EVALUATION OF THE $BC_n$ ELLIPTIC SELBERG INTEGRAL
VIA THE FUNDAMENTAL INVARIANTS

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Abstract. We give a proof of the evaluation formula for the elliptic Selberg integral of type $BC_n$ as an application of the fundamental $BC_n$-invariants.

1. Introduction

The evaluation formula of the $BC_n$ elliptic Selberg integral was proposed for the first time by van Diejen and Spiridonov [17]. Namely, under the balancing condition

$$a_1 \cdots a_6 t^{2n-2} = pq,$$

we have

$$\frac{1}{(2\pi\sqrt{-1})^n} \int_{\mathbb{T}^n} \prod_{i=1}^n \prod_{m=1}^6 \frac{\Gamma(a_m z_i^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 2}; p, q)} \prod_{1 \leq j < k \leq n} \frac{\Gamma(t^{z_j^{1/2}} z_k^{1/2}; p, q)}{\Gamma(z_j^{1/2} z_k^{1/2}; p, q)} dz_1 \cdots dz_n = \frac{2^n n!}{(p; p)_\infty^{n/2} (q; q)_\infty^{n/2}} \prod_{i=1}^n \left( \frac{\Gamma(t^i; p, q)}{\Gamma(t; p, q)} \prod_{1 \leq j < k \leq 6} \frac{\Gamma(t^{-1} a_j a_k; p, q)}{\Gamma(a_j a_k; p, q)} \right),$$

(1.1)

where $a_1, \ldots, a_6, t$ are complex parameters with $|a_m| < 1$ ($m = 1, \ldots, 6$), $|t| < 1$, and $\mathbb{T}^n$ stands for the $n$-dimensional torus. (Here $\Gamma(z; p, q)$ denotes the Ruijsenaars elliptic gamma function, and the double-signs indicate a product of all possible factors.) In the paper [17], the authors outlined a way of proving (1.1) following Anderson’s method [2], which is known as a typical derivation for the evaluation formula of the Selberg integral [15] via the other multi-dimensional integral [4] called the Dixon–Anderson integral in [5,8]. The proof outlined in [17] was eventually completed by Rains [14], proving the elliptic counterpart of the evaluation of the Dixon–Anderson integral

$$\frac{1}{(2\pi\sqrt{-1})^n} \int_{\mathbb{T}^n} \prod_{i=1}^n \prod_{m=1}^{2n+4} \frac{\Gamma(a_m z_i^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 2}; p, q)} \prod_{1 \leq j < k \leq n} \frac{1}{\Gamma(z_j^{1/2} z_k^{1/2}; p, q)} dz_1 \cdots dz_n = \frac{2^n n!}{(p; p)_\infty^{n/2} (q; q)_\infty^{n/2}} \prod_{1 \leq j < k \leq 2n+4} \Gamma(a_j a_k; p, q),$$

whose alternative proof was given by Spiridonov [16].

Besides Anderson’s method, several derivations are known for the evaluation formula of the Selberg integral. Aomoto [1] gave an alternative proof by characterizing the integral as a solution of a difference equation with some specific boundary
condition (see also [9] for the $q$-integral case). The aim of this paper is to give an alternative proof for the $BC_n$ elliptic Selberg integral [11], following Aomoto’s method as outlined below. Denoting by $I(a_1, \ldots, a_6)$ the left-hand side of (1.1), we first prove that under the balancing condition $a_1 \cdots a_6 t^{2n-2} = pq$, this integral satisfies the system of $q$-difference equations:

\[(1.2) \quad I(a_1, \ldots, a_5, a_6) = I(a_1, \ldots, aqa_k, \ldots, a_5, a^{-1} a_6) \prod_{i=1}^{n} \prod_{m=1, m \neq k}^{5} \frac{\theta(q^{-1} a_m a_6 t^{i-1}; p)}{\theta(a_m a_k t^{i-1}; p)} \]

for $k = 1, \ldots, 5$. Setting

$$\Psi(z) = \prod_{i=1}^{n} \frac{\Gamma(pa_6 z_i^{\pm 1}; p)}{\Gamma(z_i^{\pm 2}; p, q)} \prod_{1 \leq j < k \leq n} \frac{\Gamma(t z_j^{\pm 1}, z_k^{\pm 1}; p, q)}{\Gamma(z_j^{\pm 1}, z_k^{\pm 1}; p, q)},$$

we use the notation

$$\langle \varphi(z) \rangle = \int_{T^n} \varphi(z) \Psi(z) \frac{dz_1 \cdots dz_n}{z_1 \cdots z_n}$$

for any meromorphic function $\varphi(z)$ on $(\mathbb{C}^*)^n$. Then the difference equation (1.2) of the case $k = 1$ is equivalent to the equality

\[(1.3) \quad \langle E_n(a_1, a_6; z) \rangle = \langle E_0(a_1, a_6; z) \rangle \prod_{i=1}^{n} \frac{\theta(a_2 \theta(a_6 a_1^{-1} t^{i-1}; p))}{\theta(a_2 \theta(a_1 a_6^{-1} t^{i-1}; p))} \prod_{m=2}^{5} \frac{\theta(a_m a_6 t^{i-1}; p)}{\theta(a_m a_1 t^{i-1}; p)} \]

under the balancing condition $a_1 \cdots a_6 t^{2n-2} = 1$, where

$$E_0(a, b; z) = \prod_{i=1}^{n} \frac{\theta(a z_i^{\pm 1}; p)}{\theta(a b t^{i-1} z_i^{\pm 1}; p)}, \quad E_n(a, b; z) = \prod_{i=1}^{n} \frac{\theta(b z_i^{\pm 1}; p)}{\theta(b a t^{i-1} z_i^{\pm 1}; p)}.$$  

The idea of Aomoto’s method is to introduce appropriate intermediate functions which interpolate equation (1.3). We now define a set of holomorphic symmetric functions by

\[(1.4) \quad E_r(a, b; z) = \sum_{1 \leq i_1 < \cdots < i_r \leq n} \prod_{k=1}^{r} \frac{\theta(b t^{i_k-1} z_{i_k}^{\pm 1}; p)}{\theta(b t^{i_k} (a t^{i_k-1} z_{i_k})^{\pm 1}; p)} \prod_{l=1}^{n-r} \frac{\theta(a t^{l-1} z_{i_l}^{\pm 1}; p)}{\theta(a t^{l-1} (b t^{l-1})^{\pm 1}; p)} \]

for $r = 0, 1, \ldots, n$, where the summation is taken over all pairs of sequences $1 \leq i_1 < \cdots < i_r \leq n$ and $1 \leq j_1 < \cdots < j_{n-r} \leq n$ such that $\{i_1, \ldots, i_r\} \cup \{j_1, \ldots, j_{n-r}\} = \{1, 2, \ldots, n\}$. Under the condition $a_1 \cdots a_6 t^{2n-2} = 1$, one can show that the following recurrence relations hold:

\[(1.5) \quad \langle E_r(a_1, a_6; z) \rangle = C_r \langle E_{r-1}(a_1, a_6; z) \rangle \quad (r = 1, \ldots, n),\]

where the coefficients $C_r$ are given by

$$C_r = -\frac{a_2^2 t^{2r-2} \theta(t^{n-r+1}; p) \theta(a_6 a_1^{-1} t^{n-r+1}; p) \theta(a_1 a_6^{-1} t^{2r-n}; p)}{a_6^2 t^{2n-2r} \theta(t^{r}; p) \theta(a_6 a_1^{-1} t^{n-2r+2}; p) \theta(a_1 a_6^{-1} t^{r}; p)} \prod_{m=2}^{5} \frac{\theta(a_m a_6 t^{n-r}; p)}{\theta(a_m a_1 t^{r-1}; p)}.$$  

Using (1.5) repeatedly, we immediately obtain (1.3). We call these $E_r(a, b; z)$ the fundamental invariants of type $BC_n$, which thus play an essential role in this paper. The fundamental invariants (1.4) are given as a special case of the Lagrange interpolation functions of type $BC_n$ in the context of the connection problem among the independent cycles for the $BC_n$ Jackson integral; see [11] Example 2 of Theorem 1.4. See also [10] for details of the fundamental invariants (1.4). We remark that our fundamental invariants $E_r(a, b; z)$ are essentially the interpolation theta
functions of Coskun–Gustafson [3] and Rains [13] attached to single columns of partitions. In fact, $E_r(a,b;z)$ are compared explicitly with the functions of [3] and [13], respectively, as explained in [10, Introduction]. Also, the key equation (1.5) is essentially the same as [10, Theorem 4.1], which we proved in the context of a $BC_n$ elliptic summation formula. It should be mentioned that van Diejen–Spiridonov [17] already pointed out that the integral (1.1) implies the $BC_n$ elliptic summation formula via residue calculus.

Note that the integral (1.1) with $p = 0$ is known as Gustafson’s contour $q$-integral [6], which is the Nassrallah–Rahman integral in the case $n = 1$ [12]. Aomoto’s method using the fundamental invariants (1.4) with $p = 0$ leads us to the recurrence relations for Gustafson’s contour $q$-integral. This fact was previously discussed in [7, Corollary 5.2 and Eq.(5.3)].

In order to establish the evaluation formula (1.1), we need to investigate further the boundary condition for the difference equations (1.2); the precise arguments will be given later in Section 5.

This paper is organized as follows. After defining basic terminology in Section 2, we first discuss the system of $q$-difference equations (1.2) in Section 3. In Section 4 we study the analytic continuation of the integral (1.1) as a meromorphic function of the parameters $a_1,\ldots,a_5$ in a specific domain. We use this argument to show that the integral (1.1) is expressed as a product of elliptic gamma functions up to a constant. Section 5 is devoted to obtaining the boundary condition for (1.2) through asymptotic analysis of the contour integral (1.1) as $a_2 \to a_1^{-1}$ (i.e. $a_1 a_2 \to 1$). This condition determines the explicit value of the constant, which was indefinite at the time of Section 4. In the case of elliptic hypergeometric integrals, we often meet some strict restraints on parameters, which do not permit us to consider the asymptotic behavior like $a_1 \to 0$ or $\infty$ as we usually do in the rational or trigonometric ($q$-analog) cases. Thus our treatment of the boundary condition might look totally different from that of the $q$-analog case. It should be noted, however, that our method to analyze such a situation as $a_1 a_2 \to 1$ is also applicable to the case $p = 0$ of the integral (1.1), thus providing a novel insight even for the evaluation of contour $q$-integrals.

2. $BC_n$ elliptic Selberg integral

Throughout this paper we denote by $\Gamma(u;p,q)$ ($u \in \mathbb{C}^*$) the Ruijsenaars elliptic gamma function defined by

$$\Gamma(u;p,q) = \frac{(pqu^{-1};p,q)_\infty}{(u;p,q)_\infty}, \quad (u;p,q)_\infty = \prod_{\mu,\nu=0}^{\infty} \frac{1 - p^\mu q^\nu u}{1 - p^\mu q^\nu}, \quad (|p| < 1, |q| < 1).$$

Note that $\Gamma(u;p,q)$ satisfies

$$(2.1) \quad \Gamma(qu;p,q) = \theta(u;p)\Gamma(u;p,q), \quad \Gamma(pq/u;p,q) = \frac{1}{\Gamma(u;p,q)}.$$ 

We consider the meromorphic function

$$\Psi(z) = \prod_{i=1}^n \prod_{m=1}^6 \frac{\Gamma(a_m z_i^{\pm 1};p,q)}{\Gamma(z_i^{\pm 2};p,q)} \prod_{1 \leq j < k \leq n} \frac{\Gamma(t z_j^{\pm 1} z_k^{\pm 1};p,q)}{\Gamma(z_j^{\pm 1} z_k^{\pm 1};p,q)}$$

in $z = (z_1,\ldots,z_n) \in (\mathbb{C}^*)^n$ with complex parameters $a_1,\ldots,a_6,t \in \mathbb{C}^*$, assuming throughout that $|t| < 1$. We also use the notation $\Psi(a_1,\ldots,a_6;z)$ for $\Psi(z)$ when
we need to make the dependence on the parameters \( a_1, \ldots, a_6 \) explicit. For this function \( \Psi(z) \), we investigate the multiple integral

\[
I = \int_\sigma \Psi(z) \varpi(z), \quad \varpi(z) = \frac{1}{(2\pi i)^n} dz_1 \cdots dz_n
\]

over an \( n \)-cycle \( \sigma \). Since \( \Psi(z) \) is expressed as

\[
\Psi(z) = \prod_{i=1}^n (1 - z_i^{+2})(p z_i^{+2}; q) (q z_i^{+2}; q)_\infty \\
\times \prod_{1 \leq j < k \leq n} (1 - \frac{1}{z_i^{+1} z_j^{+1}}) (p z_j^{+1} z_k^{+1}; p) (q z_j^{+1} z_k^{+1}; q)_\infty \\
\times \prod_{i=1}^n \prod_{m=1}^6 (pq a_m^{-1} z_i^{+1}; p, q)_\infty \\
\times \prod_{1 \leq j < k \leq n} (pq t z_j^{+1} z_k^{+1}; p, q)_\infty \\
\times \prod_{1 \leq j < k \leq n} (pq t^{-1} z_j^{+1} z_k^{+1}; p, q)_\infty,
\]

we see that \( \Psi(z) \) has poles possibly along the divisors

\[
z_i^{+1} = a m p^\mu q^\nu \quad (1 \leq i \leq n; m = 1, \ldots, 6; \mu, \nu = 0, 1, 2, \ldots), \\
z_j^{+1} z_k^{+1} = t p^\mu q^\nu \quad (1 \leq j < k \leq n; \mu, \nu = 0, 1, 2, \ldots).
\]

Also, regarded as a function of \( z_i \) \((i = 1, \ldots, n)\), \( \Psi(z) \) has poles possibly at

\[p^\mu q^\nu a_m, \quad p^{-\mu} q^{-\nu} a_m^{-1}, \quad p^\mu q^\nu t z_j^{+1}, \quad p^{-\mu} q^{-\nu} t^{-1} z_j^{+1},\]

where \( 1 \leq m \leq 6, 1 \leq j \leq n, j \neq i \) and \( \mu, \nu = 0, 1, 2, \ldots \). If the parameters satisfy the condition \(|a_i| < 1, \ldots, |a_6| < 1\), then \( \Psi(z) \) is holomorphic in a neighborhood of the \( n \)-dimensional torus

\[
T^n = \{ z = (z_1, \ldots, z_n) \in (\mathbb{C}^*)^n \mid |z_i| = 1 \quad (i = 1, \ldots, n) \},
\]

and hence the integral

\[
I(a_1, \ldots, a_6) = \int_{T^n} \Psi(a_1, \ldots, a_6; z) \varpi(z)
\]

defines a holomorphic function on the domain

\[
U = \{(a_1, \ldots, a_6) \in (\mathbb{C}^*)^6 \mid |a_m| < 1 \quad (m = 1, \ldots, 6)\} \subset (\mathbb{C}^*)^6.
\]

This function can be continued to a holomorphic function on a larger domain by replacing \( T^n \) with an appropriate \( n \)-cycle depending on the parameters \((a_1, \ldots, a_6)\). We give below a remark on analytic continuation of this sort.

For each \((a_1, \ldots, a_6) \in (\mathbb{C}^*)^6\), we define two subsets \( S_0, S_\infty \) of \( \mathbb{C}^* \) by

\[
S_0 = \{ p^\mu q^\nu a_m \mid 1 \leq m \leq 6, \mu, \nu \in \mathbb{N}\}, \\
S_\infty = \{ p^{-\mu} q^{-\nu} a_m^{-1} \mid 1 \leq m \leq 6, \mu, \nu \in \mathbb{N}\},
\]

where \( \mathbb{N} = \{ 0, 1, 2, \ldots \} \), and suppose that \( S_0 \cap S_\infty = \phi \). Assuming that \(|t| < r^2\) for some \( r \in (0, 1) \), we choose a circle

\[
C_\rho(0) = \{ u \in \mathbb{C}^* \mid |u| = \rho \}, \quad \rho \in [r, r^{-1}],
\]

which does not intersect with \( S_0 \cup S_\infty \). Then we define a cycle \( C \) in \( \mathbb{C}^* \) by

\[
C = C_\rho(0) + \sum_{c \in S_0; |c| > \rho} C_\epsilon(c) - \sum_{c \in S_\infty; |c| < \rho} C_\epsilon(c),
\]

where \( C_\epsilon(c) \) denotes a sufficiently small circle around \( c \). Note that if \(|a_m| < 1 \quad (m = 1, \ldots, 6)\), then \( C \) is homologous to the unit circle. We now assume that
\[ |a_m| < r^{-1} \quad (m = 1, \ldots, 6). \] Then such a cycle \( C \) can be taken inside the annulus \( A_r = \{ u \in \mathbb{C}^* \mid r \leq |u| \leq r^{-1} \} \). Since \( |t| < r^2 \), the meromorphic function \( \Psi(z) \) is holomorphic in a neighborhood of the \( n \)-cycle \( C^n = C \times \cdots \times C \). Hence, the integral
\[
I = \int_{C^n} \Psi(z) \, \varpi(z)
\]
is well defined and does not depend on the choice of \( \rho \in [r, r^{-1}] \). This implies the following lemma on analytic continuation.

**Lemma 2.1.** Suppose that \( |t| < r^2 \) for some real number \( r \in (0, 1] \). Then the holomorphic function \( I(a_1, \ldots, a_6) \) on the domain \( U \) of (2.2) can be continued to a holomorphic function on
\[
(a_1, \ldots, a_6) \in (\mathbb{C}^*)^6 \quad \left| a_m \right| < r^{-1} \quad (1 \leq m \leq 6), \quad a_k a_l \notin p^{-N} q^{-N} \quad (1 \leq k, l \leq 6) \right\}
\]

As can be seen in \([10]\), under the balancing condition \( a_1 \cdots a_6 t^{2n-2} = pq \), this function \( I(a_1, \ldots, a_6) \) is eventually continued to a meromorphic function on a hypersurface in \((\mathbb{C}^*)^6\) with poles along the divisors
\[
p^{\mu} q^{\nu} t^{i-1} a_k a_l = 1 \quad (k, l \in \{1, \ldots, 6\}; i = 1, \ldots, n; \mu, \nu = 0, 1, 2, \ldots).
\]

### 3. q-Difference Equations With Respect to the Parameters

In this section we derive a system of q-difference equations for the integral \( I(a_1, \ldots, a_6) \) on the basis of the arguments in \([10]\). Our goal is to establish the following proposition.

**Proposition 3.1.** Suppose that \( |p| < |t|^{2n-2} \). Under the balancing condition \( a_1 \cdots a_6 t^{2n-2} = pq \), the integral \( I(a_1, \ldots, a_6) \) satisfies the system of q-difference equations
\[
I(a_1, \ldots, a_6) = I(a_1, \ldots, qa_k, \ldots, a_5, q^{-1} a_6) \prod_{i=1}^{n} \prod_{1 \leq m \leq 5 \atop m \neq k} \frac{\theta(q^{-1} a_m a_6 t^{i-1}; p)}{\theta(a_m a_k t^{i-1}; p)}
\]
for \( k = 1, \ldots, 5 \), provided that \( |a_1| < 1, \ldots, |a_5| < 1 \) and \( |a_6| < |q| \).

Note that the condition \( |a_6| < |q| \) is equivalent to \( |a_1 \cdots a_5| > \sqrt{p/|t|^{2n-2}} \) under the balancing condition. We need to assume that \( p \) is sufficiently small as specified above to guarantee that (3.1) holds in a nonempty region.

In order to make use of the arguments of \([10]\), we modify \( \Psi(z) \) as
\[
\tilde{\Psi}(z) = \Psi(a_1, \ldots, a_5, pa_6; z)
\]

\[
= \prod_{i=1}^{n} \frac{\Gamma(pa_6 z_i^{\pm 1}; p, q) \prod_{m=1}^{5} \Gamma(a_m z_i^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 2}; p, q)} \prod_{1 \leq j < k \leq n} \frac{\Gamma(t z_j^{\pm 1} z_k^{\pm 1}; p, q)}{\Gamma(z_j^{\pm 1} z_k^{\pm 1}; p, q)}
\]

\[
= \prod_{i=1}^{n} \frac{\prod_{m=1}^{5} \Gamma(a_m z_i^{\pm 1}; p, q)}{\Gamma(qa_6^{-1} z_i^{\pm 1}; p, q) \prod_{1 \leq j < k \leq n} \Gamma(z_j^{\pm 1} z_k^{\pm 1}; p, q)} \prod_{1 \leq j < k \leq n} \frac{\Gamma(t z_j^{\pm 1} z_k^{\pm 1}; p, q)}{\Gamma(z_j^{\pm 1} z_k^{\pm 1}; p, q)}.
\]
This function $\tilde{\Psi}(z)$ coincides with the meromorphic function $\tilde{\Phi}(z)$ in [10] up to multiplication by a $q$-periodic function in all variables $z_1, \ldots, z_n$. Namely one has

$$
T_{q,z_i} \frac{\tilde{\Psi}(z)}{\tilde{\Psi}(z)} = -\left(\frac{q^{-1}z_i^{-1}}{2}\frac{\theta(q^{-2}z_i^{-2};p)}{\theta(q^{-1}a^{-1}z_i^{-1};p)} \times \prod_{1 \leq k \leq n, k \neq i} \frac{\theta(tz_i z_k^{\pm 1};p)\theta(q^{-1}z_i^{-1}z_k^{\pm 1};p)}{\theta(q^{-1}z_i z_k^{\pm 1};p)\theta(z_i z_k^{\pm 1};p)}
\right)
$$

for $i = 1, \ldots, n$, where $T_{q,z_i}$ stands for the $q$-shift operator in $z_i$:

$$
T_{q,z_i} f(z_1, \ldots, z_n) = f(z_1, \ldots, qz_i, \ldots, z_n).
$$

As to the parameters $a_1, \ldots, a_6$, one has

$$
\frac{T_{q,a_m} \tilde{\Psi}(z)}{\tilde{\Psi}(z)} = \prod_{i=1}^{n} \theta(a_m z_i^{\pm 1};p) \quad (1 \leq m \leq 5),
$$

(3.2)

$$
\frac{T_{q,a_6} \tilde{\Psi}(z)}{\tilde{\Psi}(z)} = a_6^{-2n} \prod_{i=1}^{n} \theta(a_6 z_i^{\pm 1};p).
$$

In this paper we use the notation of expectation values to refer to the integral

$$
\langle \varphi(z) \rangle = \int_{\mathbb{T}^n} \varphi(z) \tilde{\Psi}(z) \varpi(z)
$$

for any meromorphic function $\varphi(z)$ on $(\mathbb{C}^*)^n$ such that $\varphi(z)\tilde{\Psi}(z)$ is holomorphic in a neighborhood of the $n$-dimensional torus $\mathbb{T}^n$. If we set

$$
\nabla_{q,z_i} \varphi(z) = \varphi(z) - \frac{T_{q,z_i} \tilde{\Psi}(z)}{\tilde{\Psi}(z)} T_{q,z_i} \varphi(z) \quad (i = 1, \ldots, n)
$$

as in [10], we have

$$
\langle \nabla_{q,z_i} \varphi(z) \rangle = 0 \quad (i = 1, \ldots, n)
$$

for any meromorphic function $\varphi(z)$ such that $\varphi(z)\tilde{\Psi}(z)$ is holomorphic in a neighborhood of the compact set

(3.3)

$$
|q| \leq |z_i| \leq 1, \quad |z_j| = 1 \quad (1 \leq j \leq n; \ j \neq i).
$$

In fact, by the Cauchy theorem one has

$$
\int_{\mathbb{T}^n} \tilde{\Psi}(z) \varphi(z) \varpi(z) = \int_{\mathbb{T}^n} T_{q,z_i} (\tilde{\Psi}(z) \varphi(z)) \varpi(z) \quad (i = 1, \ldots, n).
$$

We set

$$
K(a_1, \ldots, a_5, a_6) = I(a_1, \ldots, a_5, pa_6) = \langle 1 \rangle
$$
assuming that \(|a_m| < 1\) \((m = 1, \ldots, 5)\), \(|pa_6| < 1\). Then from (3.2) we have

\[
K(qa_1, a_2, \ldots, a_6) = \left( \prod_{i=1}^n \theta(a_1 z_i^{\pm 1}; p) \right)
= \langle E_0(a_1, a_6; z) \rangle \prod_{i=1}^n \theta(a_1 (a_6 t_i^{-1})^{\pm 1}; p),
\]

(3.4)

\[
K(a_1, \ldots, a_5, qa_6) = a_6^{-2n} \left( \prod_{i=1}^n \theta(a_6 z_i^{\pm 1}; p) \right)
= \langle E_n(a_1, a_6; z) \rangle \prod_{i=1}^n a_6^{-2} \theta(a_6 (a_1 t_i^{-1})^{\pm 1}; p),
\]

where \(E_r(a_1, a_6; z) = E_r^{(n)}(a_1, a_6; z)\) \((r = 0, 1, \ldots, n)\) denote the fundamental \(BC_n\)-invariants (1.4); for the basic properties of these functions, we refer the reader to [10] Section 3. On the other hand, as for the function 

\[
\varphi_{r,i}(z) = F_{i}^{-}(z) E_{r-1}^{(n-1)}(a_1, a_6; z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n) \quad (1 \leq i \leq n; 1 \leq r \leq n)
\]

of [10] Section 4, where

\[
F_{i}^{-}(z) = \prod_{m=1}^6 \frac{\theta(a_m z_i^{-1}; p)}{z_i^{-2} \theta(z_i^{-2}; p)} \prod_{1 \leq j \leq n; j \neq i} \frac{\theta(t_z^{-1} z_j^{\pm 1}; p)}{\theta(z_j^{-1} z_i^{\pm 1}; p)},
\]

one can verify that \(\varphi_{r,i}(z) \tilde{\Psi}(z)\) is holomorphic in a neighborhood of (3.3). In fact, in the product \(F_{i}^{-}(z) \tilde{\Psi}(z)\), all possible poles of each of the two functions \(F_{i}^{-}(z)\), \(\tilde{\Psi}(z)\),

\[
p^{\mu} z_i^{-2} = 1, \quad p^{\mu} z_i z_j^{\pm 1}, \quad p^{\mu} a_m z_i^{-1}, \quad p^{\mu} t_z^{-1} z_j^{\pm 1} = 1
\]

\((1 \leq j \leq n; j \neq i; m = 1, \ldots, 6; \mu = 0, 1, 2, \ldots)\),

relevant to this region are eliminated by zeros of the other. Hence we have

\[
(\nabla_{q,z} \varphi_{r,i}(z)) = 0 \quad (i = 1, \ldots, n).
\]

In the same way as we discussed in [10] Theorem 4.1, this formula implies the recurrence relation (1.5), and hence

\[
\langle E_n(a_1, a_6; z) \rangle = \langle E_0(a_1, a_6; z) \rangle \prod_{i=1}^n \left( \frac{a_i^3 \theta(a_6 a_1^{-1} t_i^{-1}; p)}{a_6^3 \theta(a_1 a_6^{-1} t_i^{-1}; p)} \right) \prod_{m=2}^5 \frac{\theta(a_m a_6 t_i^{-1}; p)}{\theta(a_m a_1 t_i^{-1}; p)}
\]

under the balancing condition \(a_1 \cdots a_6 t^{2n-2} = 1\). Combining this with (3.4) we obtain

\[
K(a_1, \ldots, a_5, qa_6) = K(qa_1, a_2, \ldots, a_6) a_6^{-n} \prod_{i=1}^n a_6^{3} \prod_{m=2}^5 \theta(a_m a_6 t_i^{-1}; p) \prod_{m=2}^5 \theta(a_m a_1 t_i^{-1}; p)
\]

\[
= K(qa_1, a_2, \ldots, a_6) (a_1 \cdots a_6 t^{2n-2}) \prod_{i=1}^n \prod_{m=2}^5 \frac{\theta(p a_m a_6 t_i^{-1}; p)}{\theta(a_m a_1 t_i^{-1}; p)}
\]

\[
= K(qa_1, a_2, \ldots, a_6) \prod_{i=1}^n \prod_{m=2}^5 \frac{\theta(p a_m a_6 t_i^{-1}; p)}{\theta(a_m a_1 t_i^{-1}; p)}.
\]
In terms of the function \( I(a_1, \ldots, a_6) \), we conclude that

\[
(3.5) \quad I(a_1, \ldots, a_5, pqa_6) = I(qa_1, a_2, \ldots, a_5, pqa_6) \prod_{i=1}^{n} \prod_{m=2}^{5} \frac{\theta(paq_{a_6}t^{i-1}; p)}{\theta(a_{m}a_{1}t^{i-1}; p)}
\]

under the conditions \( a_1 \cdots a_6 t^{2n-2} = 1 \), \( |a_m| < 1 \) \( (m = 1, \ldots, 5) \) and \( |pa_6| < 1 \). Hence, replacing \( pa_6 \) by \( a_6 \) in (3.5) and changing the balancing condition accordingly, we have

**Lemma 3.2.** Under the conditions \( a_1 \cdots a_6 t^{2n-2} = p \) and \( |a_m| < 1 \) \( (m = 1, \ldots, 6) \), one has

\[
(3.6) \quad I(a_1, \ldots, a_5, qa_6) = I(a_1, \ldots, qa_k, \ldots, a_6) \prod_{i=1}^{n} \prod_{m=2}^{5} \frac{\theta(a_{m}a_{6}t^{i-1}; p)}{\theta(a_{m}a_{k}t^{i-1}; p)}
\]

for \( k = 1, \ldots, 5 \).

Further, replacing \( a_6 \) by \( q^{-1}a_6 \) we obtain Proposition 3.1.

We now suppose that \( a_1 \cdots a_6 t^{2n-2} = pq \) and regard \( a_6 = pq/a_1 \cdots a_5 t^{2n-2} \) as a function of \( (a_1, \ldots, a_5) \). Then the integral \( I(a_1, \ldots, a_6) \), regarded as a function of \( (a_1, \ldots, a_5) \), is defined on the open subset

\[
U_0 = \{ (a_1, \ldots, a_5) \in (\mathbb{C}^*)^5 \mid |a_1| < 1, \ldots, |a_5| < 1, |a_1 \cdots a_5| > \frac{|p||q|}{|t|^{2n-2}} \}
\]

of \( (\mathbb{C}^*)^5 \). We need to assume \( |p||q| < |t|^{2n-2} \) in order to ensure that \( U_0 \) is not empty. We denote by

\[
V_0 = \{ (a_1, \ldots, a_5) \in (\mathbb{C}^*)^5 \mid |a_1| < 1, \ldots, |a_5| < 1, |a_1 \cdots a_5| > \frac{|p|}{|t|^{2n-2}} \}
\]

the nonempty open subset of \( U_0 \) where \( I(a_1, \ldots, a_6) \) satisfies the \( q \)-difference equations (3.1), assuming that \( |p| < |t|^{2n-2} \).

4. **Analytic continuation**

The integral \( I(a_1, \ldots, a_6) \), regarded as a holomorphic function in \( (a_1, \ldots, a_5) \in U_0 \), can be continued to a meromorphic function on \( (\mathbb{C}^*)^5 \). We prove this fact by means of the \( q \)-difference equations (3.1).

In view of Proposition 3.1 we consider the meromorphic function

\[
(4.1) \quad J(a_1, \ldots, a_6) = \prod_{i=1}^{n} \prod_{1 \leq j < k \leq 6} \Gamma(a_ja_k t^{i-1}; p, q).
\]

Then it turns out that \( J(a_1, \ldots, a_6) \) satisfies the same \( q \)-difference equations as (3.1). In fact, from (2.1) one has

\[
J(a_1, \ldots, a_6) = J(a_1, \ldots, qa_k, \ldots, q^{-1}a_6) \prod_{i=1}^{n} \prod_{1 \leq m \leq 5} \frac{\theta(q^{-1}a_m a_6; p)}{\theta(a_m a_k; p)}
\]

for \( k = 1, \ldots, 5 \). In the following we regard \( J(a_1, \ldots, a_6) \) as a meromorphic function in \( (a_1, \ldots, a_5) \) through \( a_6 = pq/a_1 \cdots a_5 t^{2n-2} \) as before.
Noting that the integral \( I(a_1, \ldots, a_6) \) is a holomorphic function on \( U_0 \), we consider the meromorphic function

\[
 f(a_1, \ldots, a_6) = \frac{I(a_1, \ldots, a_6)}{J(a_1, \ldots, a_6)} = \frac{I(a_1, \ldots, a_6)}{\prod_{i=1}^{n} \prod_{1 \leq j < k \leq 6} \Gamma(a_j a_k t^{i-1}; p, q)}
\]

on \( U_0 \). This ratio \( f(a_1, \ldots, a_6) \) has poles possibly along the divisors

\[
 t^{i-1} a_j a_k = p^m q^{s-r} (i = 1, \ldots, n; 1 \leq j < k \leq 6; \mu, \nu = 0, 1, 2, \ldots)
\]
in \( U_0 \). Also, \( f(a_1, \ldots, a_6) \) is \( q \)-periodic with respect to \((a_1, \ldots, a_5) \in V_0\) in the sense that

\[
 f(a_1, \ldots, a_6) = f(a_1, \ldots, q a_k, \ldots, q^{-1} a_6)
\]

for \( k = 1, \ldots, 5 \).

**Lemma 4.1.** Suppose that \(|p| < |q|^{\frac{25}{2}} |t|^{2n-2}\). Then there exists an open subset \( W_0 \subset (\mathbb{C}^*)^5 \) of the form

\[
 W_0 = \left\{ (a_1, \ldots, a_5) \in (\mathbb{C}^*)^5 \mid sr < |a_m| < r \ (1 \leq m \leq 5) \right\}
\]

\[
 (0 < r \leq 1; 0 < s < |q|)
\]
such that \( W_0 \subset V_0 \) and \( f(a_1, \ldots, a_6) \) is holomorphic on \( W_0 \).

**Proof.** Under the assumption \(|p| < |q|^{\frac{25}{2}} |t|^{2n-2}\), one can choose positive numbers \( r, s \) such that

\[
 0 < r < |q|^{\frac{1}{2}}, \quad r^4 |t|^{-n-1} \leq s < |q|, \quad |p| \leq s^5 r^5 |t|^{2n-2}.
\]

Suppose that \((a_1, \ldots, a_5) \in W_0\). Then \(|a_1 \cdots a_5| > s^5 r^5 \geq |p|/|t|^{2n-2}\) and hence \(|a_6| < |q|\). This means that \( W_0 \subset V_0 \). Note also that \(|a_6| = |pq/a_1 \cdots a_5 t^{2n-2}| > |p||q|/r^5 |t|^{2n-2}\), and hence

\[
 |p||q|/r^5 |t|^{2n-2} < |a_6| < |q|.
\]

To show that \( f(a_1, \ldots, a_6) \) is holomorphic in \( W_0 \) we verify that

\[
 |t^{n-1} a_j a_k| > |p||q| \quad (1 \leq j < k \leq 6).
\]

In fact we have for \( j = 1, \ldots, 5 \),

\[
 |t^{n-1} a_j a_6| > |t|^{n-1} sr |p||q|/r^5 |t|^{2n-2} = |p||q|s/r^4 |t|^{n-1} \geq |p||q|,
\]

and for \( 1 \leq j < k \leq 5 \),

\[
 |t^{n-1} a_j a_k| > s^2 r^2 |t|^{n-1} > s^5 r^5 |t|^{2n-2} \geq |p| > |p||q|.
\]

**Remark 4.2.** If \(|p| \leq |q|^{10} |t|^{2n-2}\), one can simply take \( r = |q|^{\frac{1}{2}} \) and \( s = |q|^{\frac{1}{2}} \) for \( f(a_1, \ldots, a_6) \) to be holomorphic on \( W_0 \subset V_0 \).

**Theorem 4.3.** Suppose that \(|p| < |q|^{\frac{25}{2}} |t|^{2n-2}\). Under the condition \( a_1 \cdots a_6 t^{2n-2} = pq \), the integral \( I(a_1, \ldots, a_6) \), regarded as a holomorphic function in \((a_1, \ldots, a_5) \in U_0\), is continued to a meromorphic function on \((\mathbb{C}^*)^5\). Furthermore, it is expressed as

\[
 I(a_1, \ldots, a_6) = c_n \prod_{i=1}^{n} \prod_{1 \leq j < k \leq 6} \Gamma(a_j a_k t^{i-1}; p, q)
\]

for some constant \( c_n \in \mathbb{C} \) independent of \( a_1, \ldots, a_6 \).
Proof. By Lemma 4.4 there exists an open subset $W_0 \subset (\mathbb{C}^*)^5$ of the form (4.2) where $f(a_1, \ldots, a_6)$ is holomorphic and satisfies the $q$-difference equations

$$f(a_1, \ldots, a_6) = f(a_1, \ldots, q a_k, \ldots, q^{-1} a_6) \quad (k = 1, \ldots, 5)$$

for $(a_1, \ldots, a_5) \in W_0$. Note that $W_0$ is the product of five copies of an annulus in which the ratio of the two radii is given by $s < |q|$. Hence, by the $q$-difference equations (4.3), the holomorphic function $f(a_1, \ldots, a_6)$ on $W_0$ is continued to a holomorphic function on the whole $(\mathbb{C}^*)^5$. It must be a constant, however, since the continued function $f(a_1, \ldots, a_6)$ is $q$-periodic with respect to the variables $a_1, \ldots, a_5$. If we denote this constant by $c_n$, we have $I(a_1, \ldots, a_6) = c_n J(a_1, \ldots, a_6)$ as a meromorphic function on $(\mathbb{C}^*)^5$. \hfill \Box

We compute the constant $c_n$ in the next section by induction on the dimension $n$. Once this constant has been determined, we see that the statement above is valid for $|p| < 1$ without any particular restriction.

5. Computation of the constant $c_n$

In order to make the dimension explicit, we use below the notation $\Psi_n(z)$, $I_n(a_1, \ldots, a_6)$, $J_n(a_1, \ldots, a_6)$ for $\Psi(z)$, $I(a_1, \ldots, a_6)$, and $J(a_1, \ldots, a_6)$ of the previous sections. As before, we assume that the parameters satisfy the balancing condition $a_1 \cdots a_6 t^{2n-2} = pq$ and regard $a_6 = qp/a_1 \cdots a_5 t^{2n-2}$ as a function of $(a_1, \ldots, a_5)$. By Theorem 4.3 we already know that two meromorphic functions $I_n(a_1, \ldots, a_6)$ and $J_n(a_1, \ldots, a_6)$ are related by the formula

$$I_n(a_1, \ldots, a_6) = c_n J_n(a_1, \ldots, a_6),$$

provided that $|p|$ is sufficiently small. To determine the constant $c_n$, we investigate the behavior of these two functions along the divisor $a_1 a_2 = 1$.

We first consider the limit of $J_n(a_1, \ldots, a_6)$ as $a_2 \to a_1^{-1}$. Noting that

$$\lim_{a_2 \to a_1^{-1}} (1 - a_1 a_2) \Gamma(a_1 a_2; p, q) = \frac{1}{(p; p)_{\infty} (q; q)_{\infty}},$$

from (4.1) we have

$$\lim_{a_2 \to a_1^{-1}} (1 - a_1 a_2) J_n(a_1, \ldots, a_6)$$

$$= \prod_{i=1}^{n-1} \frac{\Gamma(i; p, q)}{(p; p)_{\infty} (q; q)_{\infty}} \prod_{i \leq 1}^{6} \prod_{m=3}^{n} \Gamma \left( a_1^{\pm 1} a_k t_i; p, q \right) \prod_{3 \leq j < k \leq 6} \Gamma \left( a_j a_k t_i; p, q \right),$$

where $a_6$ in the right-hand side should be understood as $a_6 = pq/a_3 a_4 a_5 t^{2n-2}$. Since $a_3 a_4 a_5 a_6 t^{2n-2} = pq$ in the limit, for any permutation $(i, j, k, l)$ of $(3, 4, 5, 6)$ we have $(a_i a_j t_l; p, q) \Gamma(a_k a_l t_i; p, q) = 1$. This implies

Lemma 5.1. In the limit as $a_2 \to a_1^{-1}$, we have

$$\lim_{a_2 \to a_1^{-1}} (1 - a_1 a_2) J_n(a_1, \ldots, a_6)$$

$$= \prod_{i=1}^{n-1} \frac{\Gamma(i; p, q)}{(p; p)_{\infty} (q; q)_{\infty}} \prod_{i \leq 1}^{6} \prod_{m=3}^{n} \Gamma \left( a_1^{\pm 1} a_k t_i; p, q \right) \prod_{3 \leq j < k \leq 6} \Gamma \left( a_j a_k t_i; p, q \right). \hfill \Box$$
In this case we can choose the cycle and integral

\[ I_n(a_1, \ldots, a_6) = \int_{C^n} \Psi_n(z) \varpi_n(z) \]

over a certain -cycle \( C^n \), provided that \(| t | < r^2 \). Setting \( r = |q|^{1/2} \), we assume further that

\[ 1 < |a_1| < |q|^{-1/2}; \quad |a_m| < 1 \quad (m = 2, \ldots, 6). \]

In this case we can choose the cycle \( C \) as

\[ C = C_0 + C_1, \quad C_1 = C_1^+ - C_1^-; \quad C_0 = C_1(0), \quad C_1^+ = C_1(a_1), \quad C_1^- = C_1(a_1^{-1}). \]

Then we analyze the effect of pinching about the cycles \( C_1^+ \), \( C_1^- \) as \( a_2 \to a_1^{-1} \).

We consider the integral

\[ \frac{1}{2\pi \sqrt{-1}} \int_C \Psi_n(z_1, z_2, \ldots, z_n) \frac{dz_1}{z_1} \]

with respect to \( z_1 \). Since

\[ \Psi_n(z_1, z_2, \ldots, z_n) = \Psi_{n-1}(z_2, \ldots, z_n) \prod_{m=1}^6 \frac{\Gamma(a_m z_1 \pm 1; p, q)}{\Gamma(z_1^\mp 1; p, q)} \prod_{k=2}^n \frac{\Gamma(t z_1 \pm 1; z_k; p, q)}{\Gamma(z_1^\mp 1; z_k; p, q)}, \]

the poles \( z_1 = a_1, a_1^{-1} \) of the integrand arise only in the factor

\[ \Gamma(a_1 z_1^{\mp 1}; p, q) = \frac{(pqa_1^{-1} z_1; p, q)_{\infty} (pqa_1^{-1} z_1^{-1}; p, q)_{\infty}}{(a_1 z_1; p, q)_{\infty} (a_1 z_1^{-1}; p, q)_{\infty}}. \]

Note that

\[ \text{Res} \left( \Gamma(a_1 z_1^{\mp 1}; p, q) \frac{dz_1}{z_1}; z_1 = a_1 \right) = \frac{(pqa_1^{-2}; p, q)_{\infty}}{(a_1^2; p, q)_{\infty} (p; p)_{\infty} (q; q)_{\infty}} = \frac{\Gamma(a_1^2; p, q)}{(p; p)_{\infty} (q; q)_{\infty}} \]

and

\[ \text{Res} \left( \Gamma(a_1 z_1^{\mp 1}; p, q) \frac{dz_1}{z_1}; z_1 = a_1^{-1} \right) = -\frac{\Gamma(a_1^{-2}; p, q)}{(p; p)_{\infty} (q; q)_{\infty}}. \]

Hence we have

\[ \frac{1}{2\pi \sqrt{-1}} \int_{C_1^+} \Psi_n(z_1, \ldots, z_n) \frac{dz_1}{z_1} = \pm \frac{\Gamma(a_1^2; p, q)}{(p; p)_{\infty} (q; q)_{\infty}} \prod_{m=2}^6 \frac{\Gamma(a_m a_1^{-1}; p, q)}{\Gamma(a_1^{\mp 1}; p, q)} \prod_{k=2}^n \frac{\Gamma(t a_1^2 z_k^{\mp 1}; p, q)}{\Gamma(a_1^{\mp 1} z_k^{\mp 1}; p, q)} \Psi_{n-1}(z_2, \ldots, z_n) \]

\[ = \pm \frac{1}{(p; p)_{\infty} (q; q)_{\infty}} \prod_{m=2}^6 \frac{\Gamma(a_m a_1^{-1}; p, q)}{\Gamma(a_1^{-2}; p, q)} \prod_{k=2}^n \frac{\Gamma(t a_1^{-2} z_k^{\pm 1}; p, q)}{\Gamma(a_1^{-2} z_k^{\pm 1}; p, q)} \Psi_{n-1}(z_2, \ldots, z_n), \]
where
\[
\hat{\Psi}_{n-1}(z_2, \ldots, z_n) = \prod_{k=2}^{n} \frac{\Gamma(ta_1^{\pm 1}z_k^{\pm 1}; p, q)}{\Gamma(a_1^{\pm 1}z_k^{\pm 1}; p, q)} \Psi_{n-1}(z_2, \ldots, z_n)
\]
and hence
\[
(5.4)
\]
This implies that
\[
\frac{1}{2\pi \sqrt{-1}} \int_{C_1} \Psi_n(z) \frac{dz_1}{z_1} = \frac{1}{2\pi \sqrt{-1}} \int_{C_1^+} \Psi_n(z) \frac{dz_1}{z_1} - \frac{1}{2\pi \sqrt{-1}} \int_{C_1^-} \Psi_n(z) \frac{dz_1}{z_1}
\]
(5.3)
and hence
\[
\frac{1}{2\pi \sqrt{-1}} \int_{C} \Psi_n(z) \frac{dz_1}{z_1}
\]
\[= \frac{1}{2\pi \sqrt{-1}} \int_{C_0} \Psi_n(z) \frac{dz_1}{z_1} + 2 \prod_{m=2}^{n} \Gamma(a_m a_1^{\pm 1}; p, q) \hat{\Psi}_{n-1}(z_2, \ldots, z_n),
\]
We remark that the first term is regular at \(a_1 a_2 = 1\) and has a finite limit as \(a_2 \to a_1^{-1}\), while the second term diverges in the order \((1 - a_1 a_2)^{-1}\) because of the factor \(\Gamma(a_2 a_1; p, q)\). Since
\[
\frac{2\Gamma(2a_1^{\pm 1}; p, q)}{(p; p)_\infty(q; q)_\infty} \frac{\Gamma(6m = 3 \Gamma(a_m a_1^{\pm 1}; p, q))}{(p; p)_\infty(q; q)_\infty} \frac{1}{1 - a_1 a_2 (pa_2 a_1; p)_\infty(qa_2 a_1; q)_\infty(pqa_2 a_1; p, q)_\infty} \times \frac{2\Gamma(2a_1^{-1}; p, q) \prod_{m=3}^{n} \Gamma(a_m a_1^{\pm 1}; p, q)}{(p; p)_\infty(q; q)_\infty \Gamma(a_1^{-2}; p, q)},
\]
we have
\[
\lim_{a_2 \to a_1^{-1}} (1 - a_1 a_2) \frac{2\Gamma(2a_1^{\pm 1}; p, q)}{(p; p)_\infty(q; q)_\infty} \frac{\prod_{m=3}^{n} \Gamma(a_m a_1^{\pm 1}; p, q)}{\Gamma(a_1^{-2}; p, q)} = \frac{2 \prod_{m=3}^{n} \Gamma(a_m a_1^{\pm 1}; p, q)}{(p; p)_\infty(q; q)_\infty} \Psi_{n-1}(z_2, \ldots, z_n),
\]
where \(a_6 = pq/(a_3 a_4 a_5 l^{2n-2})\) in the right-hand side. On the other hand,
\[
\lim_{a_2 \to a_1^{-1}} \hat{\Psi}_{n-1}(z_2, \ldots, z_n) = \Psi_{n-1}(ta_1, ta_1^{-1}, a_3, \ldots, a_6; z_2, \ldots, z_n).
\]
To summarize, we obtain
\[
(5.4)
\]
\[
\frac{1}{2\pi \sqrt{-1}} \int_{C} \Psi_n(z_1, \ldots, z_n) \frac{dz_1}{z_1}
\]
\[= \frac{2 \prod_{m=3}^{n} \Gamma(a_m a_1^{\pm 1}; p, q)}{(p; p)_\infty(q; q)_\infty} \Psi_{n-1}(ta_1, ta_1^{-1}, a_3, \ldots, a_6; z_2, \ldots, z_n).
\]
We decompose the multiple integral $I_n(a_1, \ldots, a_n)$ of (5.2) as
\[
\int_{C^n} \Psi_n(z) \omega_n(z) = \int_{C_1 \times C^{n-1}} \Psi_n(z) \omega_n(z) + \int_{C_0 \times C^{n-1}} \Psi_n(z) \omega_n(z) = \int_{C_1 \times C^{n-1}} \Psi_n(z) \omega_n(z) + \int_{C_0 \times C^2} \Psi_n(z) \omega_n(z)
\]
\[
= \int_{C_1 \times C^{n-1}} \Psi_n(z) \omega_n(z) + \int_{C_0 \times C^2} \Psi_n(z) \omega_n(z) = \ldots
\]
\[
= \sum_{i=1}^{n} \int_{C_1^{n-1} \times C_1 \times C^{n-i}} \Psi_n(z) \omega_n(z) + \int_{C_0} \Psi_n(z) \omega_n(z).
\]

Regarding the integral
\[
\int_{C_1^{n-1} \times C_1 \times C^{n-i}} \Psi_n(z) \omega_n(z) = \int_{C_1^{n-1} \times C^{n-i}} \left( \frac{1}{2\pi \sqrt{-1}} \int_{C_1} \Psi_n(z) \frac{dz_i}{z_i} \right) \omega_{n-1}(z_1),
\]
\[
\omega_{n-1}(z_1) = \frac{1}{(2\pi \sqrt{-1})^{n-1}} \prod_{1 \leq j \leq n, j \neq i} \frac{dz_j}{z_j},
\]

where $z_1 = (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n)$, by (5.3) we have
\[
\frac{1}{2\pi \sqrt{-1}} \int_{C_1} \Psi_n(z) \frac{dz_i}{z_i} = \frac{2 \Gamma(a_2 a_1^{+1}; p, q) \prod_{m=3}^{6} \Gamma(a_m a_1^{+1}; p, q)}{(p; p)_\infty (q; q)_\infty (a_1 a_2^{2}; p, q)} \hat{\Psi}_{n-1}(z_1).
\]

Since $\hat{\Psi}_{n-1}(z_1) = \hat{\Psi}_{n-1}(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n)$ is regular at $z_j = a_1, a_1^{-1}$ ($j \neq i$), one can replace the $(n-1)$-cycle $C_1^{n-1} \times C^{n-i}$ by $C_0^{n-1}$ as
\[
\int_{C_1^{n-1} \times C_1 \times C^{n-i}} \Psi_n(z) \omega_n(z) = \frac{2 \Gamma(a_2 a_1^{+1}; p, q) \prod_{m=3}^{6} \Gamma(a_m a_1^{+1}; p, q)}{(p; p)_\infty (q; q)_\infty (a_1 a_2^{2}; p, q)} \int_{C_0^{n-1}} \hat{\Psi}_{n-1}(z_1) \omega_{n-1}(z_1).
\]

Hence (5.5) implies that
\[
\int_{C^n} \Psi_n(z) \omega_n(z) = \frac{2n \Gamma(a_2 a_1^{+1}; p, q) \prod_{m=3}^{6} \Gamma(a_m a_1^{+1}; p, q)}{(p; p)_\infty (q; q)_\infty (a_2 a_1^{2}; p, q)} \int_{C_0^{n-1}} \hat{\Psi}_{n-1}(z_1) \omega_{n-1}(z_1)
\]
\[
\quad + \int_{C_0} \Psi_n(z) \omega_n(z).
\]
Note that the second term of the right-hand side has a finite limit as $a_2 \to a_1^{-1}$. Multiplying (5.6) by $1 - a_1 a_2$, we can compute the limit $a_2 \to a_1^{-1}$ by (5.4) as

$$\lim_{a_2 \to a_1^{-1}} (1 - a_1 a_2) I_n(a_1, \ldots, a_6) = \lim_{a_2 \to a_1^{-1}} (1 - a_1 a_2) \int_{C^n} \Psi_n(z) \varpi_n(z)$$

This means that

$$c_n \lim_{a_2 \to a_1^{-1}} (1 - a_1 a_2) J_n(a_1, \ldots, a_6) = \frac{2n}{(p;p)_{\infty}^2(q; q)_{\infty}^2} \prod_{i=2}^{n} \prod_{k=2}^{6} \Gamma(a_i^{\pm 1} a_k t^{i-1}; p, q) \prod_{i=1}^{n-1} \prod_{3 \leq j < k \leq 6} \Gamma(a_i a_k t^{i-1}; p, q)$$

by definition (4.1). Comparing these two expressions we obtain the recurrence formula

$$c_n = c_{n-1} \frac{2n}{(p;p)_{\infty}^2(q; q)_{\infty}^2} \prod_{i=2}^{n} \prod_{k=2}^{6} \Gamma(t^{i}; p, q) \prod_{i=1}^{n-1} \prod_{3 \leq j < k \leq 6} \Gamma(a_i a_k t^{i-1}; p, q)$$

for the constants $c_n$. Starting from $c_0 = 1$, we have

$$c_n = \frac{2^n n!}{(p;p)_{\infty}^n(q; q)_{\infty}^n} \prod_{i=1}^{n} \frac{\Gamma(t^{i}; p, q)}{\Gamma(t; p, q)}$$

for $n = 0, 1, 2, \ldots$. This completes the evaluation of the $BC_n$ elliptic Selberg integral

$$I_n(a_1, \ldots, a_6) = c_n J_n(a_1, \ldots, a_6) = \frac{2^n n!}{(p;p)_{\infty}^n(q; q)_{\infty}^n} \prod_{i=1}^{n} \frac{\Gamma(t^{i}; p, q)}{\Gamma(t; p, q)} \prod_{1 \leq j < k \leq 6} \Gamma(a_j a_k t^{i-1}; p, q).$$

REFERENCES


EVALUATION OF THE BCₙ ELLIPTIC SELBERG INTEGRAL


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