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ON THE Q-CURVATURE PROBLEM ON $\mathbb{S}^3$

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ABSTRACT. Let $P_{\mathbb{S}^3} = \Delta_0^2 + \frac{1}{2}\Delta_0 - \frac{15}{16}$ denote the Panietz operator on the standard sphere $\mathbb{S}^3$. In this paper, we study the following fourth order elliptic equation with a nonlinear term of negative power type:

$$P_{\mathbb{S}^3} u = -\frac{1}{2} Q u^{-7} \text{ on } \mathbb{S}^3.$$

Here Q is a prescribed smooth function on $\mathbb{S}^3$ which is assumed to be a smooth bounded positive function. We prove the existence of positive solutions to the equation under a non-degeneracy assumption on Q.

1. INTRODUCTION

In the last two decades a great deal of research has been done to study the relationship between conformally covariant operators and the related differential equations. Important achievements on this topic have been made by applying various combinations of analytical techniques, and among all of them we mention only the variational and topological methods. For the latter, especially when the main interest is focused on the existence of positive solutions, the fundamental tool which has been used is the maximum principle. The typical example is the well-known Gidas–Ni–Nirenberg theory; see [17].

Fourth order problems are not much studied due to the lack of the maximum principle [see [13] for a reference on maximum principle]. This fact is very likely the reason why the knowledge on fourth order nonlinear problems is far from being reasonably complete, as it is in the second order case. In the current paper, we would like to make some contribution in this field. Let $(\mathbb{S}^3, g_0)$ be the standard three sphere. The Laplace operator is denoted by Δ_0. In this article, we consider the fourth order operator $P_{\mathbb{S}^3} = \Delta_0^2 + \frac{1}{2}\Delta_0 - \frac{15}{16}$ on $\mathbb{S}^3$. The equation is

$$P_{\mathbb{S}^3} u + \frac{1}{2} Q u^{-7} = 0. \tag{1.1}$$

This equation has its geometric meaning. The function Q is known to be the $Q-$curvature. For a given function Q, we are seeking a positive solution u. Geometrically this problem can be restated as follows:
Given a smooth function Q on $\mathbb{S}^3$, does there exist a metric g conformal to the standard metric g_0 such that $Q_g = Q$?

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The equation is elliptic. The natural method is a variational one due to its nice structure. One may consider the functional

$$J(u) = \left(\int_{\mathbb{S}^3} Qu^{-6}\right)^{\frac{1}{3}} \left(\int_{\mathbb{S}^3} uP_{\mathbb{S}^3}u\right). \tag{1.2}$$

In fact, any positive solution of (1.1) is a critical point of (1.2). In [25], P. Yang and M. Zhu gave a precise lower bound of the energy functional when $Q \equiv 1$:

$$\inf_{u>0, u\in H^2(\mathbb{S}^3)} \frac{\int_{\mathbb{S}^3}\left((\Delta u)^2 - \frac{1}{2}|\nabla u|^2 - \frac{15}{16}u^2\right)}{\left(\int_{\mathbb{S}^3} u^{-6}\right)^{-1/3}} = -\frac{15}{16}\cdot(2\pi)^{4/3}.$$

It is well known that equation (1.1) is not always solvable. The Kazdan-Warner condition is a necessary condition for solvability. Motivated by the conformal covariance of the equation, we assume $Q > 0$ everywhere on $\mathbb{S}^3$ satisfying the non-degeneracy condition

$$(\Delta Q(x))^2 + |\nabla Q(x)|^2 \neq 0, \text{ for all } x \in \mathbb{S}^3. \tag{1.3}$$

For such a function Q, we associate it with a vector field $\Phi : \mathbb{S}^3 \to \mathbb{R}^4$ defined by

$$\mathbf{\Phi}(x) = -\Delta Q(x)\mathbf{x} + \nabla Q(x). \tag{1.4}$$

We also assume that $deg(\frac{\mathbf{\Phi}}{|\mathbf{\Phi}|}, \mathbb{S}^3) \neq 0$. The non-degeneracy condition (1.3) is crucially required in [7], [21] and [22] to obtain a priori estimates for the solution of (2.2).

Our main statement is the following:

Theorem 1.1. *Suppose $Q > 0$ on $\mathbb{S}^3$ is non-degenerate in the sense of* (1.3) *and $deg(\frac{\mathbf{\Phi}}{|\mathbf{\Phi}|}, \mathbb{S}^3) \neq 0$. Then equation* (1.1) *admits a smooth positive solution.*

By Borsuk-Ulam theorem (see [14]), we have the following corollary:

Corollary 1.1. *Suppose $Q > 0$ on $\mathbb{S}^3$ is an even function satisfying $Q(x) = Q(-x)$, which is non-degenerate in the sense of* (1.3)*. Then* (1.1) *admits a smooth positive solution.*

A similar equation with $N \geq 5$ for the conformal Paneitz operator was studied by many authors; in particular Djadli-Hebey-Ledoux [12] studied the coercivity of the Paneitz operator and the positivity of solutions. Moreover, Djadli-Malchiodi-Ahmedou [11] and Hebey-Robert [16] studied the blow-up analysis of the Q-curvature equation.

The direct method is to minimize the functional J over the class of positive functions in the Sobolev space $H^2(\mathbb{S}^3)$. The negative exponent poses analytic difficulty associated with the conformal factor touching zero. The negative sign of the coefficient for the $Q-$curvature term in equation (1.1) makes a sharp contrast with the case when $N \geq 4$, which has been well studied in recent years. In particular, the positivity of the Paneitz operator in dimension three does not follow from the positivity of the scalar curvature. Such phenomena in the second order case only occurs in one dimension, which has been studied by Ai-Chou-Wei in [2].

The paper is organized as follows: in Section 2, we will study the geometric background of the equation and show the Kazdan-Warner condition. The key a priori estimate for positive solutions of (1.1) is provided in Section 3. In Section 4, by using the Lyapunov-Schmidt reduction method, we prove a perturbation result.

Finally, in Section 5, we use the Leray-Schauder degree theory to eliminate the perturbation assumption in Section 4.

2. Background

Let (M, g) be a smooth four-dimensional manifold and let us consider the so-called Paneitz operator on M, discovered by Paneitz [20],

$$P_g\psi = \Delta_g^2\psi - div_g\left(\frac{2}{3}S_g - 2Ric_g\right)d\psi \tag{2.1}$$

where div_g denotes the divergence, d the differential, and S_g, Ric_g denote the scalar and Ricci curvature of (M, g) respectively. Under a conformal change of metric $\tilde{g} = e^{2u}g$, the Paneitz operator becomes

$$P_{\tilde{g}} = e^{-4u}P_g. \tag{2.2}$$

Later on Branson [3] found that the notion of Paneitz operator can be extended to the case $N \neq 4$. In the case, where $N = 3$, it is given by

$$\begin{aligned} P_g\varphi &= \left[\Delta_g^2 - div_g\left(\frac{5}{4}S_g - 4Ric_g\right)d\psi - \frac{1}{2}Q_g\right]\varphi \\ &= \Delta^2\varphi + 4div(Ric(\nabla\phi, e_i), e_i) - \frac{5}{4}div(S_g\nabla\varphi) - \frac{1}{2}Q_g\varphi \end{aligned} \tag{2.3}$$

where

$$Q_g = -\frac{1}{4}\Delta S_g + \frac{23}{32}S_g^2 - 2|Ric_g|^2 \tag{2.4}$$

and $(e_i)_{i=1}^3$ is any local orthonormal frame. This operator, called the "Branson-Paneitz" operator, enjoys similar conformal covariant properties as the conformal Laplacian does in dimension $N \geq 3$.

One of the properties of the Branson-Paneitz operator is as follows: if one considers a conformal metric $\tilde{g} = u^{-4}g$ of g, the operators with respect to the metric $\tilde{g}$ and g are related by the equation

$$P_{\tilde{g}}(\varphi u^{-1}) = u^7 P_g(\varphi) \tag{2.5}$$

for all $\varphi \in C^\infty(M)$. In particular, if one sets $\varphi = u$,

$$P_{\tilde{g}}(1) = u^7 P_g(u) \tag{2.6}$$

and hence the prescribing Q-curvature equation can be written as

$$P_g u = -\frac{1}{2}Q_{\tilde{g}}u^{-7}. \tag{2.7}$$

A conformal transformation φ of a 3-dimensional closed manifold M acts on a conformal metrics $\tilde{g} = u^{-4}g$ by $\varphi^*\tilde{g} = (T_\varphi u)^{-4}g_0$ where

$$T_\varphi u = (u \circ \varphi) \cdot |det(d\varphi)|^{-1/6}. \tag{2.8}$$

Consequently, under the conformal transform φ equation (2.7) is transformed into

$$P_g(T_\varphi u) = -\frac{1}{2}(Q_{\tilde{g}} \circ \varphi)(T_\varphi u)^{-7}. \tag{2.9}$$

On the standard sphere $(\mathbb{S}^3, g_0)$. As in [8], [11] and [22], we consider the following set of conformal transformations: given a point P in $\mathbb{S}^3$ and $t > 1$, we use the stereographic projection $\pi_P : \mathbb{S}^3\backslash\{P\} \to \mathbb{R}^3$ as a chart and let $\phi_{P,t}$ be the conformal map given by $y \mapsto ty$. This set of conformal transformation can be parametrized using the unit open ball B in $\mathbb{R}^4$. Let p be the point $\frac{t-1}{t}P \in B$. We denote the conformal transform $\phi_{P,t}$ by ϕ_p.

It is well known that for conformal covariant equations, we have Kazdan-Warner type constraints (see [18], [8], [12], [22] and many others). In particular, we have

Proposition 2.1. *Let u be a positive smooth solution to* (1.1). *We have*

$$\int_{\mathbb{S}^3} \langle \nabla Q, \nabla x_i \rangle u^{-6} = 0 \tag{2.10}$$

for $i = 1, 2, 3, 4$.

Proof. We only prove for $i = 4$. Since u is a solution to (1.1), it is a critical point of the functional J given in (1.2). We introduce a one-parameter conformal transform φ_t on $\mathbb{S}^3$ by

$$\varphi_t = \phi_{N,t} \qquad t \in (0, +\infty), \tag{2.11}$$

where N is the north pole $(0, 0, 0, 1)$. We have

$$\left.\frac{d}{dt}\right|_{t=1} J[T_{\varphi_t(u)}] = 0.$$

Notice that

$$\int_{\mathbb{S}^3} (T_{\varphi_t} u) P_{\mathbb{S}^3} (T_{\varphi_t} u) = \int_{\mathbb{S}^3} u P_{\mathbb{S}^3} u.$$

It follows that

$$\begin{aligned} 0 &= \left.\frac{d}{dt}\right|_{t=1} \int_{\mathbb{S}^3} Q (T_{\varphi_t} u)^{-6} \\ &= \left.\frac{d}{dt}\right|_{t=1} \int_{\mathbb{S}^3} (Q \circ \varphi_t^{-1}) u^{-6} \\ &= \int_{\mathbb{S}^3} \left.\frac{d}{dt}\right|_{t=1} (Q \circ \varphi_t^{-1}) u^{-6}. \end{aligned}$$

We use the coordinate system (y_1, y_2, y_3) given by stereographic projection

$$\begin{cases} y_1 = \frac{x_1}{1-x_4} \\ y_2 = \frac{x_2}{1-x_4} \\ y_3 = \frac{x_3}{1-x_4} \end{cases}$$

to calculate the integrand explictly. We have

$$\left.\frac{d}{dt}\right|_{t=1} Q\left(\frac{1}{t} y_1, \frac{1}{t} y_2, \frac{1}{t} y_3\right) = -\sum_{i=1}^{3} y_i \frac{\partial Q_i}{\partial y_i} = \langle \nabla Q, \nabla x_4 \rangle.$$

□

Remark 2.1. It is well known that by the Kazdan-Warner type condition, the equation (1.1) is not always solvable for any positive function Q. A simple example is $Q = 1 + \epsilon x_i$.

3. Uniform bound

This section is devoted to obtaining a priori estimates for positive solutions of the prescribed $Q-$curvature equation (1.1) under the non-degeneracy condition (1.3). Mainly we will prove the following theorem in this section by combining the results of several lemmas below.

Theorem 3.1. *If Q is a smooth positive function defined on $\mathbb{S}^3$ satisfying the non-degeneracy condition* (1.3)*, then there exist positive constants C_1 and C_2 which only depend on* $\max Q$ *and* $\min Q$*, such that any positive solution u of*

$$P_{\mathbb{S}^3} u = -\frac{1}{2} Q u^{-7}. \tag{3.1}$$

satisfies $C_1 \leq u(x) \leq C_2$ on $\mathbb{S}^3$.

We introduce the following notation.

Lemma 3.1. *Assume u is a solution of the equation* (3.1)*. Then there exists C_1 which only depends on* $\max u$ *and* $\min Q$ *such that $u \geq C_1 > 0$.*

Proof. Since u is bounded above this implies that $\int_{\mathbb{S}^3} u^2$ is bounded. Multiplying (3.1) by u, together with Bochner's identity, we obtain the following identity:

$$\int_{\mathbb{S}^3} \left(|\nabla^2 u|^2 + \frac{3}{2} |\nabla u|^2 - \frac{15}{16} u^2 \right) = -\frac{1}{2} \int_{\mathbb{S}^3} Q u^{-6}. \tag{3.2}$$

Hence we have

$$\int_{\mathbb{S}^3} \left(|\nabla^2 u|^2 + \frac{3}{2} |\nabla u|^2 + \frac{1}{2} Q u^{-6} \right) = \frac{15}{16} \int_{\mathbb{S}^3} u^2. \tag{3.3}$$

Thus we have $\|u\|_{H^2}$ is bounded as well as $\int_{\mathbb{S}^3} u^{-6}$ is bounded since $Q \geq m > 0$. However, the volume bound of the metric $g = u^{-4} g_0$ is not enough to control the lower bound of the conformal factor. To overcome the difficulty, we just integrate the equation (3.1). We have

$$\int_{\mathbb{S}^3} u = \frac{8}{15} \int_{\mathbb{S}^3} Q(x) u^{-7} \geq \frac{8m}{15} \int_{\mathbb{S}^3} u^{-7}.$$

Hence we have

$$\int_{\mathbb{S}^3} u^{-7} \leq \frac{15C}{8m} vol(\mathbb{S}^3) < +\infty. \tag{3.4}$$

Moreover, we have

$$\begin{aligned} & \int_{\mathbb{S}^3} |\Delta u|^{\frac{84}{55}} u^{-\frac{91}{55}} \\ \leq \quad & \left(\int_{\mathbb{S}^3} (-\Delta u)^2 \right)^{\frac{42}{55}} \left(\int_{\mathbb{S}^3} u^{-7} d\sigma \right)^{\frac{13}{55}} \leq C < +\infty. \end{aligned} \tag{3.5}$$

We now consider $w = u^{-\frac{1}{12}}$. Then we obtain

$$\Delta w = -\frac{1}{12} u^{-\frac{13}{12}} \Delta u + \frac{13}{144} u^{-\frac{25}{12}} |\nabla u|^2. \tag{3.6}$$

Now we apply Green's formula for the function w:

$$\begin{aligned} w(x) &= \fint_{\mathbb{S}^3} w + \int_{\mathbb{S}^3} G(x,y)(-\Delta w)(y)d\sigma_y \\ &= \fint_{\mathbb{S}^3} w + \int_{\mathbb{S}^3} G(x,y)\left[\frac{1}{12}u^{-\frac{13}{12}}\Delta u - \frac{13}{144}u^{-\frac{25}{12}}|\nabla u|^2\right](y)d\sigma_y \\ &\leq \fint_{\mathbb{S}^3} w + \int_{\mathbb{S}^3} G(x,y)\left[\frac{1}{12}u^{-\frac{13}{12}}\Delta u\right](y)d\sigma_y \\ &\leq \fint_{\mathbb{S}^3} w + \left(\int_{\mathbb{S}^3} G(x,y)^{\frac{84}{29}}d\sigma_y\right)^{\frac{29}{84}}\left(\int_{\mathbb{S}^3}\frac{1}{12}u^{-\frac{13}{12}}|\Delta u|^{\frac{84}{55}}\right)^{\frac{55}{84}}. \qquad (3.7) \end{aligned}$$

In this estimate, we have used the fact that we can choose Green's function $G(x,y) \geq 0$ (see Theorem 4.13(d) in [1]). Now note that $\frac{84}{29} < 3$. The first factor in the far right hand side of (3.7) is finite, while we have shown in the equation (3.5) that the second factor there is also finite. Hence we conclude that w is bounded from above, which of course implies that u is bounded from below. Hence we obtain the required result. □

In fact, we also can get the upper bound in terms of the lower bound of u.

Lemma 3.2. *Assume u is a solution of the equation* (3.1) *such that $\int_{\mathbb{S}^3} u^{-7} \leq C_1$. Then there exists C which only depends on C_1 and upper bound of Q such that $u \leq C < \infty$.*

Proof. Basically by the proof of the previous lemma, we only need to bound $\int_{\mathbb{S}^3} u^2$. To this end, first of all, as we noticed before, we have

$$\int_{\mathbb{S}^3} u = \frac{8}{15}\int_{\mathbb{S}^3} Qu^{-7} \leq \frac{8(\max Q)}{15}C_1. \qquad (3.8)$$

On the other hand, we also have the identity

$$\int_{\mathbb{S}^3}(-\Delta u)^2 = \frac{1}{2}\int_{\mathbb{S}^3}|\nabla u|^2 + \frac{15}{16}\int_{\mathbb{S}^3}u^2 - \frac{1}{2}\int_{\mathbb{S}^3}Qu^{-6}. \qquad (3.9)$$

Now by Hölder's inequality and the Cauchy-Schwartz inequality we have

$$\int_{\mathbb{S}^3}|\nabla u|^2 \leq \frac{1}{2}\int_{\mathbb{S}^3}u^2 + \frac{1}{2}\int_{\mathbb{S}^3}(-\Delta u)^2.$$

Combining this fact with (3.9) and $Q > 0$ we obtain

$$\int_{\mathbb{S}^3}(-\Delta u)^2 \leq \frac{19}{12}\int_{\mathbb{S}^3}u^2. \qquad (3.10)$$

Since the first non-zero eigenvalue for the biharmonic operator Δ^2 on the standard sphere is equal to 9, we have

$$\begin{aligned} \fint_{\mathbb{S}^3} u^2 d\sigma &\leq \left(\fint_{\mathbb{S}^3} u d\sigma\right)^2 + \frac{1}{9}\fint_{\mathbb{S}^3}(-\Delta u)^2 d\sigma \\ &\leq \left(\fint_{\mathbb{S}^3} u d\sigma\right)^2 + \frac{19}{108}\fint_{\mathbb{S}^3}u^2 d\sigma. \qquad (3.11) \end{aligned}$$

Now the lemma follows from (3.11) and (3.8). □

Lemma 3.3. *If u is a solution of* (3.1), *then $\int_{\mathbb{S}^3} u^{-7}d\sigma$ is uniformly bounded.*

Proof. Lemma 3.2 says as long as $\int_{\mathbb{S}^3} u^{-7} d\sigma$ is bounded, u is bounded from above and by Lemma 3.1 u is bounded from below. Hence if possible, we assume there exists a sequence of solutions u_j such that

$$\begin{cases} \int_{\mathbb{S}^3} u_j^{-7} d\sigma & \to \infty \\ \min_{x\in\mathbb{S}^3} u_j(x) & \to 0 \end{cases}$$

as $j \to \infty$.

Let N and S be the north and south pole of $\mathbb{S}^3$ respectively. Up to a rotation, we can assume the minimum of u_j is attained at N. Let ϕ_λ be the one-parameter family of the conformal dilation which leaves the poles fixed and such that $|\det(d\phi_\lambda)|(x) \to 0$ as $\lambda \to \infty$ if $x \neq S$.

Let $v_j^{-1} = (u_j^{-1} \circ \phi_{\lambda_j})(\det(d\phi_{\lambda_j}))^{\frac{1}{6}}$. Then $⨍_{\mathbb{S}^3} v_j^{-6} = ⨍_{\mathbb{S}^3} u_j^{-6}$ and $d(\phi_{\lambda_j}) \circ d(\phi_{\lambda_j^{-1}}) = I$. Moreover, $det(d(\phi_{\lambda_j})(\pi^{-1}(x))) = [\frac{(1+|x|^2)}{(1+|\lambda_j x|^2)}]^3 \lambda_j^3$. Then we claim that there exist λ_j such that $⨍_{\mathbb{S}^3} v_j^{-7} d\sigma = 1$ for all j. The integral is a continuous function of λ. If $\lambda = 1$, then $⨍ v_j^{-7} = ⨍ u_j^{-7} \to \infty$. Using the conformal invariance of (3.1), we have

$$P_{\mathbb{S}^3} v_j = -\frac{1}{2}(Q \circ \phi_{\lambda_j}) v_j^{-7} \text{ on } \mathbb{S}^3. \tag{3.12}$$

Hence integrating (3.12) we have

$$\begin{aligned} ⨍_{\mathbb{S}^3} (Q \circ \phi_{\lambda_j}) v_j^{-7} d\sigma &= \frac{15}{8} ⨍_{\mathbb{S}^3} v_j d\sigma = \frac{15}{8} ⨍_{\mathbb{S}^3} u_j (\det(d\phi_{\lambda_j}))^{-\frac{1}{6}} d\sigma \\ &= \frac{15}{8} ⨍_{\mathbb{S}^3} u_j (\det(d\phi_{\lambda_j^{-1}}))^{\frac{1}{6}} \to 0 \text{ as } \lambda_j \to \infty. \end{aligned} \tag{3.13}$$

Noting that $⨍_{\mathbb{S}^3} v_j^{-7} d\sigma \leq \frac{15}{8m} ⨍_{\mathbb{S}^3} u_j (\det(d\phi_{\lambda_j^{-1}}))^{\frac{1}{6}}$ we have by the mean value theorem that there exists a λ_j such that $⨍_{\mathbb{S}^3} v_j^{-7} d\sigma = 1$.

Since $⨍_{\mathbb{S}^3} v_j^{-7} d\sigma = 1$, by Lemma 3.2 and Lemma 3.1, there exist two positive constants C_1 and C_2 such that $0 < C_1 \leq v_j(x) \leq C_2$ for all $x \in \mathbb{S}^3$ and all $j \geq 1$. Hence we can conclude that v_j is bounded in $W^{4,p}(\mathbb{S}^3)$ for any $p > 1$.

Hence there exists a subsequence of v_j, still denoted as v_j, such that $v_j \to v_\infty$ with $Q \circ \phi_{\lambda_j} \to Q(N)$ as $j \to \infty$ locally uniformly on any compact set $K \subset \mathbb{S}^3 \backslash \{S\}$. Thus $v_j \to v_\infty$ in $C^{3,\alpha}(K)$ for any $\alpha \in (0,1)$. Applying the standard diagonal argument we have $v_\infty \in C^\infty(\mathbb{S}^3 \backslash \{S\})$ satisfying

$$P_{\mathbb{S}^3} v_\infty = -\frac{1}{2} Q(N) v_\infty^{-7} \quad \text{on} \quad \mathbb{S}^3 \backslash \{S\}. \tag{3.14}$$

Now we choose stereographic coordinates at the north pole N. The above equation reduces to

$$\Delta^2 w_j = -\frac{1}{2}(Q \circ \phi_{\lambda_j}) w_j^{-7} \quad \text{on} \quad \mathbb{R}^3 \tag{3.15}$$

where $w_j = v_j \left(\frac{2}{1+|z|^2}\right)^{-\frac{1}{2}}$.

Since v_j are smooth functions on $\mathbb{S}^3$ and uniformly bounded both above and below, v_∞ has the same upper and lower bound. Thus $w_\infty := v_\infty \left(\frac{2}{1+|z|^2}\right)^{-\frac{1}{2}}$ will also have the linear growth. Using the classification theorem by Choi-Xu [9] and Xu [24], v_∞ is standard up to conformal transformation. By adjusting the conformal transformation, we can assume that v_∞ is a constant. Then by assumption that $⨍_{\mathbb{S}^3} v_j^{-7} d\sigma = 1$, we conclude that $v_\infty = 1$.

Thereby we conclude that

$$\|v_j - 1\|_{L^\infty} = o(1) \tag{3.16}$$

and

$$\|\nabla v_j\|_{L^\infty} = o(1) \tag{3.17}$$

and v_j satisfies (3.12) on $\mathbb{S}^3$. Now from the Kazdan-Warner identity (2.10) we have

$$\int_{\mathbb{S}^3} \langle \nabla(Q\circ\phi_{\lambda_j}), \nabla \mathbf{x}\rangle v_j^{-6} = 0 \tag{3.18}$$

where $\mathbf{x} = (x_1, x_2, x_3, x_4)$.

We denote $\mathbf{E}_j = \int_{\mathbb{S}^3} \langle \nabla(Q\circ\phi_{\lambda_j}), \nabla \mathbf{x}\rangle v_j^{-6}$. Then expanding $\mathbf{E}_j = 0$ explicitly we obtain:

$$\begin{aligned}
&\int_{\mathbb{S}^3} \langle \nabla(Q\circ\phi_{\lambda_j}), \nabla \mathbf{x}\rangle v_j^{-6} = \int_{\mathbb{S}^3} \langle \nabla(Q\circ\phi_{\lambda_j} - Q(N)), \nabla \mathbf{x}\rangle v_j^{-6} \\
= &\ -\int_{\mathbb{S}^3} (Q\circ\phi_{\lambda_j} - Q(N))\Delta \mathbf{x} v_j^{-6} - \int_{\mathbb{S}^3} (Q\circ\phi_{\lambda_j} - Q(N))\langle \nabla \mathbf{x}, \nabla v_j^{-6}\rangle \\
= &\ 3\int_{\mathbb{S}^3} (Q\circ\phi_{\lambda_j} - Q(N))\mathbf{x} v_j^{-6} - \int_{\mathbb{S}^3} (Q\circ\phi_{\lambda_j} - Q(N))\langle \nabla \mathbf{x}, \nabla v_j^{-6}\rangle \\
= &\ 3\int_{\mathbb{S}^3} (Q\circ\phi_{\lambda_j} - Q(N))\mathbf{x} + 3\int_{\mathbb{S}^3} (Q\circ\phi_{\lambda_j} - Q(N))\mathbf{x}(v_j^{-6} - 1) \\
&\ -\int_{\mathbb{S}^3} (Q\circ\phi_{\lambda_j} - Q(N))\langle \nabla \mathbf{x}, \nabla v_j^{-6}\rangle \\
= &\ M_j + N_j.
\end{aligned}$$

Let us denote $x = (x_1, x_2, x_3, x_4) \in \mathbb{S}^3$, $y = (y_1, y_2, y_3) \in \mathbb{R}^3$ and π is the stereographic projection from $\mathbb{S}^3$ to the equatorial plane $\mathbb{R}^3$ sending N to ∞. Then

$$x_i = \frac{2y_i}{1+|y|^2} \quad (1 \le i \le 3) \quad \text{and} \quad x_4 = \frac{|y|^2 - 1}{|y|^2 + 1}.$$

Now we use the Taylor expansion of Q around $N = (0,0,0,1)$ as

$$Q(x_1, x_2, x_3) = Q(N) + \sum_{\alpha=1}^{3} a_\alpha x_\alpha + \sum_{\alpha,\beta=1}^{3} b_{\alpha,\beta} x_\alpha x_\beta + O\Big(\sum_{\alpha=1}^{3} |x_\alpha|^3\Big) \tag{3.19}$$

where (3.19) holds true in the neighborhood of N; let it be

$$\mathbb{D} = \{y \in \mathbb{R}^3 : |y| \ge M\}$$

for some $M > 0$ sufficiently large. Then $\mathbb{D}_j = \{y \in \mathbb{R}^3 : |y| \ge \frac{M}{\lambda_j}\}$. Note that $\phi_{\lambda_j}(y) = \lambda_j y$ and $\phi_{\lambda_j}(\mathbb{D}_j) = \mathbb{D}$. To estimate M_j and N_j, we first notice that as $j \to \infty$,

$$Vol(\mathbb{S}^3 \backslash \pi^{-1}(\mathbb{D}_j)) = \int_{\mathbb{R}^3\backslash \mathbb{D}_j} \left(\frac{2}{1+|y|^2}\right)^3 dy = 32\pi \int_0^{M/\lambda_j} \frac{r^2}{(1+r^2)^3} dr = O\left(\frac{1}{\lambda_j^3}\right).$$

Hence we have

$$\begin{aligned} M_j &= 3\int_{\mathbb{S}^3}(Q\circ\phi_{\lambda_j}-Q(N))\mathbf{x}d\mu_{\mathbb{S}^3} \\ &= 3\int_{\mathbb{D}_j}(Q\circ\phi_{\lambda_j}-Q(N))\mathbf{x}d\sigma_y+O\Big(\frac{1}{\lambda_j^3}\Big) \end{aligned} \tag{3.20}$$

where $d\sigma$ is the volume form of $\mathbb{S}^3$ pulled back by Π. Furthermore, we notice that by spherical symmetry

$$\int_{\mathbb{D}_j} x_\alpha(\lambda_j y)x_\beta(y)d\sigma_y = 0 \quad \text{if} \quad \alpha\neq\beta; \quad 1\le\alpha,\beta\le 4. \tag{3.21}$$

Hence for $\alpha = 1,2,3,4$,

$$\begin{aligned} M_j^\alpha &= \int_{\mathbb{D}_j} a_\alpha x_\alpha(\lambda_j y)x_\alpha(y)d\sigma_y+\int_{\mathbb{D}_j}\sum_{\beta,\gamma=1}^{3} b_{\beta,\gamma}x_\beta(\lambda_j y)x_\gamma(\lambda_j y)x_\alpha(y)d\sigma_y \\ &+ O\Big(\int_{\mathbb{D}_j}\Big(\frac{|\lambda_j y|}{1+|\lambda_j y|^2}\Big)^3|x_\alpha(y)|d\sigma_y\Big)+O\Big(\frac{1}{\lambda_j^3}\Big). \end{aligned} \tag{3.22}$$

Hence arguing as in Lemma 6.1 of Chang-Gursky-Yang [7], we conclude that at N, $\nabla Q(N)=0$ and $\Delta Q(N)=0$, which contradicts the assumption of non-degeneracy of Q given by (1.3). □

Proof of Theorem 3.1. By Lemma 3.3 there exists a constant $C_1>0$ such that $\int_{\mathbb{S}^3}u^{-7}d\sigma\le C_1<\infty$. Moreover, Lemma 3.2 provides a uniform upper bound for all solutions. Thus, by Lemma 3.1, we conclude that all solutions of (3.1) are also bounded below by a positive constant. Hence the result. □

4. Perturbation result

In this section, we use Lyapunov-Schmidt's reduction method to solve the equation (1.1). We will prove the following perturbation result.

Theorem 4.1. *Let $\hat{Q}$ be a smooth positive function defined on $\mathbb{S}^3$, which satisfies:*

(1) *the non-degenerate condition $|\nabla\hat{Q}|^2+|\Delta\hat{Q}|^2>0$,*
(2) *the degree condition $deg(\frac{\Phi}{|\Phi|})\neq 0$,*
(3) $\|\hat{Q}\|_{C^0}=1$.

Then there exists some $\epsilon_0>0$ such that the equation

$$P_{\mathbb{S}^3}u+\frac{1}{2}Qu^{-7}=0 \tag{4.1}$$

has a smooth solution, where $Q=\frac{15}{8}+\epsilon\hat{Q}$ and $0<\epsilon<\epsilon_0$.

We will prove the theorem in several steps. We define an operator

$$S_p[u]=P_{\mathbb{S}^3}u+\frac{\beta^{7/6}}{2}\frac{Q_pu^{-7}}{(\int_{\mathbb{S}^3}Q_pu^{-6})^{7/6}} \quad \text{on} \quad \mathbb{S}^3 \tag{4.2}$$

where $Q_p=Q\circ\phi_p$, $\beta=\frac{15}{8}vol(\mathbb{S}^3)$. Our aim is to find a function u such that $S_p[u]=0$. In the definition of $S_p[u]$, the power $7/6$ is chosen in such a way that the second term in $S_p[u]$ is invariant under constant multiple of u.

We are looking for solution u of the form $u = 1 + w$. The linearization of the operator S around $u = 1, Q = \frac{15}{8}$ is given by

$$L[w] = P_{\mathbb{S}^3} w - \frac{105}{16} w + \frac{1575}{128\beta} \int_{\mathbb{S}^3} w. \tag{4.3}$$

We write $S[1+w] = S[1]+L[w]+N[w]$, where $N[w]$ is the remainder term. Before we can prove the perturbation result (Theorem 4.1), we need to know something about the linearized operator L. The results are standard and given in the next two lemmas.

Lemma 4.1. *Suppose that w is a smooth function with $\int_{\mathbb{S}^3} w = 0$ and $\int_{\mathbb{S}^3} w\xi_i = 0$ for $i = 1, 2, 3, 4$. Then*

$$\int_{\mathbb{S}^3} wL[w] \geq (\lambda_2 - \lambda_1)\|w\|_{L^2(S^3)} \tag{4.4}$$

where λ_1, λ_2 are the first and second eigenvalue of $P_{\mathbb{S}^3}$, respectively.

Proof. The inequality follows from a spectral decomposition of the function w. □

Remark 4.1. The inequality may be extended to $w \in W^{2,2}(\mathbb{S}^3)$ in the weak formulation. Notice that $\frac{105}{16}$ is exactly the first eigenvalue of the operator $P_{\mathbb{S}^3}$. The kernel of the operator L is exactly the first eigenspace of the Laplacian Δ. The Fredholm alternative yields the following lemma.

Lemma 4.2. *Let f be a smooth function. We consider the following equation:*

$$L(w) + f = \sum_{j=1}^{4} c_j x_j \tag{4.5}$$

for some constant c_j. Then the equation has a solution if and only if

$$\int_{\mathbb{S}^3} f x_i = c_i \int_{\mathbb{S}^3} x_i^2.$$

Moreover if the above condition holds, the solution is unique and smooth and satisfies the estimate

$$\|w\|_{W^{2,2}(\mathbb{S}^3)} \leq C\|f\|_{L^2(\mathbb{S}^3)}.$$

Remark 4.2. We denote $\mathcal{H} = \{\phi \in L^2(\mathbb{S}^3) \mid \int_{\mathbb{S}^3} \phi x_i = 0 \ (i = 1, 2, 3, 4)\}$. The operator L^{-1} may be viewed as a bounded linear operator from $\mathcal{H} \cap L^2(\mathbb{S}^3)$ into $\mathcal{H} \cap W^{2,2}(\mathbb{S}^3)$. Therefore, by Sobolev's embedding we may also consider L^{-1} as a bounded (in fact compact) linear operator from $\mathcal{H} \cap C(\mathbb{S}^3)$ into itself.

Proposition 4.1. *There exists an $\epsilon_0 > 0$, for any $p \in B$ and $0 < \epsilon < \epsilon_0$, such that the equation*

$$L[w] + S_p[1] + N[w] = \sum_{j=1}^{4} c_j x_j \tag{4.6}$$

has a smooth solution.

Proof. This is an application of the contraction mapping theorem. We write (4.6) as

$$w = -L^{-1}[S_p[1] + N[w] - \sum_{j=1}^{4} c_j x_j] = \mathcal{A}(w). \tag{4.7}$$

The remainder term can be written explicitly as

(4.8)
$$N[w] = -\frac{\beta^{7/6}}{2}\left\{\frac{Q_p}{\left(\int_{\mathbb{S}^3} Q_p\right)^{7/6}} - \frac{Q_p(1+w)^{-7}}{\left(\int_{\mathbb{S}^3} Q_p(1+w)^{-6}\right)^{7/6}}\right\} + \frac{105}{16}\left\{w - \frac{15}{8\beta}\int_{\mathbb{S}^3} w\right\}.$$

Note that $\|Q_p - \frac{15}{8}\|_{C_0} = \|Q - \frac{15}{8}\|_{C_0} = \epsilon$. We have the following estimates:

$$\begin{aligned} |N[w]| &\le C(\epsilon|w| + |w|^2), \\ \|\mathcal{A}[w_1] - \mathcal{A}[w_2]\| &\le C(\epsilon + \|w_1\|_{C_0} + \|w_2\|_{C_0})\|w_1 - w_2\|_{C_0}. \end{aligned} \tag{4.9}$$

We define a set

$$X = \{w \in (C(\mathbb{S}^3) \cap \mathcal{H}) \mid \|w\|_{C_0} \le \epsilon\}.$$

For small enough (independent of p) $\epsilon > 0$, $\mathcal{A}$ is a contraction mapping in X. As a result, by Banach's fixed point theorem, $\mathcal{A}$ has a unique fixed point in X. Smoothness of w follows from standard regularity theory for elliptic equations. □

We denote the solution constructed in the previous proposition by w_p. Let $\mathbf{C}(p)$ denote $(c_1(p), c_2(p), c_3(p), c_4(p))$ and $\mathbf{x}$ denote (x_1, x_2, x_3, x_4). In order to prove the perturbation result in this section we only need to choose p such that $\mathbf{C}(p)$ is the zero vector. Now we are going to estimate its components. We multiply the equation (4.6) by x_i and integrate on $\mathbb{S}^3$. It follows that

$$\int_{\mathbb{S}^3} S_p[1]x_i + \int_{\mathbb{S}^3} N[w_p]x_i = \frac{1}{3} vol(S^3)c_i.$$

Since $\|\phi_p\| \le \epsilon$, we have

$$\int_{\mathbb{S}^3} N[w_p]x_i = O(\epsilon^2).$$

Therefore,

$$c_i = \frac{3}{2}⨍_{\mathbb{S}^3} Q_p x_i + O(\epsilon^2).$$

We may rewrite it as

$$\mathbf{C}(p) = \frac{3}{2}⨍_{\mathbb{S}^3} Q_p \cdot \mathbf{x} + O(\epsilon^2). \tag{4.10}$$

We define a mapping

$$\mathbf{G}(p) = \int_{\mathbb{S}^3} Q_p \cdot \mathbf{x} \tag{4.11}$$

and we have the following.

Proposition 4.2. *If the non-degeneracy condition $|\nabla Q|^2 + |\Delta Q|^2 > 0$ holds, then the mapping degree $deg(\mathbf{G}, B, 0)$ is well defined.*

Proof. We only need to show that for sufficiently small $\delta > 0$, $\mathbf{G}$ does not attain 0 on the sphere $\mathbb{S}^3_{1-\delta}$ which has radius $1-\delta$. Recall that $p = \frac{t-1}{t}P$. We choose another Cartesian coordinate $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4)$ in which the $\tilde{x}_4$-axis is pointing towards the point $-P$. In this coordinate system, we write $\mathbf{C} = \sum_{i=1}^4 \tilde{c}_i \tilde{x}_i$, $\mathbf{G} = \sum_{i=1}^4 \tilde{G}_i \tilde{x}_i$.

The stereographic projection in this coordinate system is given by $y_i = \frac{\tilde{x}_i}{1-\tilde{x}_4}$. Moreover, $P = (0,0,0)$, $\phi_p(y) = \frac{1}{t}y$. As $t \to \infty$, we have

$$\begin{aligned}
\tilde{G}_i(p) &= \int_{\mathbb{S}^3} (Q \circ \phi_p)\tilde{x}_i \\
&= \int_{\mathbb{R}^3} Q\left(\frac{1}{t}y_1, \frac{1}{t}y_2, \frac{1}{t}y_3\right) \frac{2y_i}{1+|y|^2} \frac{8}{(1+|y|^2)^3} dy \\
&= \frac{a}{t} \frac{\partial Q}{\partial y_i}\bigg|_{(0,0,0)} + O\left(\frac{1}{t^2}\right), \qquad i = 1,2,3,
\end{aligned}$$

and

$$\begin{aligned}
\tilde{G}_4(p) &= \int_{\mathbb{S}^3} (Q \circ \phi_p)\tilde{x}_4 \\
&= \int_{\mathbb{R}^3} Q\left(\frac{1}{t}y_1, \frac{1}{t}y_2, \frac{1}{t}y_3\right) \frac{1-|y|^2}{1+|y|^2} \frac{8}{(1+|y|^2)^3} dy \\
&= -\frac{b}{t^2} \left(\Delta_y Q\right)|_{(0,0,0)} + O\left(\frac{1}{t^3}\right)
\end{aligned}$$

where a, b are some positive constants. By choosing different positive constants a and b, we have

$$\overrightarrow{G}(p) = \frac{a}{t}(\nabla Q)|_P + \frac{b}{t^2}(-\Delta Q)|_P \cdot p + O\left(\frac{1}{t^3}\right). \tag{4.12}$$

Under the non-degeneracy condition, $\overrightarrow{G} \neq 0$ as $t \to \infty$. □

Proof of Theorem 4.1. We restrict $\mathbf{G}$, $\mathbf{C}$ and $\mathbf{\Phi}$ on $\mathbb{S}^3_{1-\delta}$, so we have $\mathbf{G} \cdot \mathbf{C} > 0$ and $\mathbf{G} \cdot \mathbf{\Phi} > 0$. By homotopy invariance, we have

$$deg\left(\frac{\mathbf{C}}{|\mathbf{C}|}\bigg|_{\mathbb{S}^3_{1-\delta}}\right) = deg\left(\frac{\mathbf{G}}{|\mathbf{G}|}\bigg|_{\mathbb{S}^3_{1-\delta}}\right) = deg\left(\frac{\mathbf{\Phi}}{|\mathbf{\Phi}|}\bigg|_{\mathbb{S}^3_{1-\delta}}\right) \neq 0.$$

Therefore $deg(\mathbf{C}, B, 0) \neq 0$. By Kronecker's existence theorem, there exists $p \in B$ such that $\mathbf{C}(p) = 0$. □

5. Proof of Theorem 1.1

To prove the existence of a solution of (1.1), we use the Leray-Schauder degree theory (see Chang-Gursky-Yang [7] and Wei-Xu [22]). We define $Q_t(x) = tQ(x) + (1-t)Q_0(x)$ where $Q_0(x) \equiv \frac{15}{8}$. Consider the differential equation

$$P_{\mathbb{S}^3} v = -\frac{1}{2} Q_t v^{-7}. \tag{5.1}$$

From the estimates in Section 4, for any $t_0 > 0$, there is a uniform bound for the function Q_t as well as a uniform lower bound for $|\Delta Q_t(P)|$ at all critical points P of the function Q_t for all $t \in [t_0, 1]$. Hence by Theorem 3.1, fix any $\theta \in (0,1)$; there exists a $C > 0$ such that the solutions of (5.1) satisfy the following relations:

$$\|v\|_{C^{4,\theta}} < C \text{ and } \frac{1}{C} < v(x) < C. \tag{5.2}$$

We write the equation (5.1) in the following way:

$$v = -\frac{1}{2} P_{\mathbb{S}^3}^{-1}(Q_t v^{-7});$$

hence we define

$$\mathcal{F}_t[v] = v + \frac{1}{2}P_{\mathbb{S}^3}^{-1}(Q_t v^{-7}) = v - L_t(v)$$

where

$$X = \{v \in C^{4,\theta}(\mathbb{S}^3) : v \text{ satisfies } (5.2)\},$$

$$L_t(v) = -\frac{1}{2}P_{\mathbb{S}^3}^{-1}(Q_t v^{-7}).$$

In order to find a solution, we look for the zeros of the map $\mathcal{F}_t$. We have $0 \notin \mathcal{F}_t(\partial X)$ for $t \geq t_0$ by definition. Moreover, $L_t[v]$ is a family of Fredholm operators and continuous in t. Thus $deg(\mathcal{F}_t, X, 0)$ is well defined and by homotopy invariance is independent of t whenever $t \geq t_0$.

We denote $u_p = 1 + \phi_p$; it satisfies the equation

$$P_{\mathbb{S}^3} u_p = -\frac{1}{2}(Q \circ \phi_p)u_p^{-7} + (\mathbf{C}_p \cdot \mathbf{x})u_p^{-7}. \tag{5.3}$$

Furthermore, define

$$\mathcal{S} = \left\{u \in C^\infty(\mathbb{S}^3) : \int_{\mathbb{S}^3} u^{-7}x_j = 0\right\}$$

and

$$\mathcal{S}_0 = \left\{u \in \mathcal{S} : \int_{\mathbb{S}^3} u^{-6} = 1\right\}.$$

By the results of the previous section, it follows that $\|\mathbf{C}_p\|_{C_0} = O(\epsilon)$. Moreover, $deg(\mathbf{\Phi}, B, 0) = deg(\mathbf{C}, B, 0)$.

To prove Theorem 1.1, it suffices to show that $deg(\mathbf{C}, B, 0) = deg(\mathcal{F}_{t_0}, B, 0)$ for $t_0 > 0$ sufficiently small. By the continuity of degree under small perturbation we may assume that $\mathbf{C}$ and $\mathcal{F}_{t_0}$ have only isolated non-degenerate zeros such that their corresponding degrees are actually sums of local degrees of zeroes of the corresponding maps. The local degree of $\mathcal{F}_{t_0}$ at an isolated zero u_0 is given by taking a small neighborhood N of u_0 such that $0 \notin \mathcal{F}_{t_0}(\partial N)$ and taking a sequence of compact maps K_ε approximating $L_t(v)$ mapping into a finite dimensional subspace Y of $C^{4,\theta}(\mathbb{S}^3)$ such that $\mathcal{F}_{t,\varepsilon}(u) = u - K_\varepsilon(u)$ and $\mathcal{F}_{t,\varepsilon}$ does not admit a zero on the boundary of N. Then $\mathcal{F}_{t,\varepsilon} : \overline{N} \cap Y \to N$ and the local degree is

$$deg(\mathcal{F}_t, N, 0) = deg(\mathcal{F}_{t,\epsilon}|_{Y \cap \overline{N}}, N \cap Y, 0). \tag{5.4}$$

We take $Y = \bigoplus_{i=1}^k E_i$ where E_i denotes the space of i^{th}-order spherical harmonics. To study the local degree of $\mathcal{F}_t$ at u_0, we choose a conformal transformation ϕ_0 such that $\tilde{u}_0 \in \mathcal{S}_0$, where $\tilde{u}_0 = (u_0 \circ \phi_0)(det(d\phi_0))^{-1/6}$. Moreover, for any $u \in B$, we define a map T_0 by

$$T_0 u = (u \circ \phi_0) \cdot |det(d\phi_0)|^{-1/6}.$$

Then

$$P_{\mathbb{S}^3}(u) = -\frac{1}{2}Qu^{-7} \quad \text{if and only if} \quad P_{\mathbb{S}^3}(T_0 u) = -\frac{1}{2}\tilde{Q}(T_0 u)^{-7} \tag{5.5}$$

where $\tilde{Q} = Q \circ \phi_0$. Moreover let $\tilde{\mathcal{F}}_t = \mathcal{F}_t \circ T_0^{-1}$. Then we have $deg(\tilde{\mathcal{F}}_t, \tilde{N}, 0) = deg(\mathcal{F}_t, N, 0)$, where $\tilde{N} = T_0(N)$. Thus we can consider u_0 to be in a symmetric class of functions satisfying $\fint_{\mathbb{S}^3} u_0^{-6} d\sigma = 1$. Hence if $u_0 + \frac{1}{2}P_{\mathbb{S}^3}^{-1} L_t(u_0) = 0$, the linearized map $\mathcal{F}_t$ around u_0 is given by

$$\mathcal{F}_t'(u_0)[w] = w - \frac{7}{2}P_{\mathbb{S}^3}^{-1}(Q_t u_0^{-8} w) \tag{5.6}$$

where $\|Q_t - \frac{15}{8}\|_\infty = O(\varepsilon)$. Then $\|u_0 - 1\|_\infty = o(1)$ and (5.6) reduces to

$$\mathcal{F}_t'(u_0)[w] = w - \frac{7}{2}P_{\mathbb{S}^3}^{-1}(w) + (O(\varepsilon) + o(1))\|P_{\mathbb{S}^3}^{-1}(w)\|. \tag{5.7}$$

So u_0 is a unique element of B which is also in $\mathcal{S}$; hence $span\{T_{u_0}(B), T_{u_0}(\mathcal{S})\} = L^2(\mathbb{S}^3)$ where $T_{u_0}(B)$ is the tangent space to B. If 1 denotes the constant function, then $Y \cap T_1(\mathcal{S}) = E_0 \bigoplus E_2 \bigoplus \cdots \bigoplus E_k \bigoplus V$. Moreover, as $\|u_0 - 1\|_\infty = o(1)$, we have $Y \cap T_{u_0}(\mathcal{S}) = E_0 \bigoplus E_2 \bigoplus \cdots \bigoplus E_k \bigoplus V$ where $\|v\|_{L^2} = \delta(k, \varepsilon)$ for all $v \in V$. Here $\delta(k, \varepsilon) \to 0$ as $k \to \infty$ and $\varepsilon \to 0$. Hence we can calculate $\mathcal{F}_t'(u_0)$ in the direction of $v \in Y \cap T_{u_0}(\mathcal{S})$ which is straightforward. On the other hand, since the tangent space $T_{u_0}(B)$ is transverse to the space E_j $(j \neq 1)$ we can compute the derivative of (5.3) of $\mathcal{F}_t'$ in the direction of $T_{u_0}(B)$. To this end, using (5.3) we have

$$\mathcal{F}_t(u_p) = -P_{\mathbb{S}^3}^{-1}(\mathbf{C}(p) \cdot \mathbf{x} u_p^{-7}).$$

Hence in the direction $T_{u_0}(B)$, we have $\mathcal{F}_t'[u_0] = -\mathbf{C}'(u_0) \cdot P_{\mathbb{S}^3}^{-1}(\mathbf{x} u_0^{-7})$. Hence we can find a basis for $T_{u_0}(B)$, consisting of $\{\beta_i, i = 1, 2, 3, 4\}$ where $\beta_i = x_i + e_i + \varepsilon_i$ and $e_i \in E_0 \bigoplus E_2 \bigoplus \cdots \bigoplus E_k$ and $\varepsilon_i \to 0$ as $\varepsilon \to 0$. Therefore $x_i = \beta_i - e_i - \varepsilon_i$, and as a result we can express the derivative of (5.6) with respect to the natural decomposition $Y = E_0 \bigoplus E_1 \bigoplus \cdots \bigoplus E_k$ as a perturbation of the following matrix:

$$\begin{pmatrix} 71/15 & x & 0 & 0 & \cdots & 0 \\ 0 & -\mathbf{C}' & 0 & 0 & \cdots & 0 \\ 0 & x & 1 - \frac{7}{2a_3} & 0 & \cdots & 0 \\ 0 & x & 0 & 1 - \frac{7}{2a_4} & \cdots & 0 \\ . & . & . & & . & \\ 0 & x & 0 & 0 & \cdots & 1 - \frac{7}{2a_k} \end{pmatrix}$$

where $a_l := \left[l^2(l+2)^2 - \frac{l(l+2)}{2} - \frac{15}{16} \right]$ whenever $3 \leq l \leq k$. The sign of the determinant of the above matrix is completely determined by the sign of the determinant of M'. This finishes the proof of Theorem 1.1.

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