_p Minkowski problem have important applications to affine isoperimetric inequalities, see, e.g., Zhang [33], Lutwak, Yang and Zhang [27], Cianchi, Lutwak, Yang and Zhang [7], Haberl and Schuster [14, 15], Haberl, Schuster and Xiao [16]. The solutions to the L_p Minkowski problem are also related to some important flows, see, e.g., Andrews [1, 2] and Stancu [31, 32].

Clearly, Theorem 1.1 is closely related to the L_p Minkowski problem and can be restated as follows:

Theorem 1.2. *Suppose that $p \in (1,n)\cup(n,+\infty)$ and that μ_i, μ are Borel measures on S^{n-1}. If $K_i \in \mathcal{K}_o^n$ is the solution to the L_p Minkowski problem associated with μ_i and $K \in \mathcal{K}_o^n$ is the solution to the L_p Minkowski problem associated with μ, then K_i converges to K in the Hausdorff metric as μ_i converges weakly to μ.*

In [25], Lutwak proved that if μ is an even measure on S^{n-1} that is not concentrated on a closed hemisphere, then the L_p Minkowski problem associated with μ has a unique solution for each $p > 1$ with $p \neq n$ (see also, e.g., Hug, Lutwak, Yang and Zhang [19] for the non-symmetric case). It is therefore natural to ask whether the solution to the L_p Minkowski problem associated with a fixed μ is continuous with respect to p.

It is the second aim of this paper to prove the continuity of the solution to the L_p Minkowski problem for $p > 1$ with $p \neq n$.

Theorem 1.3. *Suppose that $p, q \in (1,n) \cup (n,\infty)$ and that μ is a Borel measure on S^{n-1}. If $K_p \in \mathcal{K}_o^n$ is the solution to the L_p Minkowski problem associated with μ and $K_q \in \mathcal{K}_o^n$ is the solution to the L_q Minkowski problem associated with μ, then K_q converges to K_p in the Hausdorff metric as $q \to p$.*

2. Preliminaries

In this section, we list basic facts about convex bodies. Some general references regarding convex bodies are, e.g., [8, 29].

The sets appearing in this paper are subsets of Euclidean n-space $\mathbb{R}^n$. For $x, y \in \mathbb{R}^n$, we will write $x \cdot y$ for the standard inner product. Let B^n be the unit ball of $\mathbb{R}^n$, and S^{n-1} be the unit sphere. Let $\mathcal{K}^n$ denote the set of convex bodies in $\mathbb{R}^n$ and let $\mathcal{K}_o^n$ denote the set of convex bodies that contain the origin in their interiors.

The *support function* $h_K : \mathbb{R}^n \to \mathbb{R}$ of a compact convex set K is defined, for $x \in \mathbb{R}^n$, by

$$h(K, x) = \max\{x \cdot y : y \in K\}.$$

Obviously, if $c > 0$, $x \in \mathbb{R}^n$ and $K \in \mathcal{K}^n$, then

$$h_{cK}(x) = h_K(cx) = ch_K(x).$$

The *Hausdorff distance* between two compact sets K, L in $\mathbb{R}^n$ can be defined by

$$\delta(K, L) = \max_{u \in S^{n-1}} |h_K(u) - h_L(u)|.$$

For $p > 1$, the L_p mixed volume $V_p(K, L)$ of the convex bodies $K, L \in \mathcal{K}_o^n$ was defined in [25] by Lutwak by

$$V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L(u)^p dS_p(K, u).$$

Obviously,

$$V(K) = V_p(K, K) = \frac{1}{n} \int_{S^{n-1}} h_K dS_K.$$

The following L_p mixed volume inequality established by Lutwak [25] is a fundamental inequality in the L_p Brunn-Minkowski theory: Let $p > 1$ and $K, L \in \mathcal{K}_o^n$, then

$$V_p(K, L)^n \geq V(K)^{n-p} V(L)^p, \tag{2.1}$$

with equality if and only if K and L are dilates.

3. Proof of Theorem 1.1

Let $\alpha_+ = \max\{0, \alpha\}$ for $\alpha \in \mathbb{R}$.

Lemma 3.1. *If $p > 1$, $K, K_i \in \mathcal{K}_o^n$ and $S_p(K_i, \cdot)$ converges to $S_p(K, \cdot)$ weakly, then*

$$f_i(u) = \int_{S^{n-1}} (u \cdot v)_+^p dS_p(K_i, v)$$

converges to

$$f(u) = \int_{S^{n-1}} (u \cdot v)_+^p dS_p(K, v)$$

uniformly on S^{n-1}.

Proof. Clearly, $f_i(u)^{\frac{1}{p}}$ and $f(u)^{\frac{1}{p}}$ are support functions of convex bodies. For support functions on S^{n-1}, pointwise and uniform convergence are equivalent (see, e.g., Schneider [29] p. 54). Hence, by assumption, $f_i(u)^{\frac{1}{p}}$ converges to $f(u)^{\frac{1}{p}}$ uniformly on S^{n-1}. Therefore, $f_i(u)$ converges to $f(u)$ uniformly. □

The following two lemmas from [19] will be needed.

Lemma 3.2. *Let $p > 1$, $K' \in \mathcal{K}^n$ be a convex body with $o \in K'$ and let μ be a Borel measure on S^{n-1} such that $V(K')h_{K'}^{p-1}\mu = S_{K'}$. Assume that for some constant $c_0 > 0$,*

$$\int_{S^{n-1}} (u \cdot v)_+^p d\mu(v) \geq \frac{n}{c_0^p}$$

for all $u \in S^{n-1}$. Then $K' \subset c_0 B^n$.

Lemma 3.3. *If μ is a finite Borel measure on S^{n-1} that is not concentrated on a closed hemisphere of S^{n-1}, then for any $p > 1$ with $p \neq n$, there exists a unique convex body $K \in \mathcal{K}^n$ with $o \in K$ such that*

$$h_K^{p-1}\mu = S_K.$$

Moreover, there exists a unique convex body K' in $\mathbb{R}^n$ with $o \in K'$ such that

$$V(K')h_{K'}^{p-1}\mu = S_{K'},$$

with

$$K' = V(K)^{-\frac{1}{p}}K. \tag{3.1}$$

We need one more auxiliary result.

Lemma 3.4. *If $p > 1$ with $p \neq n$, $K, K_i \in \mathcal{K}_o^n$ and $S_p(K_i, \cdot)$ converges to $S_p(K, \cdot)$ weakly, then K_i is bounded, and there exist $0 < c_1 < c_2$ and $N_0 \in \mathbb{N}$ such that*

$$c_1 < V(K_i) < c_2$$

for all $i \geq N_0$.

Proof. By replacing K by K_i and L by B^n in the L_p mixed volume inequality (2.1), we obtain

$$S_p(K_i)^n \geq V(K_i)^{n-p}\omega_n^p,$$

where

$$S_p(K_i) = \int_{S^{n-1}} dS_p(K_i, u).$$

From this and the fact that $S_p(K_i)$ converges to $S_p(K)$, we infer:

(a). when $1 < p < n$, $V(K_i)$ has a positive upper bound;

(b). when $p > n$, $V(K_i)$ has a positive lower bound.

Since $f_i(u) = \int_{S^{n-1}}(u\cdot v)_+^p dS_p(K_i, v)$ converges to $f(u) = \int_{S^{n-1}}(u\cdot v)_+^p dS_p(K, v)$ uniformly and $\int_{S^{n-1}}(u \cdot v)_+^p dS_p(K, v)$ has a positive lower bound on S^{n-1}, there exists an $N_0 \in \mathbb{N}$ such that $\int_{S^{n-1}}(u \cdot v)_+^p dS_p(K_i, v)$ has a positive lower bound for $i \geq N_0$. Thus, there exists a $c_0 > 0$ such that

$$\int_{S^{n-1}} (u \cdot v)_+^p dS_p(K_i, v) \geq \frac{n}{c_0^p} \tag{3.2}$$

for all $u \in S^{n-1}$ and $i \geq N_0$. Let $\mu_i = S_p(K_i, \cdot)$. By Lemma 3.2, Lemma 3.3 and (3.2), there exist $K_i' \in \mathcal{K}^n$ with $o \in K_i'$ such that

$$V(K_i')h_{K_i'}^{p-1}S_p(K_i, \cdot) = S(K_i', \cdot)$$

and $K_i' \subset c_0B^n$ for all $i \geq N_0$. By Lemma 3.3,

$$K_i' = V(K_i)^{-\frac{1}{p}}K_i.$$

Thus,

$$V(K_i)^{\frac{p-n}{p}} = V(K_i') \leq c_0^n\omega_n,$$

for all $i \geq N_0$. It follows that

(c). when $1 < p < n$, $V(K_i)$ has a positive lower bound;

(d). when $p > n$, $V(K_i)$ has a positive upper bound.

By **(a)-(d)**, there exist $0 < c_1 < c_2$ and $N_0 \in \mathbb{N}$ such that

$$c_1 < V(K_i) < c_2, \tag{3.3}$$

for all $i \geq N_0$.

Let $R_i := \max\{h_{K_i}(v) : v \in S^{n-1}\}$ and $v_i \in S^{n-1}$ so that $R_i = h_{K_i}(v_i)$. Since the segment $R_i[o, v_i] \subset K_i$, $R_i(v \cdot v_i)_+ \leq h_{K_i}(v)$ for all $v \in S^{n-1}$. From this, (3.2) and (3.3), it follows that when $i \geq N_0$,

$$\begin{aligned} R_i^p \frac{n}{c_0^p} &\leq R_i^p \int_{S^{n-1}} (v \cdot v_i)_+^p dS_p(K_i, v) \\ &\leq \int_{S^{n-1}} h_{K_i}(v)^p dS_p(K_i, v) \\ &= nV(K_i) \\ &\leq nc_2. \end{aligned}$$

Therefore, K_i is bounded. □

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Suppose that Theorem 1.1 is not correct. Then there exists a subsequence K_{i_j} of K_i and a real $\varepsilon_0 > 0$ such that

$$\max_{u \in S^{n-1}} |h_K(u) - h_{K_{i_j}}(u)| \geq \varepsilon_0$$

for all i_j.

By Lemma 3.4, K_i is bounded and so K_{i_j} is as well. The Blaschke selection theorem now yields a subsequence $K_{i_{j_k}}$ of K_{i_j} that converges to some compact convex set K_0, with $o \in K$. By Lemma 3.4, there exist $N_0 \in \mathbb{N}$ and $c_1 > 0$ such that $V(K_i) \geq c_1$ for all $i > N_0$. Thus, K_0 is a convex body with $K_0 \neq K$.

Let $\mu = S_p(K, \cdot)$. From the fact that

$$h_{K_{i_{j_k}}}^{1-p} S_{K_{i_{j_k}}} \to \mu$$

weakly, we infer that $h_{K_{i_{j_k}}}^{p-1} \mu$ and $S_{K_{i_{j_k}}}$ converge to the same measure $S(K_0, \cdot)$. Thus,

$$h_{K_0}^{p-1} \mu = S_{K_0}.$$

On the other hand, $h_K^{p-1} \mu = S_K$. From Lemma 3.3 and the uniqueness of the solution of the L_p Minkowski problem, it follows that $K = K_0$. This is a contradiction. Therefore, K_i converges to K. □

4. Proof of Theorem 1.2

Lemma 4.1. *Let $p, p_i \in (1, n) \cup (n, \infty)$ with $p_i \to p$ and let μ be a Borel measure on the unit sphere that is not concentrated on a closed hemisphere. If K (with o is an interior of K) is the solution to the L_p Minkowski problem associated with μ and K_i (with o is an interior of K_i) is the solution to the L_{p_i} Minkowski problem associated with μ, then K_i is bounded and there exist $0 < c_3 < c_4$ and $N_0 \in \mathbb{N}$ such that when $i \geq N_0$,*

$$c_3 < V(K_i) < c_4.$$

Proof. By the L_p mixed volume inequality and the fact that $|\mu| = S_p(K)$,

$$|\mu|^n \geq V(K_i)^{n-p_i} \omega_n^{p_i}.$$

From this and the fact that $p, p_i \in (1, n) \cup (n, \infty)$ with $p_i \to p$, it follows that there exists $N_1 \in \mathbb{N}$ such that:

(e). when $1 < p < n$ and $i \geq N_1$, $V(K_i)$ has a positive upper bound;
(f). when $p > n$ and $i \geq N_1$, $V(K_i)$ has a positive lower bound.

Since μ is not concentrated on a closed hemisphere of S^{n-1} and $p, p_i \in (1,n)\cup(n,\infty)$ with $p_i \to p$, there exist $N_2 \in \mathbb{N}$, $c_0, a_0 > 0$ and $p_0 > p > p_1 > 1$ such that when $i \geq N_2$,

$$p_0 \geq p_i \geq p_1,$$

$$\int_{S^{n-1}} (u\cdot v)_+^{p_0}\, d\mu(v) \geq \frac{n}{c_0^{p_0}},$$

for all $u \in S^{n-1}$, and

$$(c_0 + a_0)^{p_i} \geq c_0^{p_0}.$$

From this, the fact that $p_0 \geq p_i > 1$ for $i \geq N_2$ and the fact that $(u\cdot v)_+^{p_i} \geq (u\cdot v)_+^{p_0}$, we obtain

$$\begin{aligned}\int_{S^{n-1}} (u\cdot v)_+^{p_i}\, d\mu(v) &\geq \int_{S^{n-1}} (u\cdot v)_+^{p_0}\, d\mu(v)\\ &\geq \frac{n}{c_0^{p_0}}\\ &\geq \frac{n}{(c_0+a_0)^{p_i}}\end{aligned}$$

for all $u \in S^{n-1}$ and $n \geq N_2$. Thus, Lemma 3.2 and Lemma 3.3 yield

$$K_i' = V(K_i)^{-\frac{1}{p}} K_i \subset (c_0 + a_0) B^n. \tag{4.1}$$

Consequently,

$$V(K_i)^{\frac{p_i - n}{p_i}} = V(K_i') \leq (c_0 + a_0)^n \omega_n.$$

From this and the fact that $p_i \to p$, we conclude that there exists $N_3 \in \mathbb{N}$ such that:

(g). when $1 < p < n$ and $i \geq N_3$, $V(K_i)$ has a positive lower bound;

(h). when $p > n$ and $i \geq N_3$, $V(K_i)$ has a positive upper bound.

By **(e)-(h)**, there exist $0 < c_3 < c_4$ and $N_0 \in \mathbb{N}$ ($N_0 \geq \max\{N_1, N_2, N_3\}$) such that

$$c_3 < V(K_i) < c_4,$$

for all $i \geq N_0$. From this and (4.1), we infer

$$K_i \subset c_4^{\frac{1}{p}} (c_0 + a_0) B^n,$$

for all $i \geq N_0$. Therefore, K_i is bounded. □

We can now complete the proof of Theorem 1.2.

Proof of Theorem 1.2. We only need to prove that if $p_i \in (1,n)\cup(n,\infty)$ with $p_i \to p$, K is the solution to the L_p Minkowski problem associated with μ and K_i is the solution to the L_{p_i} Minkowski problem associated with μ, then K_i converges to K.

Suppose that this does not hold. Then there exists a subsequence K_{i_j} of K_i and a real $\varepsilon_0 > 0$ such that

$$\max_{u\in S^{n-1}} |h_K(u) - h_{K_{i_j}}(u)| \geq \varepsilon_0 \tag{4.2}$$

for all i_j. By Lemma 4.1 and the Blaschke selection theorem, there exists a subsequence, $K_{i_{j_k}}$, of K_{i_j} and a convex body K_0 with $o \in K_0$ such that $K_{i_{j_k}}$ converges to K_0. Thus, $S_{K_{i_{j_k}}}$ converges to S_{K_0} weakly. Since $K_{i_{j_k}}$ is the solution to the $L_{p_{i_{j_k}}}$ Minkowski problem associated with μ, we have

$$S_{K_{i_{j_k}}} = h^{p-1}_{K_{i_{j_k}}} \mu$$

for all i_{j_k}. By letting $k \to +\infty$,

$$S_{K_0} = h^{p-1}_{K_0} \mu.$$

From this, Lemma 3.3 and the uniqueness of the solution to the L_p Minkowski problem, we obtain $K = K_0$. However, by (4.2), $K \neq K_0$. This is a contradiction. Therefore, K_i converges to K. $\square$

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Department of Mathematics, Tandon School of Engineering, New York University, 6 Metrotech Center, Brooklyn, New York 11201

E-mail address: gz342@nyu.edu