PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 145, Number 2, February 2017, Pages 689–703
http://dx.doi.org/10.1090/proc/13234
Article electronically published on August 5, 2016

EVALUATION OF THE BC_n ELLIPTIC SELBERG INTEGRAL VIA THE FUNDAMENTAL INVARIANTS

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(Communicated by Mourad Ismail)

Abstract. We give a proof of the evaluation formula for the elliptic Selberg integral of type BC_n as an application of the fundamental BC_n-invariants.

1. Introduction

The evaluation formula of the BC_n elliptic Selberg integral was proposed for the first time by van Diejen and Spiridonov [17]. Namely, under the balancing condition $a_1 \cdots a_6 t^{2n-2} = pq$,

$$\begin{aligned}\frac{1}{(2\pi\sqrt{-1})^n}\int_{\mathbb{T}^n}\prod_{i=1}^{n}\frac{\prod_{m=1}^{6}\Gamma(a_m z_i^{\pm 1};p,q)}{\Gamma(z_i^{\pm 2};p,q)}\prod_{1\le j<k\le n}\frac{\Gamma(tz_j^{\pm 1}z_k^{\pm 1};p,q)}{\Gamma(z_j^{\pm 1}z_k^{\pm 1};p,q)}\frac{dz_1\cdots dz_n}{z_1\cdots z_n}\\ = \frac{2^n n!}{(p;p)_\infty^n (q;q)_\infty^n}\prod_{i=1}^{n}\left(\frac{\Gamma(t^i;p,q)}{\Gamma(t;p,q)}\prod_{1\le j<k\le 6}\Gamma(t^{i-1}a_j a_k;p,q)\right),\end{aligned} \tag{1.1}$$

where $a_1,\dots,a_6$, t are complex parameters with $|a_m|<1$ $(m=1,\dots,6)$, $|t|<1$, and $\mathbb{T}^n$ stands for the n-dimensional torus. (Here $\Gamma(z;p,q)$ denotes the Ruijsenaars elliptic gamma function, and the double-signs indicate a product of all possible factors.) In the paper [17], the authors outlined a way of proving (1.1) following Anderson's method [2], which is known as a typical derivation for the evaluation formula of the Selberg integral [15] via the other multi-dimensional integral [4] called the Dixon–Anderson integral in [5, 8]. The proof outlined in [17] was eventually completed by Rains [14], proving the elliptic counterpart of the evaluation of the Dixon–Anderson integral

$$\begin{aligned}\frac{1}{(2\pi\sqrt{-1})^n}\int_{\mathbb{T}^n}\prod_{i=1}^{n}\frac{\prod_{m=1}^{2n+4}\Gamma(a_m z_i^{\pm 1};p,q)}{\Gamma(z_i^{\pm 2};p,q)}\prod_{1\le j<k\le n}\frac{1}{\Gamma(z_j^{\pm 1}z_k^{\pm 1};p,q)}\frac{dz_1\cdots dz_n}{z_1\cdots z_n}\\ = \frac{2^n n!}{(p;p)_\infty^n (q;q)_\infty^n}\prod_{1\le j<k\le 2n+4}\Gamma(a_j a_k;p,q),\end{aligned}$$

whose alternative proof was given by Spiridonov [16].

Besides Anderson's method, several derivations are known for the evaluation formula of the Selberg integral. Aomoto [1] gave an alternative proof by characterizing the integral as a solution of a difference equation with some specific boundary

Received by the editors June 24, 2015 and, in revised form, April 6, 2016.
2010 *Mathematics Subject Classification.* Primary 33D70; Secondary 39A13.
Key words and phrases. BC_n elliptic Selberg integral, fundamental BC_n-invariants.
This work was supported by JSPS Kakenhi Grants (C)25400118 and (B)15H03626.

condition (see also [9] for the q-integral case). The aim of this paper is to give an alternative proof for the BC_n elliptic Selberg integral (1.1), following Aomoto's method as outlined below. Denoting by $I(a_1,\dots,a_6)$ the left-hand side of (1.1), we first prove that under the balancing condition $a_1\cdots a_6t^{2n-2}=pq$, this integral satisfies the system of q-difference equations:

$$I(a_1,\dots,a_5,a_6)=I(a_1,\dots,qa_k,\dots,a_5,q^{-1}a_6)\prod_{i=1}^{n}\prod_{\substack{1\le m\le 5\\ m\ne k}}\frac{\theta(q^{-1}a_ma_6t^{i-1};p)}{\theta(a_ma_kt^{i-1};p)} \tag{1.2}$$

for $k=1,\dots,5$. Setting

$$\widetilde{\Psi}(z)=\prod_{i=1}^{n}\frac{\Gamma(pa_6z_i^{\pm1};p,q)\prod_{m=1}^{5}\Gamma(a_mz_i^{\pm1};p,q)}{\Gamma(z_i^{\pm2};p,q)}\prod_{1\le j<k\le n}\frac{\Gamma(tz_j^{\pm1}z_k^{\pm1};p,q)}{\Gamma(z_j^{\pm1}z_k^{\pm1};p,q)},$$

we use the notation

$$\langle\varphi(z)\rangle=\int_{\mathbb{T}^n}\varphi(z)\widetilde{\Psi}(z)\frac{dz_1\cdots dz_n}{z_1\cdots z_n}$$

for any meromorphic function $\varphi(z)$ on $(\mathbb{C}^*)^n$. Then the difference equation (1.2) of the case $k=1$ is equivalent to the equality

$$\langle E_n(a_1,a_6;z)\rangle=\langle E_0(a_1,a_6;z)\rangle\prod_{i=1}^{n}\left(\frac{a_1^3\theta(a_6a_1^{-1}t^{i-1};p)}{a_6^3\theta(a_1a_6^{-1}t^{i-1};p)}\prod_{m=2}^{5}\frac{\theta(a_ma_6t^{i-1};p)}{\theta(a_ma_1t^{i-1};p)}\right) \tag{1.3}$$

under the balancing condition $a_1\cdots a_6t^{2n-2}=1$, where

$$E_0(a,b;z)=\prod_{i=1}^{n}\frac{\theta(az_i^{\pm1};p)}{\theta(a(bt^{i-1})^{\pm1};p)},\quad E_n(a,b;z)=\prod_{i=1}^{n}\frac{\theta(bz_i^{\pm1};p)}{\theta(b(at^{i-1})^{\pm1};p)}.$$

The idea of Aomoto's method is to introduce appropriate intermediate functions which interpolate equation (1.3). We now define a set of holomorphic symmetric functions by

$$E_r(a,b;z)=\sum_{\substack{1\le i_1<\cdots<i_r\le n\\ 1\le j_1<\cdots<j_{n-r}\le n}}\prod_{k=1}^{r}\frac{\theta(bt^{i_k-k}z_{i_k}^{\pm1};p)}{\theta(bt^{i_k-k}(at^{k-1})^{\pm1};p)}\prod_{l=1}^{n-r}\frac{\theta(at^{j_l-l}z_{j_l}^{\pm1};p)}{\theta(at^{j_l-l}(bt^{l-1})^{\pm1};p)} \tag{1.4}$$

for $r=0,1,\dots,n$, where the summation is taken over all pairs of sequences $1\le i_1<\cdots<i_r\le n$ and $1\le j_1<\cdots<j_{n-r}\le n$ such that $\{i_1,\dots,i_r\}\cup\{j_1,\dots,j_{n-r}\}=\{1,2,\dots,n\}$. Under the condition $a_1\cdots a_6t^{2n-2}=1$, one can show that the following recurrence relations hold:

$$\langle E_r(a_1,a_6;z)\rangle=C_r\langle E_{r-1}(a_1,a_6;z)\rangle\quad(r=1,\dots,n), \tag{1.5}$$

where the coefficients C_r are given by

$$C_r=-\frac{a_1^2t^{2r-2}\,\theta(t^{n-r+1};p)\theta(a_6a_1^{-1}t^{n-r+1};p)\theta(a_1a_6^{-1}t^{2r-n};p)}{a_6^2t^{2n-2r}\,\theta(t^r;p)\theta(a_6a_1^{-1}t^{n-2r+2};p)\theta(a_1a_6^{-1}t^r;p)}\prod_{m=2}^{5}\frac{\theta(a_ma_6t^{n-r};p)}{\theta(a_ma_1t^{r-1};p)}.$$

Using (1.5) repeatedly, we immediately obtain (1.3). We call these $E_r(a,b;z)$ the *fundamental invariants* of type BC_n, which thus play an essential role in this paper. The fundamental invariants (1.4) are given as a special case of the *Lagrange interpolation functions of type BC_n* in the context of the connection problem among the independent cycles for the BC_n Jackson integral; see [11, Example 2 of Theorem 1.4]. See also [10] for details of the fundamental invariants (1.4). We remark that our fundamental invariants $E_r(a,b;z)$ are essentially the interpolation theta

functions of Coskun–Gustafson [3] and Rains [13] attached to single columns of partitions. In fact, $E_r(a,b;z)$ are compared explicitly with the functions of [3] and [13], respectively, as explained in [10, Introduction]. Also, the key equation (1.5) is essentially the same as [10, Theorem 4.1], which we proved in the context of a BC_n elliptic summation formula. It should be mentioned that van Diejen–Spiridonov [17] already pointed out that the integral (1.1) implies the BC_n elliptic summation formula via residue calculus.

Note that the integral (1.1) with $p=0$ is known as Gustafson's contour q-integral [6], which is the Nassrallah–Rahman integral in the case $n=1$ [12]. Aomoto's method using the fundamental invariants (1.4) with $p=0$ leads us to the recurrence relations for Gustafson's contour q-integral. This fact was previously discussed in [7, Corollary 5.2 and Eq. (5.3)].

In order to establish the evaluation formula (1.1), we need to investigate further the boundary condition for the difference equations (1.2); the precise arguments will be given later in Section 5.

This paper is organized as follows. After defining basic terminology in Section 2, we first discuss the system of q-difference equations (1.2) in Section 3. In Section 4 we study the analytic continuation of the integral (1.1) as a meromorphic function of the parameters $a_1,\dots,a_5$ in a specific domain. We use this argument to show that the integral (1.1) is expressed as a product of elliptic gamma functions up to a constant. Section 5 is devoted to obtaining the boundary condition for (1.2) through asymptotic analysis of the contour integral (1.1) as $a_2 \to a_1^{-1}$ (i.e. $a_1a_2 \to 1$). This condition determines the explicit value of the constant, which was indefinite at the time of Section 4. In the case of elliptic hypergeometric integrals, we often meet some strict restraints on parameters, which do not permit us to consider the asymptotic behavior like $a_i \to 0$ or ∞ as we usually do in the rational or trigonometric (q-analog) cases. Thus our treatment of the boundary condition might look totally different from that of the q-analog case. It should be noted, however, that our method to analyze such a situation as $a_1a_2 \to 1$ is also applicable to the case $p=0$ of the integral (1.1), thus providing a novel insight even for the evaluation of contour q-integrals.

2. BC_n elliptic Selberg integral

Throughout this paper we denote by $\Gamma(u;p,q)$ ($u \in \mathbb{C}^*$) the Ruijsenaars elliptic gamma function defined by

$$\Gamma(u;p,q) = \frac{(pqu^{-1};p,q)_\infty}{(u;p,q)_\infty}, \quad (u;p,q)_\infty = \prod_{\mu,\nu=0}^{\infty}(1-p^\mu q^\nu u) \quad (|p|<1, |q|<1).$$

Note that $\Gamma(u;p,q)$ satisfies

$$\Gamma(qu;p,q) = \theta(u;p)\Gamma(u;p,q), \quad \Gamma(pq/u;p,q) = \frac{1}{\Gamma(u;p,q)}. \tag{2.1}$$

We consider the meromorphic function

$$\Psi(z) = \prod_{i=1}^{n} \frac{\prod_{m=1}^{6}\Gamma(a_m z_i^{\pm 1};p,q)}{\Gamma(z_i^{\pm 2};p,q)} \prod_{1\le j<k\le n} \frac{\Gamma(t z_j^{\pm 1} z_k^{\pm 1};p,q)}{\Gamma(z_j^{\pm 1} z_k^{\pm 1};p,q)}$$

in $z=(z_1,\dots,z_n) \in (\mathbb{C}^*)^n$ with complex parameters $a_1,\dots,a_6,t \in \mathbb{C}^*$, assuming throughout that $|t|<1$. We also use the notation $\Psi(a_1,\dots,a_6;z)$ for $\Psi(z)$ when

we need to make the dependence on the parameters $a_1, \dots, a_6$ explicit. For this function $\Psi(z)$, we investigate the multiple integral

$$I = \int_\sigma \Psi(z)\varpi(z), \quad \varpi(z) = \frac{1}{(2\pi\sqrt{-1})^n}\frac{dz_1\cdots dz_n}{z_1\cdots z_n}$$

over an n-cycle σ. Since $\Psi(z)$ is expressed as

$$\begin{aligned}\Psi(z) = &\prod_{i=1}^n (1-z_i^{\pm 2})(pz_i^{\pm 2};p)_\infty(qz_i^{\pm 2};q)_\infty \\ &\times \prod_{1\le j<k\le n}(1-z_j^{\pm 1}z_k^{\pm 1})(pz_j^{\pm 1}z_k^{\pm 1};p)_\infty(qz_j^{\pm 1}z_k^{\pm 1};q)_\infty \\ &\times \prod_{i=1}^n\prod_{m=1}^6 \frac{(pqa_m^{-1}z_i^{\pm 1};p,q)_\infty}{(a_m z_i^{\pm 1};p,q)_\infty}\prod_{1\le j<k\le n}\frac{(pqt^{-1}z_j^{\pm 1}z_k^{\pm 1};p,q)_\infty}{(tz_j^{\pm 1}z_k^{\pm 1};p,q)_\infty},\end{aligned}$$

we see that $\Psi(z)$ has poles possibly along the divisors

$$\begin{aligned} z_i^{\pm 1} &= a_m p^\mu q^\nu \qquad (1\le i\le n;\ m=1,\dots,6;\ \mu,\nu=0,1,2,\dots), \\ z_j^{\pm 1}z_k^{\pm 1} &= t\,p^\mu q^\nu \qquad (1\le j<k\le n;\ \mu,\nu=0,1,2,\dots).\end{aligned}$$

Also, regarded as a function of z_i $(i=1,\dots,n)$, $\Psi(z)$ has poles possibly at

$$p^\mu q^\nu a_m, \quad p^{-\mu}q^{-\nu}a_m^{-1}, \quad p^\mu q^\nu t z_j^{\pm 1}, \quad p^{-\mu}q^{-\nu}t^{-1}z_j^{\pm 1},$$

where $1\le m\le 6$, $1\le j\le n$, $j\ne i$ and $\mu,\nu=0,1,2,\dots$. If the parameters satisfy the condition $|a_1|<1,\dots,|a_6|<1$, then $\Psi(z)$ is holomorphic in a neighborhood of the n-dimensional torus

$$\mathbb{T}^n = \bigl\{z=(z_1,\dots,z_n)\in(\mathbb{C}^*)^n \bigm| |z_i|=1 \quad (i=1,\dots,n)\bigr\},$$

and hence the integral

$$I(a_1,\dots,a_6) = \int_{\mathbb{T}^n}\Psi(a_1,\dots,a_6;z)\varpi(z)$$

defines a holomorphic function on the domain

$$U = \bigl\{(a_1,\dots,a_6)\in(\mathbb{C}^*)^6 \bigm| |a_m|<1 \quad (m=1,\dots,6)\bigr\}\subset(\mathbb{C}^*)^6. \tag{2.2}$$

This function can be continued to a holomorphic function on a larger domain by replacing $\mathbb{T}^n$ with an appropriate n-cycle depending on the parameters $(a_1,\dots,a_6)$. We give below a remark on analytic continuation of this sort.

For each $(a_1,\dots,a_6)\in(\mathbb{C}^*)^6$, we define two subsets S_0, S_∞ of $\mathbb{C}^*$ by

$$\begin{aligned} S_0 &= \bigl\{\ p^\mu q^\nu a_m \bigm| 1\le m\le 6;\ \mu,\nu\in\mathbb{N}\ \bigr\}, \\ S_\infty &= \bigl\{\ p^{-\mu}q^{-\nu}a_m^{-1} \bigm| 1\le m\le 6;\ \mu,\nu\in\mathbb{N}\ \bigr\},\end{aligned}$$

where $\mathbb{N}=\{0,1,2,\dots\}$, and suppose that $S_0\cap S_\infty=\phi$. Assuming that $|t|<r^2$ for some $r\in(0,1)$, we choose a circle

$$C_\rho(0) = \bigl\{u\in\mathbb{C}^* \bigm| |u|=\rho\ \bigr\}, \quad \rho\in[r,r^{-1}],$$

which does not intersect with $S_0\cup S_\infty$. Then we define a cycle C in $\mathbb{C}^*$ by

$$C = C_\rho(0) + \sum_{c\in S_0;|c|>\rho} C_\varepsilon(c) - \sum_{c\in S_\infty;|c|<\rho} C_\varepsilon(c),$$

where $C_\varepsilon(c)$ denotes a sufficiently small circle around c. Note that if $|a_m|<1$ $(m=1,\dots,6)$, then C is homologous to the unit circle. We now assume that

$|a_m| < r^{-1}$ $(m = 1, \ldots, 6)$. Then such a cycle C can be taken inside the annulus $A_r = \{u \in \mathbb{C}^* \mid r \le |u| \le r^{-1}\}$. Since $|t| < r^2$, the meromorphic function $\Psi(z)$ is holomorphic in a neighborhood of the n-cycle $C^n = C \times \cdots \times C$. Hence, the integral

$$I = \int_{C^n} \Psi(z)\varpi(z)$$

is well defined and does not depend on the choice of $\rho \in [r, r^{-1}]$. This implies the following lemma on analytic continuation.

Lemma 2.1. *Suppose that $|t| < r^2$ for some real number $r \in (0, 1]$. Then the holomorphic function $I(a_1, \ldots, a_6)$ on the domain U of (2.2) can be continued to a holomorphic function on*

$$\left\{(a_1, \ldots, a_6) \in (\mathbb{C}^*)^6 \;\middle|\; \begin{array}{c} |a_m| < r^{-1} \ (1 \le m \le 6), \\ a_k a_l \notin p^{-\mathbb{N}} q^{-\mathbb{N}} \ (1 \le k, l \le 6) \end{array} \right\}. \tag{2.3}$$

□

As can be seen in (1.1), under the balancing condition $a_1 \cdots a_6 t^{2n-2} = pq$, this function $I(a_1, \ldots, a_6)$ is eventually continued to a meromorphic function on a hypersurface in $(\mathbb{C}^*)^6$ with poles along the divisors

$$p^\mu q^\nu t^{i-1} a_k a_l = 1 \quad (k, l \in \{1, \ldots, 6\}; i = 1, \ldots, n; \mu, \nu = 0, 1, 2, \ldots).$$

3. q-difference equations with respect to the parameters

In this section we derive a system of q-difference equations for the integral $I(a_1, \ldots, a_6)$ on the basis of the arguments in [10]. Our goal is to establish the following proposition.

Proposition 3.1. *Suppose that $|p| < |t|^{2n-2}$. Under the balancing condition $a_1 \cdots a_6 t^{2n-2} = pq$, the integral $I(a_1, \ldots, a_6)$ satisfies the system of q-difference equations*

$$I(a_1, \ldots, a_5, a_6) = I(a_1, \ldots, qa_k, \ldots, a_5, q^{-1}a_6) \prod_{i=1}^{n} \prod_{\substack{1 \le m \le 5 \\ m \ne k}} \frac{\theta(q^{-1}a_m a_6 t^{i-1}; p)}{\theta(a_m a_k t^{i-1}; p)} \tag{3.1}$$

for $k = 1, \ldots, 5$, provided that $|a_1| < 1, \ldots, |a_5| < 1$ and $|a_6| < |q|$.

Note that the condition $|a_6| < |q|$ is equivalent to $|a_1 \cdots a_5| > |p|/|t|^{2n-2}$ under the balancing condition. We need to assume that p is sufficiently small as specified above to guarantee that (3.1) holds in a nonempty region.

In order to make use of the arguments of [10], we modify $\Psi(z)$ as

$$\begin{aligned}
\widetilde{\Psi}(z) &= \Psi(a_1, \ldots, a_5, pa_6; z) \\
&= \prod_{i=1}^{n} \frac{\Gamma(pa_6 z_i^{\pm 1}; p, q) \prod_{m=1}^{5} \Gamma(a_m z_i^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 2}; p, q)} \prod_{1 \le j < k \le n} \frac{\Gamma(t z_j^{\pm 1} z_k^{\pm 1}; p, q)}{\Gamma(z_j^{\pm 1} z_k^{\pm 1}; p, q)} \\
&= \prod_{i=1}^{n} \frac{\prod_{m=1}^{5} \Gamma(a_m z_i^{\pm 1}; p, q)}{\Gamma(q a_6^{-1} z_i^{\pm 1}; p, q) \Gamma(z_i^{\pm 2}; p, q)} \prod_{1 \le j < k \le n} \frac{\Gamma(t z_j^{\pm 1} z_k^{\pm 1}; p, q)}{\Gamma(z_j^{\pm 1} z_k^{\pm 1}; p, q)}.
\end{aligned}$$

This function $\widetilde{\Psi}(z)$ coincides with the meromorphic function $\widetilde{\Phi}(z)$ in [10] up to multiplication by a q-periodic function in all variables $z_1, \dots, z_n$. Namely one has

$$\frac{T_{q,z_i}\widetilde{\Psi}(z)}{\widetilde{\Psi}(z)} = -\frac{(q^{-1}z_i^{-1})^2\theta(q^{-2}z_i^{-2};p)}{z_i^2\,\theta(z_i^2;p)}\prod_{m=1}^{6}\frac{\theta(a_m z_i;p)}{\theta(q^{-1}a_m z_i^{-1};p)} \times \prod_{\substack{1\le k\le n\\ k\ne i}}\frac{\theta(tz_i z_k^{\pm 1};p)\theta(q^{-1}z_i^{-1}z_k^{\pm 1};p)}{\theta(q^{-1}tz_i^{-1}z_k^{\pm 1};p)\theta(z_i z_k^{\pm 1};p)}$$

for $i = 1, \dots, n$, where T_{q,z_i} stands for the q-shift operator in z_i:

$$T_{q,z_i}f(z_1, \dots, z_n) = f(z_1, \dots, qz_i, \dots, z_n).$$

As to the parameters $a_1, \dots, a_6$, one has

$$\begin{aligned} \frac{T_{q,a_m}\widetilde{\Psi}(z)}{\widetilde{\Psi}(z)} &= \prod_{i=1}^{n}\theta(a_m z_i^{\pm 1};p) \quad (1 \le m \le 5), \\ \frac{T_{q,a_6}\widetilde{\Psi}(z)}{\widetilde{\Psi}(z)} &= a_6^{-2n}\prod_{i=1}^{n}\theta(a_6 z_i^{\pm 1};p). \end{aligned} \tag{3.2}$$

In this paper we use the notation of expectation values to refer to the integral

$$\langle\varphi(z)\rangle = \int_{\mathbb{T}^n}\varphi(z)\widetilde{\Psi}(z)\varpi(z)$$

for any meromorphic function $\varphi(z)$ on $(\mathbb{C}^*)^n$ such that $\varphi(z)\widetilde{\Psi}(z)$ is holomorphic in a neighborhood of the n-dimensional torus $\mathbb{T}^n$. If we set

$$\nabla_{q,z_i}\varphi(z) = \varphi(z) - \frac{T_{q,z_i}\widetilde{\Psi}(z)}{\widetilde{\Psi}(z)}T_{q,z_i}\varphi(z) \qquad (i = 1, \dots, n)$$

as in [10], we have

$$\langle\nabla_{q,z_i}\varphi(z)\rangle = 0 \qquad (i = 1, \dots, n)$$

for any meromorphic function $\varphi(z)$ such that $\varphi(z)\widetilde{\Psi}(z)$ is holomorphic in a neighborhood of the compact set

$$|q| \le |z_i| \le 1, \quad |z_j| = 1 \quad (1 \le j \le n;\ j \ne i). \tag{3.3}$$

In fact, by the Cauchy theorem one has

$$\int_{\mathbb{T}^n}\widetilde{\Psi}(z)\varphi(z)\varpi(z) = \int_{\mathbb{T}^n}T_{q,z_i}(\widetilde{\Psi}(z)\varphi(z))\varpi(z) \quad (i = 1, \dots, n).$$

We set

$$K(a_1, \dots, a_5, a_6) = I(a_1, \dots, a_5, pa_6) = \langle 1\rangle$$

assuming that $|a_m| < 1$ $(m = 1, \dots, 5)$, $|pa_6| < 1$. Then from (3.2) we have

$$
\begin{aligned}
K(qa_1, a_2, \dots, a_6) &= \langle \prod_{i=1}^{n} \theta(a_1 z_i^{\pm 1}; p) \rangle \\
&= \langle E_0(a_1, a_6; z) \rangle \prod_{i=1}^{n} \theta(a_1 (a_6 t^{i-1})^{\pm 1}; p), \\
K(a_1, \dots, a_5, qa_6) &= a_6^{-2n} \langle \prod_{i=1}^{n} \theta(a_6 z_i^{\pm 1}; p) \rangle \\
&= \langle E_n(a_1, a_6; z) \rangle \prod_{i=1}^{n} a_6^{-2} \theta(a_6 (a_1 t^{i-1})^{\pm 1}; p),
\end{aligned}
\tag{3.4}
$$

where $E_r(a_1, a_6; z) = E_r^{(n)}(a_1, a_6; z)$ $(r = 0, 1, \dots, n)$ denote the fundamental BC_n-invariants (1.4); for the basic properties of these functions, we refer the reader to [10, Section 3]. On the other hand, as for the function

$$\varphi_{r,i}(z) = F_i^-(z) E_{r-1}^{(n-1)}(a_1, a_6; z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_n) \quad (1 \le i \le n; 1 \le r \le n)$$

of [10, Section 4], where

$$F_i^-(z) = \frac{\prod_{m=1}^{6} \theta(a_m z_i^{-1}; p)}{z_i^{-2}\, \theta(z_i^{-2}; p)} \prod_{\substack{1 \le j \le n \\ j \ne i}} \frac{\theta(t z_i^{-1} z_j^{\pm 1}; p)}{\theta(z_i^{-1} z_j^{\pm 1}; p)},$$

one can verify that $\varphi_{r,i}(z)\widetilde{\Psi}(z)$ is holomorphic in a neighborhood of (3.3). In fact, in the product $F_i^-(z)\widetilde{\Psi}(z)$, all possible poles of each of the two functions $F_i^-(z)$, $\widetilde{\Psi}(z)$,

$$
\begin{gathered}
p^\mu z_i^{-2} = 1, \quad p^\mu z_i z_j^{\pm 1}, \quad p^\mu a_m z_i^{-1}, \quad p^\mu t z_i^{-1} z_j^{\pm 1} = 1 \\
(1 \le j \le n;\ j \ne i;\ m = 1, \dots, 6;\ \mu = 0, 1, 2, \dots),
\end{gathered}
$$

relevant to this region are eliminated by zeros of the other. Hence we have

$$\langle \nabla_{q, z_i} \varphi_{r,i}(z) \rangle = 0 \qquad (i = 1, \dots, n).$$

In the same way as we discussed in [10, Theorem 4.1], this formula implies the recurrence relation (1.5), and hence

$$\langle E_n(a_1, a_6; z) \rangle = \langle E_0(a_1, a_6; z) \rangle \prod_{i=1}^{n} \left(\frac{a_1^3 \theta(a_6 a_1^{-1} t^{i-1}; p)}{a_6^3 \theta(a_1 a_6^{-1} t^{i-1}; p)} \prod_{m=2}^{5} \frac{\theta(a_m a_6 t^{i-1}; p)}{\theta(a_m a_1 t^{i-1}; p)} \right)$$

under the balancing condition $a_1 \cdots a_6 t^{2n-2} = 1$. Combining this with (3.4) we obtain

$$
\begin{aligned}
K(a_1, \dots, a_5, qa_6) &= K(qa_1, a_2, \dots, a_6) \frac{a_1^n}{a_6^{3n}} \prod_{i=1}^{n} \prod_{m=2}^{5} \frac{\theta(a_m a_6 t^{i-1}; p)}{\theta(a_m a_1 t^{i-1}; p)} \\
&= K(qa_1, a_2, \dots, a_6)(a_1 \cdots a_6 t^{2n-2})^n \prod_{i=1}^{n} \prod_{m=2}^{5} \frac{\theta(p a_m a_6 t^{i-1}; p)}{\theta(a_m a_1 t^{i-1}; p)} \\
&= K(qa_1, a_2, \dots, a_6) \prod_{i=1}^{n} \prod_{m=2}^{5} \frac{\theta(p a_m a_6 t^{i-1}; p)}{\theta(a_m a_1 t^{i-1}; p)}.
\end{aligned}
$$

In terms of the function $I(a_1,\dots,a_6)$, we conclude that

$$I(a_1,\dots,a_5,pqa_6)=I(qa_1,a_2,\dots,a_5,pa_6)\prod_{i=1}^{n}\prod_{m=2}^{5}\frac{\theta(pa_ma_6t^{i-1};p)}{\theta(a_ma_1t^{i-1};p)} \tag{3.5}$$

under the conditions $a_1\cdots a_6t^{2n-2}=1$, $|a_m|<1$ $(m=1,\dots,5)$ and $|pa_6|<1$. Hence, replacing pa_6 by a_6 in (3.5) and changing the balancing condition accordingly, we have

Lemma 3.2. *Under the conditions $a_1\cdots a_6t^{2n-2}=p$ and $|a_m|<1$ $(m=1,\dots,6)$, one has*

$$I(a_1,\dots,a_5,qa_6)=I(a_1,\dots,qa_k,\dots,a_6)\prod_{i=1}^{n}\prod_{\substack{1\le m\le 5\\ m\ne k}}\frac{\theta(a_ma_6t^{i-1};p)}{\theta(a_ma_kt^{i-1};p)} \tag{3.6}$$

for $k=1,\dots,5$. □

Further, replacing a_6 by $q^{-1}a_6$ we obtain Proposition 3.1.

We now suppose that $a_1\cdots a_6t^{2n-2}=pq$ and regard $a_6=pq/a_1\cdots a_5t^{2n-2}$ as a function of $(a_1,\dots,a_5)$. Then the integral $I(a_1,\dots,a_6)$, regarded as a function of $(a_1,\dots,a_5)$, is defined on the open subset

$$U_0=\left\{\ (a_1,\dots,a_5)\in(\mathbb{C}^*)^5\ \middle|\ |a_1|<1,\dots,|a_5|<1,|a_1\cdots a_5|>\frac{|p||q|}{|t|^{2n-2}},\ \right\}$$

of $(\mathbb{C}^*)^5$. We need to assume $|p||q|<|t|^{2n-2}$ in order to ensure that U_0 is not empty. We denote by

$$V_0=\left\{\ (a_1,\dots,a_5)\in(\mathbb{C}^*)^5\ \middle|\ |a_1|<1,\dots,|a_5|<1,|a_1\cdots a_5|>\frac{|p|}{|t|^{2n-2}},\ \right\}$$

the nonempty open subset of U_0 where $I(a_1,\dots,a_6)$ satisfies the q-difference equations (3.1), assuming that $|p|<|t|^{2n-2}$.

4. Analytic continuation

The integral $I(a_1,\dots,a_6)$, regarded as a holomorphic function in $(a_1,\dots,a_5)\in U_0$, can be continued to a meromorphic function on $(\mathbb{C}^*)^5$. We prove this fact by means of the q-difference equations (3.1).

In view of Proposition 3.1 we consider the meromorphic function

$$J(a_1,\dots,a_6)=\prod_{i=1}^{n}\prod_{1\le j<k\le 6}\Gamma(a_ja_kt^{i-1};p,q). \tag{4.1}$$

Then it turns out that $J(a_1,\dots,a_6)$ satisfies the same q-difference equations as (3.1). In fact, from (2.1) one has

$$J(a_1,\dots,a_6)=J(a_1,\dots,qa_k,\dots,q^{-1}a_6)\prod_{i=1}^{n}\prod_{\substack{1\le m\le 5\\ m\ne k}}\frac{\theta(q^{-1}a_ma_6;p)}{\theta(a_ma_k;p)}$$

for $k=1,\dots,5$. In the following we regard $J(a_1,\dots,a_6)$ as a meromorphic function in $(a_1,\dots,a_5)$ through $a_6=pq/a_1\cdots a_5t^{2n-2}$ as before.

Noting that the integral $I(a_1,\dots,a_6)$ is a holomorphic function on U_0, we consider the meromorphic function

$$f(a_1,\dots,a_6)=\frac{I(a_1,\dots,a_6)}{J(a_1,\dots,a_6)}=\frac{I(a_1,\dots,a_6)}{\prod_{i=1}^n\prod_{1\le j<k\le 6}\Gamma(a_ja_kt^{i-1};p,q)}$$
$$=I(a_1,\dots,a_6)\prod_{i=1}^n\prod_{1\le j<k\le 6}\frac{(a_ja_kt^{i-1};p,q)_\infty}{(pq/a_ja_kt^{i-1};p,q)_\infty}$$

on U_0. This ratio $f(a_1,\dots,a_6)$ has poles possibly along the divisors

$$t^{i-1}a_ja_k=p^{\mu+1}q^{\nu+1}\quad(i=1,\dots,n;\ 1\le j<k\le 6;\ \mu,\nu=0,1,2,\dots)$$

in U_0. Also, $f(a_1,\dots,a_6)$ is q-periodic with respect to $(a_1,\dots,a_5)\in V_0$ in the sense that

$$f(a_1,\dots,a_6)=f(a_1,\dots,qa_k,\dots,q^{-1}a_6)$$

for $k=1,\dots,5$.

Lemma 4.1. *Suppose that $|p|<|q|^{\frac{25}{4}}|t|^{2n-2}$. Then there exists an open subset $W_0\subset(\mathbb{C}^*)^5$ of the form*

$$W_0=\left\{\ (a_1,\dots,a_5)\in(\mathbb{C}^*)^5\ \middle|\ sr<|a_m|<r\ (1\le m\le 5)\ \right\}\qquad(0<r\le 1;\ 0<s<|q|)\tag{4.2}$$

such that $W_0\subset V_0$ and $f(a_1,\dots,a_6)$ is holomorphic on W_0.

Proof. Under the assumption $|p|<|q|^{\frac{25}{4}}|t|^{2n-2}$, one can choose positive numbers r,s such that

$$0<r<|q|^{\frac14},\quad r^4|t|^{n-1}\le s<|q|,\quad |p|\le s^5r^5|t|^{2n-2}.$$

Suppose that $(a_1,\dots,a_5)\in W_0$. Then $|a_1\cdots a_5|>s^5r^5\ge|p|/|t|^{2n-2}$ and hence $|a_6|<|q|$. This means that $W_0\subset V_0$. Note also that $|a_6|=|pq/a_1\cdots a_5t^{2n-2}|>|p||q|/r^5|t|^{2n-2}$, and hence

$$|p||q|/r^5|t|^{2n-2}<|a_6|<|q|.$$

To show that $f(a_1,\dots,a_6)$ is holomorphic in W_0 we verify that

$$|t^{n-1}a_ja_k|>|p||q|\quad(1\le j<k\le 6).$$

In fact we have for $j=1,\dots,5$,

$$|t^{n-1}a_ja_6|>|t|^{n-1}sr|p||q|/r^5|t|^{2n-2}=|p||q|s/r^4|t|^{n-1}\ge|p||q|,$$

and for $1\le j<k\le 5$,

$$|t^{n-1}a_ja_k|>s^2r^2|t|^{n-1}>s^5r^5|t|^{2n-2}\ge|p|>|p||q|.$$

□

Remark 4.2. If $|p|\le|q|^{10}|t|^{2n-2}$, one can simply take $r=|q|^{\frac12}$ and $s=|q|^{\frac32}$ for $f(a_1,\dots,a_6)$ to be holomorphic on $W_0\subset V_0$.

Theorem 4.3. *Suppose that $|p|<|q|^{\frac{25}{4}}|t|^{2n-2}$. Under the condition $a_1\cdots a_6t^{2n-2}=pq$, the integral $I(a_1,\dots,a_6)$, regarded as a holomorphic function in $(a_1,\dots,a_5)\in U_0$, is continued to a meromorphic function on $(\mathbb{C}^*)^5$. Furthermore, it is expressed as*

$$I(a_1,\dots,a_6)=c_n\prod_{i=1}^n\prod_{1\le j<k\le 6}\Gamma(a_ja_kt^{i-1};p,q)$$

for some constant $c_n\in\mathbb{C}$ independent of $a_1,\dots,a_6$.

Proof. By Lemma 4.1, there exists an open subset $W_0 \subset (\mathbb{C}^*)^5$ of the form (4.2) where $f(a_1,\dots,a_6)$ is holomorphic and satisfies the q-difference equations

$$f(a_1,\dots,a_6) = f(a_1,\dots,qa_k,\dots,q^{-1}a_6) \qquad (k=1,\dots,5) \tag{4.3}$$

for $(a_1,\dots,a_5) \in W_0$. Note that W_0 is the product of five copies of an annulus in which the ratio of the two radii is given by $s < |q|$. Hence, by the q-difference equations (4.3), the holomorphic function $f(a_1,\dots,a_6)$ on W_0 is continued to a *holomorphic* function on the whole $(\mathbb{C}^*)^5$. It must be a constant, however, since the continued function $f(a_1,\dots,a_6)$ is q-periodic with respect to the variables $a_1,\dots,a_5$. If we denote this constant by c_n, we have $I(a_1,\dots,a_6) = c_n J(a_1,\dots,a_6)$ as a meromorphic function on $(\mathbb{C}^*)^5$. □

We compute the constant c_n in the next section by induction on the dimension n. Once this constant has been determined, we see that the statement above is valid for $|p| < 1$ without any particular restriction.

5. Computation of the constant c_n

In order to make the dimension explicit, we use below the notation $\Psi_n(z)$, $I_n(a_1,\dots,a_6)$, $J_n(a_1,\dots,a_6)$ for $\Psi(z)$, $I(a_1,\dots,a_6)$, and $J(a_1,\dots,a_6)$ of the previous sections. As before, we assume that the parameters satisfy the balancing condition $a_1\cdots a_6 t^{2n-2} = pq$ and regard $a_6 = qp/a_1\cdots a_5 t^{2n-2}$ as a function of $(a_1,\dots,a_5)$. By Theorem 4.3 we already know that two meromorphic functions $I_n(a_1,\dots,a_6)$ and $J_n(a_1,\dots,a_6)$ are related by the formula

$$I_n(a_1,\dots,a_6) = c_n J_n(a_1,\dots,a_6), \tag{5.1}$$

provided that $|p|$ is sufficiently small. To determine the constant c_n, we investigate the behavior of these two functions along the divisor $a_1a_2 = 1$.

We first consider the limit of $J_n(a_1,\dots,a_6)$ as $a_2 \to a_1^{-1}$. Noting that

$$\lim_{a_2\to a_1^{-1}} (1-a_1a_2)\Gamma(a_1a_2;p,q) = \frac{1}{(p;p)_\infty (q;q)_\infty},$$

from (4.1) we have

$$\begin{aligned}&\lim_{a_2\to a_1^{-1}} (1-a_1a_2) J_n(a_1,\dots,a_6)\\ &= \frac{\prod_{i=1}^{n-1}\Gamma(t^i;p,q)}{(p;p)_\infty(q;q)_\infty}\prod_{i=1}^{n}\left(\prod_{m=3}^{6}\Gamma(a_1^{\pm1}a_kt^{i-1};p,q)\prod_{3\le j<k\le 6}\Gamma(a_ja_kt^{i-1};p,q)\right),\end{aligned}$$

where a_6 in the right-hand side should be understood as $a_6 = pq/a_3a_4a_5t^{2n-2}$. Since $a_3a_4a_5a_6t^{2n-2} = pq$ in the limit, for any permutation (i,j,k,l) of $(3,4,5,6)$ we have $(a_ia_jt^{n-1})(a_ka_lt^{n-1}) = pq$, and hence

$$\Gamma(a_ia_jt^{n-1};p,q)\Gamma(a_ka_lt^{n-1};p,q) = 1.$$

This implies

Lemma 5.1. *In the limit as $a_2 \to a_1^{-1}$, we have*

$$\begin{aligned}&\lim_{a_2\to a_1^{-1}} (1-a_1a_2) J_n(a_1,\dots,a_6)\\ &= \frac{\prod_{i=1}^{n-1}\Gamma(t^i;p,q)}{(p;p)_\infty(q;q)_\infty}\prod_{i=1}^{n}\prod_{m=3}^{6}\Gamma(a_1^{\pm1}a_kt^{i-1};p,q)\prod_{i=1}^{n-1}\prod_{3\le j<k\le 6}\Gamma(a_ja_kt^{i-1};p,q). \quad \square\end{aligned}$$

We next investigate the behavior of $I_n(a_1,\dots,a_6)$ as $a_2 \to a_1^{-1}$, assuming that $|p|$ is sufficiently small so that equality (5.1) holds. Here we suppose that $|p| < |q|^7|t|^{2n-2}$ for convenience. As we remarked in Lemma 2.1, in the region (2.3) the integral $I_n(a_1,\dots,a_6)$ is expressed as the integral

$$I_n(a_1,\dots,a_6) = \int_{C^n} \Psi_n(z)\varpi_n(z) \tag{5.2}$$

over a certain n-cycle C^n, provided that $|t| < r^2$. Setting $r = |q|^{\frac{1}{2}}$, we assume further that

$$1 < |a_1| < |q|^{-\frac{1}{2}}; \quad |a_m| < 1 \quad (m = 2,\dots,6).$$

In this case we can choose the cycle C as

$$C = C_0 + C_1, \quad C_1 = C_1^+ - C_1^-; \quad C_0 = C_1(0), \quad C_1^+ = C_\epsilon(a_1), \quad C_1^- = C_\epsilon(a_1^{-1}).$$

Then we analyze the effect of pinching about the cycles C_1^+, C_1^- as $a_2 \to a_1^{-1}$.

We consider the integral

$$\frac{1}{2\pi\sqrt{-1}}\int_C \Psi_n(z_1,z_2,\dots,z_n)\frac{dz_1}{z_1}$$

with respect to z_1. Since

$$\Psi_n(z_1,z_2,\dots,z_n) = \Psi_{n-1}(z_2,\dots,z_n)\frac{\prod_{m=1}^6 \Gamma(a_m z_1^{\pm 1};p,q)}{\Gamma(z_1^{\pm 2};p,q)}\prod_{k=2}^n \frac{\Gamma(tz_1^{\pm 1}z_k^{\pm 1};p,q)}{\Gamma(z_1^{\pm 1}z_k^{\pm 1};p,q)},$$

the poles $z_1 = a_1, a_1^{-1}$ of the integrand arise only in the factor

$$\Gamma(a_1 z_1^{\pm 1};p,q) = \frac{(pqa_1^{-1}z_1;p,q)_\infty (pqa_1^{-1}z_1^{-1};p,q)_\infty}{(a_1z_1;p,q)_\infty (a_1z_1^{-1};p,q)_\infty}.$$

Note that

$$\operatorname{Res}\Big(\Gamma(a_1z_1^{\pm 1};p,q)\frac{dz_1}{z_1}; z_1 = a_1\Big) = \frac{(pqa_1^{-2};p,q)_\infty}{(a_1^2;p,q)_\infty (p;p)_\infty (q;q)_\infty} = \frac{\Gamma(a_1^2;p,q)}{(p;p)_\infty (q;q)_\infty}$$

and

$$\operatorname{Res}\Big(\Gamma(a_1z_1^{\pm 1};p,q)\frac{dz_1}{z_1}; z_1 = a_1^{-1}\Big) = -\frac{\Gamma(a_1^2;p,q)}{(p;p)_\infty (q;q)_\infty}.$$

Hence we have

$$\begin{aligned}
&\frac{1}{2\pi\sqrt{-1}}\int_{C_1^\pm} \Psi_n(z_1,\dots,z_n)\frac{dz_1}{z_1}\\
&= \pm\frac{\Gamma(a_1^2;p,q)}{(p;p)_\infty (q;q)_\infty}\frac{\prod_{m=2}^6 \Gamma(a_m a_1^{\pm 1};p,q)}{\Gamma(a_1^{\pm 2};p,q)}\prod_{k=2}^n \frac{\Gamma(ta_1^{\pm 1}z_k^{\pm 1};p,q)}{\Gamma(a_1^{\pm 1}z_k^{\pm 1};p,q)}\Psi_{n-1}(z_2,\dots,z_n)\\
&= \pm\frac{1}{(p;p)_\infty (q;q)_\infty}\frac{\prod_{m=2}^6 \Gamma(a_m a_1^{\pm 1};p,q)}{\Gamma(a_1^{-2};p,q)}\prod_{k=2}^n \frac{\Gamma(ta_1^{\pm 1}z_k^{\pm 1};p,q)}{\Gamma(a_1^{\pm 1}z_k^{\pm 1};p,q)}\Psi_{n-1}(z_2,\dots,z_n)\\
&= \pm\frac{1}{(p;p)_\infty (q;q)_\infty}\frac{\prod_{m=2}^6 \Gamma(a_m a_1^{\pm 1};p,q)}{\Gamma(a_1^{-2};p,q)}\widehat{\Psi}_{n-1}(z_2,\dots,z_n),
\end{aligned}$$

where

$$\begin{aligned}\widehat{\Psi}_{n-1}(z_2,\dots,z_n) &= \prod_{k=2}^{n}\frac{\Gamma(ta_1^{\pm1}z_k^{\pm1};p,q)}{\Gamma(a_1^{\pm1}z_k^{\pm1};p,q)}\Psi_{n-1}(z_2,\dots,z_n)\\ &= \prod_{i=2}^{n}\frac{\Gamma(ta_1^{\pm1}z_i^{\pm1};p,q)\prod_{m=2}^{6}\Gamma(a_mz_i^{\pm1};p,q)}{\Gamma(a_1^{-1}z_i^{\pm1};p,q)\Gamma(z_i^{\pm2};p,q)}\prod_{2\le j<k\le n}\frac{\Gamma(tz_j^{\pm1}z_k^{\pm1};p,q)}{\Gamma(z_j^{\pm1}z_k^{\pm1};p,q)}.\end{aligned}$$

This implies that

$$\begin{aligned}\frac{1}{2\pi\sqrt{-1}}\int_{C_1}\Psi_n(z)\frac{dz_1}{z_1} &= \frac{1}{2\pi\sqrt{-1}}\int_{C_1^+}\Psi_n(z)\frac{dz_1}{z_1}-\frac{1}{2\pi\sqrt{-1}}\int_{C_1^-}\Psi_n(z)\frac{dz_1}{z_1}\\ &= \frac{2\prod_{m=2}^{6}\Gamma(a_ma_1^{\pm1};p,q)}{(p;p)_\infty(q;q)_\infty\Gamma(a_1^{-2};p,q)}\widehat{\Psi}_{n-1}(z_2,\dots,z_n),\end{aligned}\tag{5.3}$$

and hence

$$\begin{aligned}&\frac{1}{2\pi\sqrt{-1}}\int_{C}\Psi_n(z)\frac{dz_1}{z_1}\\ &= \frac{1}{2\pi\sqrt{-1}}\int_{C_0}\Psi_n(z)\frac{dz_1}{z_1}+\frac{2\prod_{m=2}^{6}\Gamma(a_ma_1^{\pm1};p,q)}{(p;p)_\infty(q;q)_\infty\Gamma(a_1^{-2};p,q)}\widehat{\Psi}_{n-1}(z_2,\dots,z_n).\end{aligned}$$

We remark that the first term is regular at $a_1a_2=1$ and has a finite limit as $a_2\to a_1^{-1}$, while the second term diverges in the order $(1-a_1a_2)^{-1}$ because of the factor $\Gamma(a_2a_1;p,q)$. Since

$$\begin{aligned}&\frac{2\Gamma(a_2a_1^{\pm1};p,q)\prod_{m=3}^{6}\Gamma(a_ma_1^{\pm1};p,q)}{(p;p)_\infty(q;q)_\infty\Gamma(a_1^{-2};p,q)}\\ &= \frac{1}{1-a_1a_2}\frac{(pqa_2^{-1}a_1^{-1};p,q)}{(pa_2a_1;p)_\infty(qa_2a_1;q)_\infty(pqa_2a_1;p,q)_\infty}\\ &\quad\times\frac{2\Gamma(a_2a_1^{-1};p,q)\prod_{m=3}^{6}\Gamma(a_ma_1^{\pm1};p,q)}{(p;p)_\infty(q;q)_\infty\Gamma(a_1^{-2};p,q)},\end{aligned}$$

we have

$$\lim_{a_2\to a_1^{-1}}(1-a_1a_2)\frac{2\Gamma(a_2a_1^{\pm1};p,q)\prod_{m=3}^{6}\Gamma(a_ma_1^{\pm1};p,q)}{(p;p)_\infty(q;q)_\infty\Gamma(a_1^{-2};p,q)}=\frac{2\prod_{m=3}^{6}\Gamma(a_ma_1^{\pm1};p,q)}{(p;p)_\infty^2(q;q)_\infty^2},$$

where $a_6=pq/(a_3a_4a_5t^{2n-2})$ in the right-hand side. On the other hand,

$$\lim_{a_2\to a_1^{-1}}\widehat{\Psi}_{n-1}(z_2,\dots,z_n)=\Psi_{n-1}(ta_1,ta_1^{-1},a_3,\dots,a_6;z_2,\dots,z_n).$$

To summarize, we obtain

$$\begin{aligned}&\lim_{a_2\to a_1^{-1}}(1-a_2a_1)\frac{1}{2\pi\sqrt{-1}}\int_{C}\Psi_n(z_1,\dots,z_n)\frac{dz_1}{z_1}\\ &= \frac{2\prod_{m=3}^{6}\Gamma(a_ma_1^{\pm1};p,q)}{(p;p)_\infty^2(q;q)_\infty^2}\Psi_{n-1}(ta_1,ta_1^{-1},a_3,\dots,a_6;z_2,\dots,z_n).\end{aligned}\tag{5.4}$$

We decompose the multiple integral $I_n(a_1,\dots,a_6)$ of (5.2) as

$$\begin{aligned}\int_{C^n}\Psi_n(z)\varpi_n(z) &= \int_{C_1\times C^{n-1}}\Psi_n(z)\varpi_n(z)+\int_{C_0\times C^{n-1}}\Psi_n(z)\varpi_n(z)\\ &= \int_{C_1\times C^{n-1}}\Psi_n(z)\varpi_n(z)+\int_{C_0\times C_1\times C^{n-2}}\Psi_n(z)\varpi_n(z)\\ &\qquad+\int_{C_0\times C_0\times C^{n-2}}\Psi_n(z)\varpi_n(z)\\ &= \cdots\\ &= \sum_{i=1}^{n}\int_{C_0^{i-1}\times C_1\times C^{n-i}}\Psi_n(z)\varpi_n(z)+\int_{C_0^n}\Psi_n(z)\varpi_n(z).\end{aligned}\tag{5.5}$$

Regarding the integral

$$\int_{C_0^{i-1}\times C_1\times C^{n-i}}\Psi_n(z)\varpi_n(z)=\int_{C_0^{i-1}\times C^{n-i}}\left(\frac{1}{2\pi\sqrt{-1}}\int_{C_1}\Psi_n(z)\frac{dz_i}{z_i}\right)\varpi_{n-1}(z_{\widehat{i}}),$$
$$\varpi_{n-1}(z_{\widehat{i}})=\frac{1}{(2\pi\sqrt{-1})^{n-1}}\prod_{\substack{1\le j\le n\\ j\ne i}}\frac{dz_j}{z_j},$$

where $z_{\widehat{i}}=(z_1,\dots,z_{i-1},z_{i+1},\dots,z_n)$, by (5.3) we have

$$\begin{aligned}&\frac{1}{2\pi\sqrt{-1}}\int_{C_1}\Psi_n(z)\frac{dz_i}{z_i}\\ &=\frac{2\,\Gamma(a_2a_1^{\pm1};p,q)\prod_{m=3}^{6}\Gamma(a_ma_1^{\pm1};p,q)}{(p;p)_\infty(q;q)_\infty\Gamma(a_1^{-2};p,q)}\widehat{\Psi}_{n-1}(z_{\widehat{i}}).\end{aligned}$$

Since $\widehat{\Psi}_{n-1}(z_{\widehat{i}})=\widehat{\Psi}_{n-1}(z_1,\dots,z_{i-1},z_{i+1},\dots,z_n)$ is regular at $z_j=a_1,a_1^{-1}$ $(j\ne i)$, one can replace the $(n-1)$-cycle $C_0^{i-1}\times C^{n-i}$ by C_0^{n-1} as

$$\begin{aligned}&\int_{C_0^{i-1}\times C_1\times C^{n-i}}\Psi_n(z)\varpi_n(z)\\ &=\frac{2\,\Gamma(a_2a_1^{\pm1};p,q)\prod_{m=3}^{6}\Gamma(a_ma_1^{\pm1};p,q)}{(p;p)_\infty(q;q)_\infty\Gamma(a_1^{-2};p,q)}\int_{C_0^{n-1}}\widehat{\Psi}_{n-1}(z_{\widehat{i}})\varpi(z_{\widehat{i}}).\end{aligned}$$

Hence (5.5) implies that

$$\begin{aligned}&\int_{C^n}\Psi_n(z)\varpi_n(z)\\ &\quad=\frac{2n\Gamma(a_2a_1^{\pm1};p,q)\prod_{m=3}^{6}\Gamma(a_ma_1^{\pm1};p,q)}{(p;p)_\infty(q;q)_\infty\Gamma(a_1^{-2};p,q)}\int_{C_0^{n-1}}\widehat{\Psi}_{n-1}(z_{\widehat{1}})\varpi_{n-1}(z_{\widehat{1}})\\ &\qquad+\int_{C_0^n}\Psi_n(z)\varpi_n(z).\end{aligned}\tag{5.6}$$

Note that the second term of the right-hand side has a finite limit as $a_2 \to a_1^{-1}$. Multiplying (5.6) by $1 - a_1 a_2$, we compute the limit $a_2 \to a_1^{-1}$ by (5.4) as

$$\begin{aligned}
&\lim_{a_2 \to a_1^{-1}} (1 - a_1 a_2) I_n(a_1, \dots, a_6) = \lim_{a_2 \to a_1^{-1}} (1 - a_1 a_2) \int_{C^n} \Psi_n(z) \varpi_n(z) \\
&= \frac{2n \prod_{m=3}^{6} \Gamma(a_m a_1^{\pm 1}; p, q)}{(p;p)_\infty^2 (q;q)_\infty^2} \int_{C_0^{n-1}} \Psi_{n-1}(t a_1, t a_1^{-1}, a_3, \dots, a_6; z_{\hat{1}}) \varpi_{n-1}(z_{\hat{1}}) \\
&= \frac{2n \prod_{m=3}^{6} \Gamma(a_m a_1^{\pm 1}; p, q)}{(p;p)_\infty^2 (q;q)_\infty^2} I_{n-1}(t a_1, t a_1^{-1}, a_3, \dots, a_6).
\end{aligned}$$

This means that

$$\begin{aligned}
&c_n \lim_{a_2 \to a_1^{-1}} (1 - a_1 a_2) J_n(a_1, \dots, a_6) \\
&\quad = \frac{2n \prod_{m=3}^{6} \Gamma(a_m a_1^{\pm 1}; p, q)}{(p;p)_\infty^2 (q;q)_\infty^2} \, c_{n-1} \, J_{n-1}(t a_1, t a_1^{-1}, a_3, \dots, a_6).
\end{aligned}$$

The left-hand side is already given by Lemma 5.1, while the right-hand side is computed as

$$\frac{2n \, c_{n-1}}{(p;p)_\infty^2 (q;q)_\infty^2} \prod_{i=2}^{n} \Gamma(t^i; p, q) \prod_{i=1}^{n} \prod_{k=3}^{6} \Gamma(a_1^{\pm 1} a_k t^{i-1}; p, q) \prod_{i=1}^{n-1} \prod_{3 \le j < k \le 6} \Gamma(a_j a_k t^{i-1}; p, q)$$

by definition (4.1). Comparing these two expressions we obtain the recurrence formula

$$c_n = c_{n-1} \frac{2n}{(p;p)_\infty (q;q)_\infty} \frac{\Gamma(t^n; p, q)}{\Gamma(t; p, q)} \qquad (n = 1, 2, \dots)$$

for the constants c_n. Starting from $c_0 = 1$, we have

$$c_n = \frac{2^n n!}{(p;p)_\infty^n (q;q)_\infty^n} \prod_{i=1}^{n} \frac{\Gamma(t^i; p, q)}{\Gamma(t; p, q)} \qquad (n = 0, 1, 2, \dots).$$

This completes the evaluation of the BC_n elliptic Selberg integral

$$\begin{aligned}
I_n(a_1, \dots, a_6) &= c_n J_n(a_1, \dots, a_6) \\
&= \frac{2^n n!}{(p;p)_\infty^n (q;q)_\infty^n} \prod_{i=1}^{n} \frac{\Gamma(t^i; p, q)}{\Gamma(t; p, q)} \prod_{i=1}^{n} \prod_{1 \le j < k \le 6} \Gamma(a_j a_k t^{i-1}; p, q).
\end{aligned}$$

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