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SIMPLE AND LARGE EQUIVALENCE RELATIONS

LEWIS BOWEN

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Abstract. We construct ergodic discrete probability-measure-preserving equivalence relations $\mathcal{R}$ that have no proper ergodic normal subequivalence relations and no proper ergodic finite-index subequivalence relations. We show that every treeable equivalence relation satisfying a mild ergodicity condition and cost > 1 surjects onto every countable group with ergodic kernel. Lastly, we provide a simple characterization of normality for subequivalence relations and an algebraic description of the quotient.

1. Introduction

Let (X, μ) denote a standard Borel probability space and $\mathcal{R} \subset X \times X$ a Borel equivalence relation. We say that $\mathcal{R}$ is **discrete** if for all $x \in X$, the $\mathcal{R}$-class of x, denoted $[x]_{\mathcal{R}}$, is countable. All equivalence relations considered in this note are discrete regardless of whether this is mentioned explicitly. We endow $\mathcal{R}$ with two measures μ_L and μ_R satisfying:

$$\mu_L(S) = \int |S_x| \, d\mu(x), \quad \mu_R(S) = \int |S^y| \, d\mu(y),$$

where

$$S_x = \{y \in X : \ (x, y) \in S\}, \quad S^y = \{x \in X : \ (x, y) \in S\}.$$

We say that μ is **$\mathcal{R}$-quasi-invariant** if μ_L and μ_R are in the same measure class. We say μ is **$\mathcal{R}$-invariant** or $\mathcal{R}$ is **measure-preserving** if $\mu_L = \mu_R$. A subset $A \subset X$ is **$\mathcal{R}$-invariant** or **$\mathcal{R}$-saturated** if it is a union of $\mathcal{R}$-classes. We say $\mathcal{R}$ is **ergodic** if for every measurable $\mathcal{R}$-invariant subset $A \subset X$, $\mu(A) \in \{0, 1\}$. In the sequel we will always assume μ is $\mathcal{R}$-quasi-invariant.

A Borel subset $\mathcal{S} \subset \mathcal{R}$ is a **subequivalence relation** if it is a Borel equivalence relation in its own right. If $\mathcal{S}, \mathcal{S}' \subset \mathcal{R}$ are two subequivalence relations whose symmetric difference is null with respect to μ_L (or equivalently μ_R), then we say $\mathcal{S}$ and $\mathcal{S}'$ agree almost everywhere (a.e.). From here on we will not distinguish between relations that agree almost everywhere. We say $\mathcal{S}$ is **proper** if it does not equal $\mathcal{R}$ a.e. We usually write $\mathcal{S} \le \mathcal{R}$ to mean $\mathcal{S} \subset \mathcal{R}$ (a.e. with respect to μ_L) when $\mathcal{S}$ is a subequivalence relation.

The concept of a normal subequivalence relation was introduced in [FSZ88, FSZ89] where it was shown that if $\mathcal{S} \subset \mathcal{R}$ is normal, then there is a natural quotient

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object, denoted $\mathcal{R}/\mathcal{S}$, which is a discrete measured groupoid (groupoids are defined in §2). Moreover, if $\mathcal{S}$ is ergodic, then $\mathcal{R}/\mathcal{S}$ is a countable group and in this case we say $\mathcal{R}$ **surjects** onto $\mathcal{R}/\mathcal{S}$ and $\mathcal{R}/\mathcal{S}$ is a quotient of $\mathcal{R}$.

Unfortunately, the definition of normality given in [FSZ88, FSZ89] is rather complicated. In §2 we provide a simple characterization: $\mathcal{S}$ is **normal** if and only if it is the kernel of a Borel morphism $c : \mathcal{R} \to \mathcal{G}$ where $\mathcal{G}$ is a discrete Borel groupoid. We also show in §3 that when $\mathcal{S}$ is normal and ergodic in $\mathcal{R}$, then $\mathcal{R}/\mathcal{S}$ is isomorphic with the full group $[\mathcal{R}]$ modulo the normalizer of $[\mathcal{S}]$ in $[\mathcal{R}]$. This fact was probably known to the authors of [FSZ88, FSZ89] but it is not explicitly stated.

If $\mathcal{S} \subset \mathcal{R}$ is an arbitrary subequivalence relation and $\mathcal{R}$ is ergodic, then, as shown in [FSZ89], there exists a number $N \in \mathbb{N} \cup \{\infty\}$ such that for a.e. $x \in X$, $[x]_{\mathcal{R}}$ contains exactly N $\mathcal{S}$-classes. This number N is called the **index** of $\mathcal{S}$ in $\mathcal{R}$ and is denoted $N = [\mathcal{R} : \mathcal{S}]$. Our first main result:

Theorem 1.1. *There exists an ergodic discrete probability-measure-preserving equivalence relation $\mathcal{R}$ such that $\mathcal{R}$ does not contain any proper ergodic normal subequivalence relations. Moreover, $\mathcal{R}$ does not contain any proper finite-index ergodic subequivalence relations.*

The proof of Theorem 1.1 is based on Popa's Cocycle Superrigidity Theorem [Po07]. The ergodicity condition is necessary because if $\mathcal{P}$ is any finite measurable partition of X, then the subequivalence relation $\mathcal{S}$ defined by: $(x, y) \in \mathcal{S}$ if and only if $(x, y) \in \mathcal{R}$ and x, y are in the same part of $\mathcal{P}$, has finite-index and is normal in $\mathcal{R}$. Of course, $\mathcal{S}$ is not ergodic if $\mathcal{P}$ is nontrivial.

Remark 1. Stefaan Vaes constructed the first explicit examples of type II_1 von Neumann algebras having only trivial finite-index subfactors in [Va08] by a twisted group-measure space construction. Moreover it follows from [Va08, Theorem 6.4] that the orbit-equivalence relation of the generalized Bernoulli shift action $SL(2, \mathbb{Q}) \ltimes \mathbb{Q}^2 \curvearrowright (X_0, \mu_0)^{\mathbb{Q}^2}$ has no finite-index ergodic subequivalence relations and no nontrivial finite extensions. Here (X_0, μ_0) is any nontrivial atomic probability space with unequal weights (so that it has a trivial automorphism group). More generally, the proof of [Va08, Theorem 6.4] shows how to describe all finite-index subequivalence relations, extensions and bimodules whenever cocycle superrigidity applies.

We say that a measured equivalence relation $\mathcal{R}$ is **large** if for every countable group G there exists an ergodic normal subequivalence relation $\mathcal{N} \le \mathcal{R}$ such that $\mathcal{R}/\mathcal{N} \cong G$.

Next we prove that some treeable equivalence relations are large:

Theorem 1.2. *Suppose $\mathcal{R}$ is a treeable ergodic equivalence relation on (X, μ) of cost > 1 and there exists an ergodic primitive proper subequivalence relation $\mathcal{S} \le \mathcal{R}$. Then $\mathcal{R}$ is large.*

The terms treeable and primitive are explained in §6 below. (Primitive means the same as free factor; this notion was studied by Damien Gaboriau [Ga00, Ga05]). For example, the orbit-equivalence relation of any Bernoulli shift over a rank ≥ 2 free group satisfies the hypothesis above and hence is large. It is an open question whether every ergodic treeable equivalence relation with cost > 1 satisfies the hypotheses of Theorem 1.2.[1] In particular it is unknown whether every such

[1] Robin Tucker-Drob recently proved that the answer to this question is positive. Thus every ergodic treeable pmp equivalence relation of cost > 1 is large.

equivalence relation surjects every countable group. In unpublished work, Clinton Conley, Damien Gaboriau, Andrew Marks and Robin Tucker-Drob have proven that any treeable *strongly ergodic* pmp equivalence relation satisfies the hypothesis of Theorem 1.2 and therefore surjects onto every countable group.

Organization

In §2 we provide a simple characterization for normality of a subequivalence relation. §3 provides an algebraic description for the quotient $\mathcal{R}/\mathcal{N}$. §4 reviews generalized Bernoulli shifts and Popa's Cocycle Superrigidity Theorem [Po07]. The latter is used in §5 to prove Theorem 1.1. The last section §6 proves Theorem 1.2. The proof is mostly independent of the rest of the paper.

2. A simple criterion for normality

The purpose of this section is to give a simple criterion for normality. To begin, we need some definitions.

Definition 1 (Choice functions)**.** Let $\mathcal{R}$ be an ergodic discrete Borel equivalence relation on (X,μ). Let $\mathcal{S} \subset \mathcal{R}$ be a Borel subequivalence relation and let $N = [\mathcal{R} : \mathcal{S}]$. A **family of choice functions** is a set $\{\phi_j\}_{j=1}^N$ of Borel functions $\phi_j : X \to X$ such that for each $x \in X$, $\{[\phi_j(x)]_{\mathcal{S}}\}_{j=1}^N$ is a partition of $[x]_{\mathcal{R}}$. [FSZ89, Lemma 1.1] shows that a family of choice functions exists.

Definition 2 (Discrete probability groupoid)**.** A **groupoid** is a category in which all morphisms are invertible. To be precise, a groupoid consists of the following data. There is a set $\mathcal{G} = \mathcal{G}^1$ of **morphisms** and a subset $\mathcal{G}^0 \subset \mathcal{G}$ of **objects**.

There are source and range maps, denoted $\mathfrak{s} : \mathcal{G} \to \mathcal{G}^0$ and $\mathfrak{r} : \mathcal{G} \to \mathcal{G}^0$. We let $\mathcal{G}^2 \subset \mathcal{G} \times \mathcal{G}$ be the set of all "composable pairs". Precisely, $\mathcal{G}^2$ is the set of all $(f,g) \in \mathcal{G} \times \mathcal{G}$ such that $\mathfrak{r}(g) = \mathfrak{s}(f)$. Then there is a **composition map** $c : \mathcal{G}^2 \to \mathcal{G}$ satisfying the associative property: $c(f, c(g,h)) = c(c(f,g),h)$ (for any $f,g,h \in \mathcal{G}$). Moreover, there is an **inverse map** $\iota : \mathcal{G} \to \mathcal{G}$ satisfying $c(\iota(f), f) = \mathfrak{s}(f)$ and $c(f, \iota(f)) = \mathfrak{r}(f)$ for all f. Moreover, $c(\mathfrak{r}(f), f) = f = c(f, \mathfrak{s}(f))$ for any f. The maps $\mathfrak{s}, \mathfrak{r}, \iota, c$ are called the **structure maps**.

To ease notation, we write $c(f,g) = fg$ and $\iota(f) = f^{-1}$ for $f, g \in \mathcal{G}$. We also denote the groupoid by $\mathcal{G}$ leaving the structure maps implicit.

A groupoid is **discrete** if for every $x \in \mathcal{G}^0$, $\mathfrak{s}^{-1}(x)$ and $\mathfrak{r}^{-1}(x)$ are at most countable. A groupoid is **Borel** if there is a Borel structure on $\mathcal{G}$ such that $\mathcal{G}^0$ is a Borel subset of $\mathcal{G}$, $\mathcal{G}^2$ is a Borel subset of $\mathcal{G} \times \mathcal{G}$ with the product structure and all of the structure maps are Borel. Let μ be a Borel probability measure on $\mathcal{G}^0$. Define two measures $\mu_{\mathfrak{s}}, \mu_{\mathfrak{r}}$ on $\mathcal{G}$ by

$$\mu_{\mathfrak{s}}(K) = \int |\mathfrak{s}^{-1}(x) \cap K| \; d\mu(x), \quad \mu_{\mathfrak{r}}(K) = \int |\mathfrak{r}^{-1}(x) \cap K| \; d\mu(x).$$

We say $(\mathcal{G}, \mu)$ is a **discrete probability groupoid** if $\mu_{\mathfrak{s}}$ and $\mu_{\mathfrak{r}}$ are equivalent measures. A subset $X \subset \mathcal{G}^0$ is **invariant** if for every $x \in X$ and $f, g \in \mathcal{G}$ with $\mathfrak{s}(f) = \mathfrak{r}(g) = x$ we have $\mathfrak{r}(f), \mathfrak{s}(g) \in X$. The measure μ is **ergodic** if every measurable invariant set X has $\mu(X) \in \{0, 1\}$.

Example 1. Let G be a countable group. We may think of G as a groupoid with $G^0 = \{e\}$ and composition being group composition. In this case the trivial probability measure on G^0 makes G into a discrete probability groupoid. This measure is ergodic.

Example 2. Let $\mathcal{R}$ be a discrete equivalence relation on a probability space (X, μ). Assume μ is $\mathcal{R}$-quasi-invariant. We may think of $\mathcal{R}$ as a probability groupoid with $\mathcal{R}^0 = \{(x,x) : x \in X\}$, $\mathfrak{s}(x,y) = y, \mathfrak{r}(x,y) = x, \iota(x,y) = (y,x)$ and $c((x,y)(y,z)) = (x,z)$.

Definition 3. Let $\mathcal{G}, \mathcal{H}$ be groupoids. A map $\alpha : \mathcal{G} \to \mathcal{H}$ is a **morphism** (or **functor**) if $\alpha(fg) = \alpha(f)\alpha(g)$ for all composable $f, g \in \mathcal{G}$. In particular, if $\mathcal{R}$ is an equivalence relation, then a morphism is also called a **cocycle**. Moreover a map $\alpha : \mathcal{R} \to \mathcal{G}$ is a morphism if $\alpha(x,z) = \alpha(x,y)\alpha(y,z)$ for all $(x,y),(y,z) \in \mathcal{R}$. The **kernel** of α is the subgroupoid $\ker(\alpha) \subset \mathcal{G}$ consisting of all $g \in \mathcal{G}$ such that $\alpha(g) \in \mathcal{H}^0$. In particular, if $\mathcal{G}$ is an equivalence relation, then $\ker(\alpha)$ is a subequivalence relation.

Theorem 2.1. *Let $\mathcal{R}$ be an ergodic discrete Borel equivalence relation on (X, μ). Let $\mathcal{S} \subset \mathcal{R}$ be a subequivalence relation. The following are equivalent:*

(1) $\mathcal{S}$ is normal in $\mathcal{R}$ in the sense of [FSZ89, Definition 2.1].

(2) There are choice functions $\{\phi_j\}$ for $\mathcal{S} \subset \mathcal{R}$ with $\phi_j \in \mathrm{End}_{\mathcal{R}}(\mathcal{S})$ for all j. This means that if $(x,y) \in \mathcal{S}$, then $(\phi_j(x), \phi_j(y)) \in \mathcal{S}$.

(3) The extension $\rho : \hat{\mathcal{R}} = \mathcal{S} \times_\sigma J \to \mathcal{S}$ is normal in the sense of Zimmer [Zi76];

(4) There is a discrete, ergodic measured groupoid $(\mathcal{H}, \nu)$ and a morphism $\theta : \mathcal{R} \to \mathcal{H}$ such that

- *(a) $\ker(\theta) = \mathcal{S}$;*
- *(b) θ is class-surjective in the following sense: for any $h \in \mathcal{H}$ and $x \in X$ with $\theta(x)$ equal to the source of h, there exists $y \in [x]_{\mathcal{R}}$ with $\theta(y,x) = h$;*
- *(c) for any discrete ergodic measured groupoid $(\mathcal{H}', \nu')$ and homomorphism $\theta' : \mathcal{R} \to \mathcal{H}'$ with $\ker(\theta') \supset \mathcal{S}$ there is a morphism $\kappa : \mathcal{H} \to \mathcal{H}'$ with $\kappa\theta = \theta'$;*

(5) there is a discrete Borel groupoid $\mathcal{G}$ and a Borel morphism $c : \mathcal{R} \to \mathcal{G}$ with $\mathcal{S} = \ker(c)$.

Proof. The equivalence of the first four statements is [FSZ89, Theorem 2.2]. Clearly (4) implies (5). So we need only show that (5) implies (2). So let $\mathcal{G}$ be a discrete Borel groupoid with unit space $\mathcal{G}^0$, source and range maps $\mathfrak{s}, \mathfrak{r} : \mathcal{G} \to \mathcal{G}^0$. Also let $c : \mathcal{R} \to \mathcal{G}$ be a Borel morphism. Without loss of generality, we may assume c is surjective. Because $\mathcal{R}$ is ergodic there exists $N \in \mathbb{N} \cup \{\infty\}$ such that $\{c(x,y) : y \in [x]_{\mathcal{R}}\}$ has cardinality N for a.e. x. After removing a measure zero subset of X, we may assume $\{c(x,y) : y \in [x]_{\mathcal{R}}\}$ has cardinality N for every x.

By the Lusin-Novikov Theorem [Ke95, Theorem 18.10], there exists a countable family of Borel functions $\{f_j\}_{j=1}^N$, $f_j : \mathcal{G}^0 \to \mathcal{G}$ such that for every $x \in \mathcal{G}^0$,

$$\{f_j(x)\}_{j=1}^N = \mathfrak{s}^{-1}(x).$$

Since each f_j is injective and Borel, the image of f_j is a Borel subset of $\mathcal{G}$. Let $\mathrm{Image(f_j)}$ denote this subset. Then $c^{-1}(\mathrm{Image(f_j)}) \subset \mathcal{R}$ is Borel. Also its left projection is X (equivalently, $\mathfrak{s}(c^{-1}(\mathrm{Image})) = \mathrm{X}$). So for each j there exists a Borel map $\psi_j : X \to X$ with a graph contained in $c^{-1}(\mathrm{Image(f_j)})$.

For each $x \in X$, define $\phi_1(x) = \psi_1(x)$. For $i > 1$ inductively define $\phi_i(x) = \psi_j(x)$ where j is the smallest number such that there does not exist $k < i$ with $(\phi_k(x), \psi_j(x)) \in \mathcal{S}$. Then $\{\phi_j\}_{j=1}^N$ is a family of choice functions satisfying (2). $\square$

3. The quotient group

The purpose of this section is to provide an algebraic description of the quotient $\mathcal{R}/\mathcal{N}$ when $\mathcal{N} \lhd \mathcal{R}$ is normal.

To be precise, let $\mathcal{R}$ denote an ergodic probability-measure-preserving discrete equivalence relation on a probability space (X, μ). Let $\mathrm{Aut}(X, \mu)$ be the group of all measure-preserving automorphisms $\phi : X \to X$. We implicitly identify two automorphisms that agree almost everywhere. Let $\mathrm{Aut}(\mathcal{R})$ be the subgroup of all $\phi \in \mathrm{Aut}(X, \mu)$ such that $x\mathcal{R}y \Rightarrow \phi(x)\mathcal{R}\phi(y)$ (for μ_L-a.e. (x, y)). Also let $[\mathcal{R}] = \mathrm{Inn}(\mathcal{R})$ be the subgroup of all $\phi \in \mathrm{Aut}(\mathcal{R})$ such that $x\mathcal{R}\phi(x)$ for a.e. x. Then $[\mathcal{R}]$ is normal in $\mathrm{Aut}(\mathcal{R})$, so we may consider the quotient $\mathrm{Out}(\mathcal{R}) := \mathrm{Aut}(\mathcal{R})/[\mathcal{R}]$.

In the sequel we use the word 'countable' to mean 'countable or finite'.

Let Γ be a countable subgroup of $\mathrm{Aut}(\mathcal{R})$. Let $\mathcal{R}_\Gamma := \langle \mathcal{R}, \Gamma \rangle$ denote the smallest equivalence relation on X such that $\mathcal{R} \subset \mathcal{R}_\Gamma$ and $(x, \gamma x) \in \mathcal{R}_\Gamma$ for all $x \in X$ and $\gamma \in \Gamma$. Observe that this is a discrete probability-measure-preserving ergodic equivalence relation and $\mathcal{R}$ is normal in $\mathcal{R}_\Gamma$.

Lemma 3.1. *If $\Gamma, \Lambda \le \mathrm{Aut}(\mathcal{R})$ are countable subgroups and $\Gamma[\mathcal{R}] = \Lambda[\mathcal{R}]$, then $\mathcal{R}_\Gamma = \mathcal{R}_\Lambda$. Here, $\Gamma[\mathcal{R}] \subset \mathrm{Out}(\mathcal{R})$ denotes the image of Γ in $\mathrm{Out}(\mathcal{R}) = \mathrm{Aut}(\mathcal{R})/[\mathcal{R}]$.*

Proof. This is straightforward. □

So if $\Gamma \le \mathrm{Out}(\mathcal{R})$ is any countable subgroup, then we may define $\mathcal{R}_\Gamma := \mathcal{R}_{\Gamma'}$ where $\Gamma' \le \mathrm{Aut}(\mathcal{R})$ is any countable subgroup such that $\Gamma = \Gamma'[\mathcal{R}]$. The next lemma follows immediately.

Lemma 3.2. *If $\Gamma \le \Lambda \le \mathrm{Out}(\mathcal{R})$ are countable subgroups, then $\mathcal{R}_\Gamma \le \mathcal{R}_\Lambda$.*

Theorem 3.3. *Let $\mathcal{R} \le \mathcal{U}$ be ergodic discrete probability-measure-preserving equivalence relations on (X, μ). Suppose $\mathcal{R}$ is normal in $\mathcal{U}$. Then there exists a countable subgroup $\Gamma \le \mathrm{Out}(\mathcal{R})$ such that $\mathcal{U} = \mathcal{R}_\Gamma$. Moreover, $\mathcal{R}_\Gamma/\mathcal{R}$ is isomorphic to Γ. In particular, $[\mathcal{R}_\Gamma : \mathcal{R}] = |\Gamma|$.*

Proof. This follows from [FSZ89, Theorems 2.12 and 2.13]. □

Corollary 3.4. *Let $\mathcal{R} \le \mathcal{U}$ be ergodic discrete probability-measure-preserving equivalence relations on (X, μ). Suppose $\mathcal{R}$ is normal in $\mathcal{U}$. Then*

$$\mathcal{U}/\mathcal{R} \cong N_\mathcal{U}(\mathcal{R})/[\mathcal{R}]$$

where

$$N_\mathcal{U}(\mathcal{R}) := \{g \in [\mathcal{U}] : \; g[\mathcal{R}]g^{-1} = [\mathcal{R}]\}.$$

Moreover $N_\mathcal{U}(\mathcal{R}) = [\mathcal{U}] \cap \mathrm{Aut}(\mathcal{R})$.

Proof. We prove the last claim first. So suppose $g \in N_\mathcal{U}(\mathcal{R})$ and $x \in X$. Then for any $\phi \in [\mathcal{R}]$ we must have $g\phi g^{-1} \in [\mathcal{R}]$. This implies $(x, g\phi g^{-1}x) \in \mathcal{R}$ which implies, by replacing x with gx, that $(gx, g\phi x) \in \mathcal{R}$. Since ϕ is arbitrary and $[\mathcal{R}]$ acts transitively on each $\mathcal{R}$-class, this implies $g \in \mathrm{Aut}(\mathcal{R})$. Thus $N_\mathcal{U}(\mathcal{R}) \subset [\mathcal{U}] \cap \mathrm{Aut}(\mathcal{R})$.

Now suppose $g \in [\mathcal{U}] \cap \mathrm{Aut}(\mathcal{R})$. If $\phi \in [\mathcal{R}]$, then $(x, \phi x) \in \mathcal{R}$. This implies $(gx, g\phi x) \in \mathcal{R}$. By replacing x with $g^{-1}x$ we obtain $(x, g\phi g^{-1}x) \in \mathcal{R}$ which implies $g\phi g^{-1} \in [\mathcal{R}]$ (since x is arbitrary). Thus $g \in N_\mathcal{U}(\mathcal{R})$. This proves $N_\mathcal{U}(\mathcal{R}) = [\mathcal{U}] \cap \mathrm{Aut}(\mathcal{R})$.

By Theorem 3.3 there exists a countable subgroup $\Gamma \le \mathrm{Out}(\mathcal{R})$ such that $\mathcal{R}_\Gamma = \mathcal{U}$ and $\mathcal{U}/\mathcal{R} \cong \Gamma$. Let $\tilde{\Gamma} \le \mathrm{Aut}(\mathcal{R})$ be the inverse image of Γ under the quotient map

$\mathrm{Aut}(\mathcal{R}) \to \mathrm{Aut}(\mathcal{R})/[\mathcal{R}] = \mathrm{Out}(\mathcal{R})$. Since $\mathcal{U} = \mathcal{R}_\Gamma$ we must have $\tilde{\Gamma} \le [\mathcal{U}]$ and therefore $\tilde{\Gamma} \le N_{\mathcal{U}}(\mathcal{R})$ which implies $\Gamma \le N_{\mathcal{U}}(\mathcal{R})/[\mathcal{R}]$.

On the other hand, we clearly have $\mathcal{R}_{N_{\mathcal{U}}(\mathcal{R})} \le \mathcal{U} = \mathcal{R}_\Gamma$. So Lemma 3.2 implies $N_{\mathcal{U}}(\mathcal{R})/[\mathcal{R}] \le \Gamma$. Theorem 3.3 now implies $N_{\mathcal{U}}(\mathcal{R})/[\mathcal{R}] = \Gamma \cong \mathcal{U}/\mathcal{R}$. □

4. Generalized Bernoulli shifts and cocycle superrigidity

Let G be a countable group, I a countable set on which G acts and (X_0, μ_0) a standard probability space. We let X_0^I be the set of all functions $x : I \to X_0$ and μ_0^I the product measure on X_0^I. Then $G \curvearrowright X_0^I$ by $(gx)(i) = x(g^{-1}i)$. This action preserves the measure μ_0^I. The action $G \curvearrowright (X_0, \mu_0)^I$ is a **generalized Bernoulli shift**.

Our interest in these actions stems from Popa's Cocycle Superrigidity Theorem. To explain, let $G \curvearrowright (X, \mu)$ be a probability-measure-preserving action. A **cocycle** into a countable group H is a Borel map $c : G \times X \to H$ such that

$$c(g_1 g_2, x) = c(g_1, g_2 x) c(g_2, x).$$

Alternatively, if $G \curvearrowright (X, \mu)$ is essentially free (this means $gx \neq x$ for every $g \in G - \{1_G\}$ and a.e. $x \in X$), then we can identify $G \times X$ with the orbit-equivalence relation, denoted $\mathcal{R}$, via $(g, x) \mapsto (gx, x)$. In this way, we can think of the cocycle as a map from $\mathcal{R}$ to H. We say the action is **cocycle superrigid** if for every such cocycle there is a homomorphism $\rho : G \to H$ and a Borel map $\phi : X \to H$ such that

$$c(g, x) = \phi(gx)^{-1} \rho(g) \phi(x).$$

The next result is a special case of a celebrated theorem due to S. Popa [Po07] (see also [PV08, Theorem 3.2 and Proposition 2.3]).

Theorem 4.1. *Suppose every orbit of $G \curvearrowright I$ is infinite. If G has property (T), then the generalized Bernoulli shift $G \curvearrowright (X_0, \mu_0)^I$ is cocycle superrigid.*

5. A simple equivalence relation

Theorem 5.1. *Suppose $G \curvearrowright (X, \mu)$ is an essentially free measure-preserving ergodic action of a countably infinite group G on a standard probability space (X, μ). Let $\mathcal{R} = \{(x, gx) : \ x \in X, g \in G\}$ be the orbit equivalence relation. Suppose $G \curvearrowright (X, \mu)$ is cocycle superrigid. If G is simple, then $\mathcal{R}$ has no proper ergodic normal subequivalence relations. If G has no nontrivial finite quotients, then $\mathcal{R}$ has no proper ergodic normal finite-index subequivalence relations.*

Proof. Let $\mathcal{R}$ be the orbit-equivalence relation of the action $G \curvearrowright (X, \mu)$. Let $\mathcal{N} \lhd \mathcal{R}$ be an ergodic normal subequivalence relation and $c : \mathcal{R} \to \mathcal{R}/\mathcal{N}$ the canonical cocycle. Since the action is cocycle superrigid, there exists a Borel map $\phi : X \to H := \mathcal{R}/\mathcal{N}$ and a homomorphism $\rho : G \to H$ such that

$$c(gx, x) = \phi(gx)^{-1} \rho(g) \phi(x).$$

Claim 1. If ρ is trivial, then $\mathcal{N} = \mathcal{R}$.

Proof of Claim 1. For every $g \in \mathcal{R}/\mathcal{N}$, $\phi^{-1}(g) \subset X$ is $\mathcal{N}$-invariant. Because $\mathcal{N}$ is ergodic, this implies ϕ is essentially constant which implies $\mathcal{N} = \mathcal{R}$. □

Claim 2. ρ is noninjective.

Proof of Claim 2. To obtain a contradiction, suppose ρ is injective. We claim that $\mathcal{N}$ is finite. To see this, let

$$X_g = \{x \in X : (gx, x) \in \mathcal{N}\} = \{x \in X : \phi(gx)\phi(x)^{-1} = \rho(g)\}.$$

Also, for $g \in G, h \in \mathcal{R}/\mathcal{N}$ let

$$X_{g,h} = \{x \in X_g : \phi(x) = h\} = \{x \in X : \phi(x) = h, \phi(gx) = \rho(g)h\}.$$

Because ρ is injective, for any fixed h, the sets $\{gX_{g,h} : g \in G\}$ are pairwise disjoint. Therefore $\sum_{g\in G} \mu(X_{g,h}) \leq 1$. By the Borel-Cantelli Lemma, almost every x is contained in at most finitely many of the $X_{g,h}$'s (for fixed h). However for each $x \in X$ there is exactly one h such that x is contained in some $X_{g,h}$. So, in fact, x is contained in at most finitely many $X_{g,h}$'s (allowing g and h to vary). Since $X_g = \bigcup_h X_{g,h}$ this implies that a.e. x is contained in at most finitely many X_g's which implies that for a.e. x, the $\mathcal{N}$-equivalence class $[x]_{\mathcal{N}}$ is finite.

Because G is infinite and $G \curvearrowright (X, \mu)$ is essentially free and ergodic, μ is nonatomic. Because $\mathcal{N}$ is finite and μ is nonatomic, $\mathcal{N}$ is not ergodic. This contradiction proves that ρ is noninjective. □

If G is simple, either ρ is trivial or injective and so the claims above finish the proof. If G does not have any nontrivial finite quotients and $\mathcal{R}/\mathcal{N}$ is finite, then $\rho : G \to \mathcal{R}/\mathcal{N}$ must be trivial. So Claim 1 finishes the proof. □

Definition 4 (Restrictions)**.** Let $\mathcal{R} \subset X \times X$ be a probability-measure-preserving Borel equivalence relation on a probability space (X, μ). If $Y \subset X$ has positive measure, then we let $\mathcal{R}_Y := \mathcal{R} \cap (Y \times Y)$ denote the **restriction of** $\mathcal{R}$ **to** Y. It is an equivalence relation on Y.

Lemma 5.2. *Let $Y \subset X$ be a Borel set with positive measure. Let $\mathcal{S} \leq \mathcal{R}_Y$ be a subequivalence relation. If $\mathcal{R}$ is ergodic, then there exists a subequivalence relation $\mathcal{T} \leq \mathcal{R}$ such that $\mathcal{T}_Y = \mathcal{S}$. Moreover, if $\mathcal{S}$ is ergodic, then $\mathcal{T}$ is ergodic and if $\mathcal{S}$ is normal in $\mathcal{R}_Y$, then $\mathcal{T}$ is normal in $\mathcal{R}$.*

Remark 2. $\mathcal{T}$ is not unique. Moreover, even if $\mathcal{S}$ is ergodic and normal there may exist subequivalence relations $\mathcal{T}'$ such that $\mathcal{T}'_Y = \mathcal{S}$ but $\mathcal{T}'_Y$ is neither ergodic nor normal.

Proof. Because $\mathcal{R}$ is ergodic there exists a Borel map $\phi : X \to Y$ with graph contained in $\mathcal{R}$ such that $\phi(y) = y$ for every $y \in Y$. Define the subequivalence relation $\mathcal{T}$ by $x\mathcal{T}y$ iff $\phi(x)\mathcal{S}\phi(y)$. In other words, if $\Phi : \mathcal{R} \to \mathcal{R}_Y$ is the map $\Phi(x, y) = (\phi(x), \phi(y))$, then $\mathcal{T} = \Phi^{-1}(\mathcal{S})$. This implies that $\mathcal{T}$ is Borel. It is easy to check that $\mathcal{T}$ is a subequivalence relation and $\mathcal{T}_Y = \mathcal{S}$.

Suppose that $\mathcal{S}$ is ergodic. Let $A \subset X$ be a $\mathcal{T}$-saturated set of positive measure. Observe that $A = \phi^{-1}(\phi(A))$ by definition of $\mathcal{T}$. Also $\phi(A)$ is $\mathcal{S}$-saturated. Therefore $\phi(A) = Y$ since $\mathcal{S}$ is ergodic. So $A = \phi^{-1}\phi(A) = X$. Because A is arbitrary, $\mathcal{T}$ is ergodic.

Suppose that $\mathcal{S}$ is normal. Then there exists a groupoid morphism $c : \mathcal{R}_Y \to \mathcal{G}$ such that $\mathcal{S} = \ker(c)$. Define $c' : \mathcal{R} \to \mathcal{G}$ by $c'(x, y) = c(\phi(x), \phi(y))$. Observe that

$$c'(x, y)c'(y, z) = c(\phi(x), \phi(y))c(\phi(y), \phi(z)) = c(\phi(x), \phi(z)) = c'(x, z).$$

So c' is a cocycle. If $(x, y) \in \ker(c')$, then $c(\phi(x), \phi(y)) \in \mathcal{G}^0$ which implies $(\phi(x), \phi(y)) \in \ker(c) = \mathcal{S}$ which implies $(x, y) \in \mathcal{T}$. So $\ker(c') \subset \mathcal{T}$. On the other hand, if $(x,y) \in \mathcal{T}$, then $(\phi(x),\phi(y)) \in \mathcal{S} = \ker(c)$ which implies $(x,y) \in \ker(c')$. So $\mathcal{T} = \ker(c')$ is normal by Theorem 2.1. □

Proposition 5.3. *Let $\mathcal{R}$ be an ergodic probability-measure-preserving equivalence relation with a finite-index ergodic subequivalence relation $\mathcal{S} \le \mathcal{R}$. Then $\mathcal{R}$ has a finite-index ergodic normal subequivalence relation $\mathcal{N}$ with $\mathcal{N} \le \mathcal{S}$.*

Proof. Let $n = [\mathcal{R} : \mathcal{S}]$ denote the index of $\mathcal{S}$ in $\mathcal{R}$. Let $\phi : \mathcal{R} \to \{1,\dots,n\}$ be any Borel function satisfying

- for a.e. $x \in X$, $\phi(x,x) = 1$;
- for a.e. $(x,y),(x,z) \in \mathcal{R}$ with $(y,z) \in \mathcal{S}$, $\phi(x,y) = \phi(x,z)$;
- for a.e. $x \in X$, the map $y \mapsto \phi(x,y)$ surjects onto $\{1,\dots,n\}$. So this map is a bijection from the set of $\mathcal{S}$-classes in $[x]_{\mathcal{R}}$ to $\{1,\dots,n\}$.

Define a cocycle $\alpha : \mathcal{R} \to \mathrm{Sym}(n)$ (the symmetric group of $\{1,\dots,n\}$) by

$$\alpha(x,y)(k) = \phi(x,z)$$

where $z \in [x]_{\mathcal{R}}$ is any element satisfying $\phi(y,z) = k$. Let $\mathcal{K}$ be the kernel of this cocycle. This is a finite-index normal subequivalence relation and $\mathcal{K} \le \mathcal{S}$ but $\mathcal{K}$ might not be ergodic. However, it can have at most finitely many ergodic components (this is true for any finite-index subequivalence relation). Let $Y \subset X$ be an ergodic component of $\mathcal{K}$. So $\mathcal{K}_Y$ is an ergodic finite-index normal subequivalence relation of $\mathcal{R}_Y$. By Lemma 5.2 there exists an ergodic normal finite-index subequivalence relation $\mathcal{N} \le \mathcal{R}$ such that $\mathcal{N}_Y = \mathcal{K}_Y$. Since $\mathcal{N}$ and $\mathcal{S}$ are ergodic and $\mathcal{N}_Y \le \mathcal{S}_Y$ we must have that $\mathcal{N} \le \mathcal{S}$. □

Proof of Theorem 1.1. Let G be a simple property (T) group. Quoting from [Th10]: there are two sources of simple groups with Kazhdan's property (T). Such groups appear for example as lattices in certain Kac-Moody groups; see [CR06]. Much earlier, it was also shown by Gromov ([Gr87]) that every hyperbolic group surjects onto a Tarski monster, i.e., every proper subgroup of this quotient is finite cyclic; in particular: this quotient group is simple and is a Kazhdan group if the hyperbolic group was a Kazhdan group.

Let (X_0,μ_0) be a nontrivial Borel probability space and $G \curvearrowright (X,\mu) := (X_0,\mu_0)^G$ the Bernoulli shift action. By Popa's Cocycle Superrigidity Theorem 4.1, $G \curvearrowright (X,\mu)$ is cocycle superrigid. So Theorem 5.1 implies $\mathcal{R}$ has no ergodic proper normal subequivalence relations. Proposition 5.3 implies $\mathcal{R}$ has no ergodic proper finite-index subequivalence relations □

6. Treeable equivalence relations

Definition 5. A **graphing** of an equivalence relation $\mathcal{R} \subset X \times X$ is a Borel subset $G \subset X \times X$ such that $\mathcal{R}$ is the smallest equivalence relation containing G and G is symmetric: $(x,y) \in G \Rightarrow (y,x) \in G$. The **local graph** of G at x is denoted by G_x. It has vertex set $[x]_{\mathcal{R}}$ and edges $\{y,z\}$ where $y,z \in [x]_{\mathcal{R}}$ and $(y,z) \in G$. So G is a graphing if and only if it is symmetric and all local graphs are connected. A graphing is a **treeing** if all of its local graphs are trees.

Definition 6. Let $\mathcal{R}$ be an ergodic treeable equivalence relation. A subequivalence relation $\mathcal{S} \le \mathcal{R}$ is **primitive** if there exist treeings $G_{\mathcal{S}}, G_{\mathcal{R}}$ of $\mathcal{S}$ and $\mathcal{R}$ such that $G_{\mathcal{S}} \subset G_{\mathcal{R}}$. This means the same as free factor as used in [Ga00, Ga05].

Example 3. If $F = \langle S \rangle$ is a free group with free generating set $S \subset F$ and $F \curvearrowright (X, \mu)$ is an essentially free action, then $G_F = \{(x, sx), (sx, x) : x \in X, s \in S\}$ is a treeing of the orbit-equivalence relation $\mathcal{R}$. Moreover if $g \in S$ and $\mathcal{S}$ is the orbit-equivalence relation generated by $\{g^n\}_{n \in \mathbb{Z}}$, then $\mathcal{S}$ is primitive in $\mathcal{R}$ since $G_{\mathcal{S}} = \{(x, gx), (gx, x) : x \in X\}$ is a treeing of $\mathcal{S}$ and $G_{\mathcal{S}} \subset G_F$. More generally, if g is primitive in F (this means that it is contained in some free generating set of F) and $\mathcal{S}$ is the orbit-equivalence relation of $\{g^n\}_{n \in \mathbb{Z}}$, then $\mathcal{S}$ is primitive in $\mathcal{R}$.

Before proving Theorem 1.2 we need a lemma.

Lemma 6.1. *Suppose* Γ *is a countable group and* $c : \mathcal{R} \to \Gamma$ *is a cocycle such that* $\ker(c) \le \mathcal{R}$ *is ergodic. Let*

$$\Lambda = \{g \in \Gamma : \ \mu_L(\{(x, y) \in \mathcal{R} : \ c(x, y) = g\}) > 0\}.$$

Then Λ *is a subgroup of* Γ *and* $\mathcal{R}/\ker(c)$ *is isomorphic to* Λ.

Proof. For $x \in X$, let $\Gamma_x = \{c(x, y) : \ y \in [x]_{\mathcal{R}}\}$. If $(x, z) \in \ker(c)$, then $c(x, y) = c(z, y)$. So $\Gamma_x = \Gamma_z$. Since $\ker(c)$ is ergodic, this implies the existence of a subset $\Gamma' \subset \Gamma$ such that $\Gamma' = \Gamma_x$ for a.e. x. Observe that since $c(x, y) \in \Gamma_x$, $c(y, x) = c(x, y)^{-1} \in \Gamma_y$. Thus Γ' is invariant under inverse. Also if $c(x, y) \in \Gamma_x$ and $c(y, z) \in \Gamma_y$, then $c(x, z) = c(x, y)c(y, z) \in \Gamma_x$. So Γ' is a subgroup. By ergodicity again, $\Lambda = \Gamma'$. So without loss of generality, we may assume $\Lambda = \Gamma$.

It follows from [FSZ89, Theorem 2.2] that there is a homomorphism $\theta' : \mathcal{R}/\ker(c) \to \Gamma$ such that if $\theta : \mathcal{R} \to \mathcal{R}/\ker(c)$ is the canonical morphism, then

$$\theta'\theta = c.$$

Since c and θ have the same kernel, θ' must be injective. Since $\Lambda = \Gamma$, it is also surjective and so $\mathcal{R}/\ker(c)$ is isomorphic to Λ. □

Theorem 6.2. *Suppose* $\mathcal{R}$ *is a treeable ergodic probability-measure-preserving equivalence relation on* (X, μ) *of cost* > 1 *and there exists a subequivalence relation* $\mathcal{S} \le \mathcal{R}$ *that is primitive, ergodic and proper. Then* $\mathcal{R}$ *surjects onto every countable group.*

Proof. Because $\mathcal{S} \le \mathcal{R}$ is primitive, there exist treeings $G_{\mathcal{S}} \subset G_{\mathcal{R}}$ of $\mathcal{S}$ and $\mathcal{R}$. Because $\mathcal{S}$ is proper, $\mu_L(\mathcal{R} \setminus \mathcal{S}) > 0$ and therefore $\mu_L(G_{\mathcal{R}} \setminus G_{\mathcal{S}}) > 0$. Let $c : G_{\mathcal{R}} \setminus G_{\mathcal{S}} \to \mathbb{F}_\infty$ be any measurable map such that

- for every $g \in \mathbb{F}_\infty$, $\mu_L(c^{-1}(g)) > 0$;
- $c(x, y) = c(y, x)^{-1}$ wherever this is defined.

Here $\mathbb{F}_\infty$ denotes the free group of countably infinite rank. We extend c to $G_{\mathcal{S}}$ by $c(x, y) = e$ for any $(x, y) \in \mathcal{S}$. Now c is defined on all of $G_{\mathcal{R}}$. Because $G_{\mathcal{R}}$ is a treeing there is a unique extension of c to a cocycle $c : \mathcal{R} \to \mathbb{F}_\infty$.

By definition $\ker(c)$ contains $\mathcal{S}$. Because $\mathcal{S}$ is ergodic, this implies $\ker(c)$ is ergodic. Lemma 6.1 now implies $\mathcal{R}/\ker(c) \cong \mathbb{F}_\infty$.

Now let Λ be an arbitrary countable group and $\phi : \mathbb{F}_\infty \to \Lambda$ a surjective homomorphism. Let $c' : \mathcal{R} \to \Lambda$ be the cocycle $c'(x, y) = \phi(c(x, y))$. Since $\ker(c)$ is ergodic, $\ker(c')$ is ergodic. So Lemma 6.1 implies $\mathcal{R}/\ker(c') \cong \Lambda$. □

Example 4. If $\mathcal{R}$ is the orbit-equivalence relation of a Bernoulli shift action of $\mathbb{F}_n$ $(n \ge 2)$, then every generator of $\mathbb{F}_n$ acts ergodically. Therefore, $\mathcal{R}$ satisfies the hypotheses of Theorem 6.2.

Conjecture 1. *Let $\mathcal{R}$ be an ergodic treeable equivalence relation of cost > 1. Then there exists an ergodic element $f \in [\mathcal{R}]$ such that the subequivalence relation generated by f is primitive in $\mathcal{R}$.*

Remark 3. Robin Tucker-Drob recently announced a proof of Conjecture 1. Together with Theorem 6.2 this implies that every ergodic pmp treeable equivalence relation $\mathcal{R}$ with cost > 1 is large. So every countable group is a quotient of $\mathcal{R}$.

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Department of Mathematics, University of Hawai'i–Manoa, Honolulu, Hawaii