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# DEGREE OF COMMUTATIVITY OF INFINITE GROUPS

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Abstract. We prove that, in a finitely generated residually finite group of subexponential growth, the proportion of commuting pairs is positive if and only if the group is virtually abelian. In particular, this covers the case where the group has polynomial growth (i.e., virtually nilpotent groups). We also show that for non-elementary hyperbolic groups, the proportion of commuting pairs is always zero.

## 1. Introduction

In a finite group $G$, a way of studying a property one is interested in is by counting, or estimating, the probability that this property holds among the elements of $G$. For example, we can look at the proportion of pairs of elements in $G$ which commute to each other,

$$\mathrm{dc}(G) = \frac{|\{(u, v) \in G^2 \ : \ uv = vu\}|}{|G|^2},$$

and call it the *degree of commutativity* of $G$. Of course, $\mathrm{dc}(G)$ is a rational number between 0 and 1, $\mathrm{dc}(G) = 1$ if and only if $G$ is abelian, and the closer $\mathrm{dc}(G)$ is to 1 the "more abelian" $G$ will be. An interesting result due to Gustafson [7] states that there is no finite group with $\mathrm{dc}(G)$ strictly between $5/8$ and 1; i.e.,

**Theorem 1.1** (Gustafson, [7])**.** *Let $G$ be a finite group. If $\mathrm{dc}(G) > 5/8$, then $G$ is abelian.*

The bound in Gustafson's result is tight, since easy computations show that the degree of commutativity of the quaternion group $Q_8$ is, precisely, $\mathrm{dc}(Q_8) = 5/8$.

The aim of the present paper is to define *degree of commutativity* in the context of infinite discrete groups and to prove there is a generalization of Gustafson's result.

**Definition 1.2.** Let $G$ be a finitely generated group and $X$ a finite generating set. The *degree of commutativity of $G$ with respect to $X$*, denoted $\mathrm{dc}_X(G)$, is defined as

$$\mathrm{dc}_X(G) = \limsup_{n\to\infty} \frac{|\{(u, v) \in (\mathbb{B}_X(n))^2 \ : \ uv = vu\}|}{|\mathbb{B}_X(n)|^2},$$

where $\mathbb{B}_X(n)$ denotes the *ball of radius $n$* centered at 1 on the Cayley graph $\Gamma(G, X)$ of $G$ with respect to $X$.

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We keep the subindex to specify the dependency of this notion, in general, from the set of generators and to distinguish it from the corresponding notion for finite groups. Of course, if $G$ is finite, then $\mathrm{dc}_X(G) = \mathrm{dc}(G)$ for every generating set $X$ of $G$. Also, we maintain the $\limsup$ because we do not know, in general, whether it is always a real limit or not.

The main result in the paper is the following one.

**Theorem 1.3.** *Let $G$ be a finitely generated residually finite group of subexponential growth, and let $X$ be a finite generating set for $G$. Then:*

(i) $\mathrm{dc}_X(G) > 0$ *if and only if $G$ is virtually abelian;*
(ii) $\mathrm{dc}_X(G) > 5/8$ *if and only if $G$ is abelian.*

As a corollary we obtain that the positivity of $\mathrm{dc}_X(G)$ is independent from $X$.

**Corollary 1.4.** *Let $G$ be a finitely generated residually finite group of subexponential growth, and let $X$ and $Y$ be two finite generating sets for $G$. Then,* $\mathrm{dc}_X(G) = 0 \Leftrightarrow \mathrm{dc}_Y(G) = 0$.

Note that, as a special case, Theorem 1.3 and Corollary 1.4 both apply to polynomially growing groups: by Gromov's Theorem [6] these are precisely the virtually nilpotent groups, and it is well known that all of them are residually finite.

**Corollary 1.5.** *Let $G$ be a finitely generated group of polynomial growth, and let $X$ be a finite generating set for $G$. Then: (i)* $\mathrm{dc}_X(G) > 0$ *if and only if $G$ is virtually abelian; and (ii)* $\mathrm{dc}_X(G) > 5/8$ *if and only if $G$ is abelian.*

We conjecture that the same result is true without any hypothesis on the growth of $G$.

**Conjecture 1.6.** *Let $G$ be a finitely generated group, and let $X$ be a finite generating set for $G$. Then: (i)* $\mathrm{dc}_X(G) > 0$ *if and only if $G$ is virtually abelian; and (ii)* $\mathrm{dc}_X(G) > 5/8$ *if and only if $G$ is abelian.*

In view of Theorem 1.3 (and since virtually abelian groups are polynomially growing), one could approach Conjecture 1.6 by showing that exponentially growing groups and non-residually finite subexponentially growing groups $G$ all satisfy $\mathrm{dc}_X(G) = 0$, for every finite generating set $X$. We can show the following particular case.

**Theorem 1.7.** *Let $G$ be a non-elementary hyperbolic group, and let $X$ be a finite generating set for $G$. Then* $\mathrm{dc}_X(G) = 0$.

## 2. Groups of subexponential growth and the proof of Theorem 1.3

In the finite realm, the degree of commutativity behaves well with respect to normal subgroups and quotients. The first statement in this direction is the following one due to Gallagher (and meaning that a group $G$ is *at most* as abelian as any normal subgroup and any quotient of itself).

**Lemma 2.1** (Gallagher, [5]). *Let $G$ be a finite group and $N \trianglelefteq G$ a normal subgroup. Then,*

$$\mathrm{dc}(G) \leqslant \mathrm{dc}(N) \cdot \mathrm{dc}(G/N).$$

We now develop a simpler version of this result for infinite groups, which will be enough to prove Theorem 1.3. First, we recall the following natural result (which is, however, false if one deletes the subexponential growth condition; see [3, Example 1.5]).

**Proposition 2.2** (Burillo–Ventura, [3]). *Let $G$ be a finitely generated group with subexponential growth, and let $X$ be a finite generating set for $G$. For every finite index subgroup $H \leqslant G$ and every $g \in G$, we have*

$$\lim_{n\to\infty} \frac{|gH \cap \mathbb{B}_X(n)|}{|\mathbb{B}_X(n)|} = \lim_{n\to\infty} \frac{|Hg \cap \mathbb{B}_X(n)|}{|\mathbb{B}_X(n)|} = \frac{1}{[G:H]}.$$

From this we may deduce the following.

**Proposition 2.3.** *Let $G$ be a finitely generated subexponentially growing group, and let $X$ be a finite generating set for $G$. Then, for any finite quotient $G/N$, we have*

$$\mathrm{dc}_X(G) \leqslant \mathrm{dc}(G/N).$$

*Proof.* Let $N \trianglelefteq G$ be a normal subgroup of $G$ of index, say, $[G : N] = d$.

By Proposition 2.2, for every $g \in G$ we have

$$\lim_{n\to\infty} \frac{|gN \cap \mathbb{B}_X(n)|}{|\mathbb{B}_X(n)|} = \frac{1}{d},$$

independently from $X$ and $g$; additionally, this is a real limit, not just a lim sup. Since there are finitely many classes modulo $N$, the previous limit is *uniform* on $g$; i.e., for every $\varepsilon > 0$ there exists $n_0$ such that, for every $n \geqslant n_0$ and *all* $g \in G$,

$$\left(\frac{1}{d} - \varepsilon\right) |\mathbb{B}_X(n)| \leqslant |gN \cap \mathbb{B}_X(n)| \leqslant \left(\frac{1}{d} + \varepsilon\right) |\mathbb{B}_X(n)|. \tag{1}$$

Now, suppose $\mathrm{dc}_X(G) > \mathrm{dc}(G/N)$ and let us find a contradiction. By definition, this means that there exist $\delta > 0$ for which

$$\frac{|\{(u,v) \in (\mathbb{B}_X(n))^2 \ : \ uv = vu\}|}{|\mathbb{B}_X(n)|^2} > \mathrm{dc}(G/N) + \delta$$

for infinitely many $n$'s. In view of this $\delta$, take $\varepsilon > 0$ small enough so that $\varepsilon d(2+\varepsilon d) \leqslant \delta$, and we have (1) for all but finitely many $n$'s. Combining both assertions, there is a big enough $n$ such that

$$\begin{aligned}
\mathrm{dc}(G/N) + \delta &< \frac{|\{(u,v) \in (\mathbb{B}_X(n))^2 \ : \ uv = vu\}|}{|\mathbb{B}_X(n)|^2} \\
&\leqslant \frac{1}{|\mathbb{B}_X(n)|^2} |\{(\overline{u},\overline{v}) \in (G/N)^2 \ : \ \overline{u}\,\overline{v} = \overline{v}\,\overline{u}\}| \left(\frac{1}{d} + \varepsilon\right)^2 |\mathbb{B}_X(n)|^2 \\
&= \frac{|\{(\overline{u},\overline{v}) \in (G/N)^2 \ : \ \overline{u}\,\overline{v} = \overline{v}\,\overline{u}\}|}{d^2} (1+\varepsilon d)^2 \\
&\leqslant \frac{|\{(\overline{u},\overline{v}) \in (G/N)^2 \ : \ \overline{u}\,\overline{v} = \overline{v}\,\overline{u}\}|}{d^2} + 2\varepsilon d + \varepsilon^2 d^2 \\
&= \mathrm{dc}(G/N) + \varepsilon d(2+\varepsilon d),
\end{aligned}$$

where the second inequality comes from the facts that $uv =_G vu$ implies $\overline{u}\,\overline{v} =_{G/N} \overline{v}\,\overline{u}$ and that, according to (1), every class in $G/N$ has, at most, $(1/d+\varepsilon)\,|\mathbb{B}_X(n)|$ representatives in $\mathbb{B}_X(n)$. We deduce that $\delta < \varepsilon d(2+\varepsilon d)$, which is a contradiction. □

*Proof of Theorem* 1.3. Recall that a group $G$ is *residually finite* if, for every non-trivial element $1 \neq g \in G$, there is a finite quotient of $G$ where the image of $g$ is non-trivial. Equivalently, for every $g \neq 1$, there exists a finite index normal subgroup $N \trianglelefteq G$ such that $g \notin N$. When $G$ is finitely generated, we can always additionally take this subgroup to be *characteristic in* $G$ (i.e., invariant under every automorphism of $G$): take, for instance, $K \leqslant G$ to be the intersection of all subgroups of $G$ whose index is the same as that of $N$ (there are finitely many, so $K \leqslant N$ is still of finite index in $G$).

Assertion (ii) follows directly from Proposition 2.3: one implication is trivial; for the other, if $G$ satisfies $\mathrm{dc}_X(G) > 5/8$, then all its finite quotients do and so, by Gustafson's Theorem 1.1, they all are abelian. This already implies that $G$ is itself abelian: if $g_1, g_2 \in G$ do not commute, then $1 \neq [g_1, g_2]$ would survive in some finite quotient $Q = G/N$, contradicting the fact that such $Q$ is abelian.

For part (i), let $H \leqslant G$ be an abelian subgroup of finite index. Then,

$$\mathrm{dc}_X(G) \geq \limsup_{n\to\infty} \frac{|(H\times H)\cap(\mathbb{B}_X(n)\times\mathbb{B}_X(n))|}{|\mathbb{B}_X(n)|^2} = \frac{1}{|G\times G : H\times H|} = \frac{1}{|G:H|^2},$$

since $H \times H$ is a finite index subgroup of $G \times G$, which also has subexponential growth, so we can invoke Proposition 2.2 (a technical detail is left to the reader here: the sets $\mathbb{B}_X(n)\times\mathbb{B}_X(n)$ are not exactly balls in the group $G\times G$; however, the value of the above limits is not affected by this difference).

Conversely, assume that $G$ is not virtually abelian, and let us prove that $\mathrm{dc}_X(G) = 0$. Knowing that $G$ is finitely generated, residually finite, and not virtually abelian, we can choose two non-commuting elements $g_1, g_2 \in G$ and a characteristic subgroup $K_1$ of finite index in $G$ such that $[g_1, g_2] \notin K_1$ (hence, $G/K_1$ is non-abelian and so $\mathrm{dc}(G/K_1) \leq 5/8$). Clearly, these three properties go to finite index subgroups, and so we can repeat the construction and get a descending sequence of subgroups,

$$\cdots \trianglelefteq K_i \trianglelefteq K_{i-1} \trianglelefteq \cdots \trianglelefteq K_2 \trianglelefteq K_1 \trianglelefteq K_0 = G,$$

each characteristic and of finite index in the previous one and such that $\mathrm{dc}(K_i/K_{i+1}) \leq 5/8$. Then, for every $i$, $[G : K_i] < \infty$ and $K_i$ is characteristic (and so normal) in $G$. Now, since $(G/K_i)/(K_{i-1}/K_i) = G/K_{i-1}$, Lemma 2.1 tells us that

$$\mathrm{dc}(G/K_i) \leqslant \mathrm{dc}(K_{i-1}/K_i)\cdot \mathrm{dc}(G/K_{i-1}) \leqslant \frac{5}{8}\,\mathrm{dc}(G/K_{i-1}),$$

for every $i \geqslant 1$. By induction, $\mathrm{dc}(G/K_i) \leqslant (5/8)^i$ and then, by Proposition 2.3, we get

$$\mathrm{dc}_X(G) \leqslant \mathrm{dc}(G/K_i) \leqslant (5/8)^i.$$

Since this is true for every $i \geqslant 1$, we conclude that $\mathrm{dc}_X(G) = 0$. □

## 3. Hyperbolic groups

We now give another criterion to show that certain groups have degree of commutativity equal to zero. It will apply to (many) exponentially growing groups not contained in the results from the previous sections.

**Lemma 3.1.** *Let $G$ be a finitely generated group, and let $X$ be a finite generating set for $G$. Suppose that there exists a subset $\mathcal{N} \subseteq G$ satisfying the following conditions:*

(i) *$\mathcal{N}$ is $X$-negligible, i.e.,* $\lim_{n\to\infty} \dfrac{|\mathcal{N} \cap \mathbb{B}_X(n)|}{|\mathbb{B}_X(n)|} = 0$*;*

(ii) $\lim_{n\to\infty} \dfrac{|C(g) \cap \mathbb{B}_X(n)|}{|\mathbb{B}_X(n)|} = 0$ *uniformly in $g \in G \setminus \mathcal{N}$.*

*Then,* $\mathrm{dc}_X(G) = 0$.

*Proof.* By the two hypotheses, given $\varepsilon > 0$ there exist $n_0$ and $n_1$ such that, for $n \geqslant \max\{n_0, n_1\}$, and for all $g \in G \setminus \mathcal{N}$, we have $|\mathcal{N} \cap \mathbb{B}_X(n)| < \frac{\varepsilon}{2}|\mathbb{B}_X(n)|$ and $|C(g) \cap \mathbb{B}_X(n)| < \frac{\varepsilon}{2}|\mathbb{B}_X(n)|$. Hence,

$$\begin{aligned}
|\{(u,v) \in (\mathbb{B}_X(n))^2 \mid uv = vu\}| &= \sum_{u \in \mathbb{B}_X(n)} |C(u) \cap \mathbb{B}_X(n)| \\
&= \sum_{u \in \mathbb{B}_X(n)\setminus\mathcal{N}} |C(u) \cap \mathbb{B}_X(n)| + \sum_{u \in \mathcal{N}\cap\mathbb{B}_X(n)} |C(u) \cap \mathbb{B}_X(n)| \\
&\leqslant \sum_{u \in \mathbb{B}_X(n)\setminus\mathcal{N}} \frac{\varepsilon}{2}|\mathbb{B}_X(n)| + \sum_{u \in \mathcal{N}\cap\mathbb{B}_X(n)} |\mathbb{B}_X(n)| \\
&\leqslant \frac{\varepsilon}{2}|\mathbb{B}_X(n)|^2 + \frac{\varepsilon}{2}|\mathbb{B}_X(n)|^2 \\
&= \varepsilon|\mathbb{B}_X(n)|^2.
\end{aligned}$$

Therefore, $\mathrm{dc}_X(G) = 0$ (with existence of the real limit, not just a lim sup). ☐

*Proof of Theorem* 1.7. Fix any finite generating set $X$ for $G$. Since $G$ is non-elementary hyperbolic it has exponential growth, i.e., $\lim_{n\to\infty} |\mathbb{B}_X(n+1)|/|\mathbb{B}_X(n)| = \lambda > 1$.

For the whole proof, fix $\varepsilon_0 > 0$ so that $\lambda - \varepsilon_0 > 1$.

The above limit means that $\lambda - \varepsilon_0 < |\mathbb{B}_X(n+1)|/|\mathbb{B}_X(n)| < \lambda + \varepsilon_0$, for $n \gg 0$. Hence, using induction, there exist constants $C_1$ and $C_2$ such that $C_1(\lambda-\varepsilon_0)^n \leqslant |\mathbb{B}_X(n)| \leqslant C_2(\lambda+\varepsilon_0)^n$ for all $n \in \mathbb{N}$. By induction on $m$, it also follows that $|\mathbb{B}_X(n)|/|\mathbb{B}_X(n+m)| \leqslant 1/(\lambda-\varepsilon_0)^m$, for $n \gg 0$ and for all $m \geqslant 1$.

Let $\mathcal{N}$ be the set of torsion elements in $G$; we shall see that the two conditions in Lemma 3.1 are satisfied with respect to this set. To start, it was proved by Dani (see [4, Theorem 1.1]) that there exist constants $D_1$ and $D_2$ such that $|\mathcal{N}\cap\mathbb{B}_X(n)| \leqslant D_1|\mathbb{B}_X(\lceil \frac{n}{2} \rceil + D_2)|$ (this also follows easily from [1, Proposition 3], together with the fact that in a hyperbolic group there are finitely many conjugacy classes of torsion elements). Therefore, for $n \gg 0$ we have

$$\frac{|\mathcal{N} \cap \mathbb{B}_X(n)|}{|\mathbb{B}_X(n)|} \leqslant \frac{D_1|\mathbb{B}_X(\lceil \frac{n}{2} \rceil + D_2)|}{|\mathbb{B}_X(n)|} \leqslant \frac{D_1}{(\lambda-\varepsilon_0)^{n-\lceil \frac{n}{2} \rceil - D_2}} = \frac{D_1}{(\lambda-\varepsilon_0)^{\lfloor \frac{n}{2} \rfloor - D_2}},$$

so $\mathcal{N}$ is negligible and condition (i) in Lemma 3.1 is satisfied.

Before proving (ii), let us recall a few well-known facts about hyperbolic groups. For $g \in G$, $\tau(g) = \lim_{n\to\infty} |g^n|_X/n$ denotes the *stable translation length* of $g$. Since $|g^{n+m}|_X \leqslant |g^n|_X + |g^m|_X$, Fekete's lemma gives that $\tau(g) = \inf_{n\in\mathbb{N}}\{|g^n|_X/n\}$, and so $|n|\tau(g) \leqslant |g^n|_X$ for all $n \in \mathbb{Z}$. The hyperbolicity of $G$ implies that these translation lengths are discrete (see [2, III.Γ.3.17]); in particular, there is a positive integer $p$ such that $\tau(g) \geqslant 1/p$ for all $g \in G \setminus \mathcal{N}$. Also, centralizers of elements of infinite order in hyperbolic groups are virtually cyclic (see [2, Corollary III.Γ.3.10]), and there is a bound $M > 0$ on the size of finite subgroups of $G$ (depending only on the hyperbolicity constant of $G$); see [2, Theorem III.Γ.3.2].

Now let $C = \langle g \rangle$ be an infinite cyclic subgroup of $G$. Then, $g^k \in C \cap \mathbb{B}_X(n)$ implies that $|k|/p \leq |k|\tau(g) \leq |g^k|_X \leq n$, and hence $|k| \leq pn$; therefore,

$$|C \cap \mathbb{B}_X(n)| \leqslant 2pn + 1.$$

Furthermore, we can also deduce that, for any $x \in G$, the coset of $C$ containing $x$ also grows linearly: this is because if $xC \cap \mathbb{B}_X(n)$ is non-empty, it will contain some $w$ of length at most $n$, $w^{-1}(xC \cap \mathbb{B}_X(n)) \subseteq C \cap \mathbb{B}_X(2n)$ and hence,

$$|xC \cap \mathbb{B}_X(n)| = |w^{-1}(xC \cap \mathbb{B}_X(n))| \leqslant |C \cap \mathbb{B}_X(2n)| \leqslant 4pn + 1.$$

To check condition (ii) from Lemma 3.1, take $g \in G \setminus \mathcal{N}$. The centralizer $C_G(g)$ is virtually cyclic and, by a classical result, it is also of type finite-by-$\mathbb{Z}$ or finite-by-$\mathbb{D}_\infty$, where $\mathbb{D}_\infty$ is the infinite dihedral group (see [8, Lemma 4.1]). Passing to a subgroup of index two $H \leqslant C_G(g)$ if necessary, we have a short exact sequence $1 \to K \to H \to \mathbb{Z} \to 1$, with $K$ (inside $H \leqslant C_G(g) \leqslant G$) finite. Since this sequence splits, $\mathbb{Z}$ is a subgroup of $H$ of index $|K| \leqslant M$, and so $C_G(g)$ has a subgroup of index at most $2M$ which is infinite cyclic. Putting these results together, we get that

$$\frac{|C_G(g) \cap \mathbb{B}_X(n)|}{|\mathbb{B}_X(n)|} \leqslant \frac{2M(4pn+1)}{C_1(\lambda - \varepsilon_0)^n} \longrightarrow 0$$

uniformly on $g \in G \setminus \mathcal{N}$, when $n \to \infty$. So, condition (ii) in Lemma 3.1 is satisfied. □

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