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A NOTE ON THE AKEMANN-DONER AND FARAH-WOFSEY CONSTRUCTIONS

TRISTAN BICE AND PIOTR KOSZMIDER

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Abstract. We remove the assumption of the continuum hypothesis from the Akemann-Doner construction of a non-separable C^*-algebra A with only separable commutative C^*-subalgebras. We also extend a result of Farah and Wofsey's, constructing $\aleph_1$ commuting projections in the Calkin algebra with no commutative lifting. This removes the assumption of the continuum hypothesis from a version of a result of Anderson. Both results are based on Luzin's almost disjoint family construction.

Background

Recall that an almost disjoint family is a family $\mathcal{F}$ of infinite subsets of $\mathbb{N}$ such that $A \cap B$ is finite for any distinct $A, B \in \mathcal{F}$. Uncountable almost disjoint families, known already to Hausdorff, Luzin and Sierpiński in the second decade of the 20th century carry sophisticated combinatorics. Since the times of Alexandroff and Urysohn's memoir [1] this combinatorics has been employed in constructions of interesting mathematical structures. Applications in topology include, for example, compact spaces of countable tightness which are not Frechet, or two Frechet compact spaces whose product is not Frechet (for a recent survey see [11]). The use of almost disjoint families in Banach space theory was initiated by Johnson and Lindenstrauss in [12] and followed by many authors (e.g., [15], [10]).

In [2], Akemann and Doner considered C^*-subalgebras of the C^*-algebra $\ell_\infty(\mathcal{M}_2)$ of bounded sequences of 2×2 complex matrices obtained from almost disjoint families. Assuming the continuum hypothesis (abbreviated later as CH) they constructed an uncountable almost disjoint family which yielded the first example of a non-separable C^*-algebra with only separable commutative C^*-subalgebras.

Later Popa (see [14], Corollary 6.7) proved that the reduced C^*-algebra of an uncountable free group is an example of such a C^*-algebra whose existence does not require CH or any other set-theoretic assumption beyond the usual axioms ZFC.

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We show in Theorem 5 that CH can in fact already be removed from the Akemann-Doner construction by considering a so-called Luzin family (see [13]), putting it on a more equal footing with Popa's example. Indeed, while Popa's example is highly non-commutative (being simple, for example), the Akemann-Doner example is barely non-commutative (being 2-subhomogeneous, for example). Thus we see that, even in ZFC, a C^*-algebra can be nearly commutative and yet only have small commutative C^*-subalgebras. Our version of the Akemann-Doner construction has many other interesting features which are the consequence of its relative elementarity; for example it is a subalgebra A of the algebra $\mathcal{B}(H)$ of all bounded operators on a separable Hilbert space H which includes a separable ideal $J = A \cap \mathcal{K}(H)$, where $\mathcal{K}(H)$ denotes the ideal of compact operators on H, such that its quotient A/J by J is the commutative C^*-algebra $c_0(\omega_1)$ of all continuous functions on the discrete uncountable space ω_1 vanishing at the infinity. In particular it is a scattered C^*-algebra in the sense of [8], while Popa's example, as a simple C^*-algebra, has the opposite properties, for example has no minimal projections.

The second application of Luzin's family which we present in this note is related to a topic concerning the Calkin algebra $\mathcal{B}(H)/\mathcal{K}(H)$ of bounded operators on the separable Hilbert space H modulo the ideal of compact operators on H. This topic can be traced back to the paper [4] of Anderson, where assuming CH he constructed a maximal selfadjoint abelian subalgebra (masa) of $\mathcal{B}(H)/\mathcal{K}(H)$ which cannot be lifted to a masa in $\mathcal{B}(H)$. In his proof Anderson constructed under CH an uncountable family $\mathcal{P}$ of commuting projections in the Calkin algebra such that no uncountable $\mathcal{P}_1 \subseteq \mathcal{P}$ can be lifted to a family of commuting projections in $\mathcal{B}(H)$.[1]

Echoing Luzin's construction, in Theorem 5.35 of [7], Farah and Wofsey constructed an $\aleph_1$-sized family of projections $\mathcal{P}$ in the Calkin algebra $\mathcal{C}(H)$ which can not be simultaneously diagonalized. In fact, the proof shows that $\pi[A] \cap \mathcal{P}$ is countable, for all C^*-subalgebras A of $\mathcal{B}(H)$ isomorphic to l_∞, where π is the canonical homomorphism from $\mathcal{B}(H)$ onto the Calkin algebra $\mathcal{C}(H) = \mathcal{B}(H)/\mathcal{K}(H)$. They conjectured that this could be extended to arbitrary commutative C^*-subalgebras A of $\mathcal{B}(H)$. Our main result of Section 3, Theorem 6 proves this conjecture and removes the assumption of CH from the above version of Anderson's result. That is, without any additional set-theoretic assumptions we construct in the Calkin algebra an uncountable family $\mathcal{P}$ of commuting projections such that no uncountable $\mathcal{P}_1 \subseteq \mathcal{P}$ can be lifted to a family of commuting projections in $\mathcal{B}(H)$. We would like to thank both Ilijas Farah and Joerg Brendle for various discussions related to this part of our work.

We should mention that Akemann and Weaver noted at the end of [3] that, regardless of CH, there must be $2^{2^{\aleph_0}}$ masas in the Calkin algebra which do not lift to masas in $\mathcal{B}(H)$. Each of these masas is a maximal extension of a commutative C*-subalgebra generated by $2^{\aleph_0}$ projections, although projections may not generate the maximal extensions as in Anderson's construction (e.g. under $\mathfrak{p} = \mathfrak{c}$, a commutative C*-subalgebra generated by projections was constructed in [16], Example 1.6, which has no maximal extension generated by projections). And by our construction, any

[1] If $\mathcal{P}$ is the almost central collection of projections from Theorem 4 of [4], then any of its uncountable subsets is almost central as well. If an almost central collection $\mathcal{P}$ could be lifted to a commuting collection of projections $\mathcal{P}'$ in $\mathcal{B}(H)$, one could consider in $\mathcal{B}(H)$ a masa $\mathcal{A} \supseteq \mathcal{P}'$. But then, by a theorem of Johnson and Parrott from [9], the algebra $\pi[\mathcal{A}]$ would be a masa in the Calkin algebra containing $\mathcal{P}$ and lifting to a masa in $\mathcal{B}$, which would contradict Proposition 3 of [4].

masa containing a certain $\aleph_1$ commuting projection does not lift to a masa in $\mathcal{B}(H)$. Also Luzin families have been used recently to construct subalgebras of $\ell_\infty(\mathcal{M}_2)$ with other interesting properties; see [6] and [18].

1. Luzin families

Definition 1. A Luzin family is an almost disjoint family $\mathcal{L} = \{D_\alpha : \alpha < \omega_1\}$ such that, for every $\alpha < \omega_1$ and every $k \in \mathbb{N}$, the following set is finite:

$$\{\beta < \alpha : D_\beta \cap D_\alpha \subseteq \{0, \dots, k\}\}.$$

For a ZFC construction of a Luzin family, see [6], Appendix B, [11], Theorem 3.1, or [17], Theorem 4.1.

Whenever $A, B \subseteq \mathbb{N}$, we write respectively $A \subseteq^* B$ or $A \cap B =^* \emptyset$ if $A \setminus B$ is finite or $A \cap B$ is finite. For the convenience of the reader let us recall a fundamental property of a Luzin family:

Proposition 2. *Suppose that $\mathcal{L}$ is a Luzin family and $\mathcal{L}', \mathcal{L}'' \subseteq \mathcal{L}$ are uncountable and disjoint. Then there is no $A \subseteq \mathbb{N}$ such that, for all $D' \in \mathcal{L}'$ and $D'' \in \mathcal{L}''$,*

$$D' \subseteq^* A \text{ and } D'' \cap A =^* \emptyset.$$

Proof. If there is such an $A \subseteq \mathbb{N}$, then $X' = \{\alpha < \omega : D_\alpha \setminus \{1, \dots, k'\} \subseteq A\}$ is uncountable, for some $k' \in \mathbb{N}$. Likewise $X'' = \{\alpha < \omega : D_\alpha \setminus \{1, \dots, k'\} \cap A = \emptyset\}$ is uncountable, for some $k'' \in \mathbb{N}$. Let $k = \max(k', k'')$ and take $\alpha \in X'$ such that $X'' \cap \alpha$ is infinite. But $D_\beta \cap D_\alpha \subseteq \{1, \dots, k\}$ for every $\beta \in X'' \cap \alpha$, which contradicts the definition of a Luzin family. □

2. The Akemann-Doner construction

First note the following elementary C^*-algebra result.

Lemma 3. *If projections p and q commute and $||p - q|| < 1$, then $p = q$.*

Proof. As p and q commute, $pq^\perp = p(1 - q) = p - pq$ is a projection such that $||pq^\perp|| = ||p(p - q)|| \leq ||p - q|| < 1$ so $pq^\perp = 0$. Likewise, $p^\perp q$ is a projection with $||p^\perp q|| = ||(p - q)q|| \leq ||p - q|| < 1$ so $p^\perp q = 0$. Thus $p = pq = q$. □

In fact, for any $p, q \in \mathcal{P}^1 =$ the rank one projections in $\mathcal{M}_2$, we have

$$||p - q|| = ||pq^\perp|| = ||p^\perp q||.$$

Thus using the fact that for any a in the algebra $||ap||$ is the supremum of all $||av||$ taken over unit vectors v in the range of p, for any unit vector $v \in \mathcal{H}_2$ with $pv = v$,

$$1 = ||v|| = ||qv||^2 + ||q^\perp v||^2 = ||pq||^2 + ||pq^\perp||^2 = ||p - q^\perp||^2 + ||p - q||^2 \tag{2.1}$$

(as the involution is isometric, $||pq|| = ||(pq)^*|| = ||qp||$). We now make the following assumptions.

Definition 4.

- $\mathbf{p} \in \mathcal{P}^1$ is fixed throughout.
- $\mathcal{L}$ is a Luzin almost disjoint family on $\mathbb{N}$.
- $(p_D)_{D \in \mathcal{L}} \subseteq \mathcal{P}^1$ are distinct with $||p_D - \mathbf{p}|| < \frac{1}{4}$, for all $D \in \mathcal{L}$.
- π is the canonical homomorphism from $\ell_\infty(\mathcal{M}_2)$ to $\ell_\infty(\mathcal{M}_2)/c_0(\mathcal{M}_2)$.

- For $D \subseteq \mathbb{N}$, c_D denotes the central projection in $\ell_\infty(\mathcal{M}_2)$ defined by
$$c_D(n) = \begin{cases} 1, & n \in D, \\ 0, & n \notin D. \end{cases}$$
- Elements of $\mathcal{M}_2$ are identified with constant functions in $\ell_\infty(\mathcal{M}_2)$.
- L is the C^*-subalgebra of $\ell_\infty(\mathcal{M}_2)$ generated by $(p_D c_D)_{D\in\mathcal{L}}$ and $c_0(\mathcal{M}_2)$.

Theorem 5. *L is non-separable but only has separable commutative C^*-subalgebras.*

Proof. As $\pi(p_D c_D)$ is an uncountable pairwise orthogonal collection of projections, $\pi[L] \cong c_0(\aleph_1)$ so L is non-separable. This also means, for any $a \in L$, that
$$\pi(a) = \sum \lambda_D^a \pi(p_D c_D)$$
for unique $(\lambda_D^a)_{D\in\mathcal{L}} \subseteq \mathbb{C}$ with $\lambda_D^a \to 0$ (on the countable subset $\{D \in \mathcal{L} : \lambda_D^a \neq 0\}$).

Now suppose that A is a non-separable C^*-subalgebra of L which is commutative. We will get a contradiction with the property of a Luzin family from Lemma 2. Let θ_n be the evaluation homomorphism at n, i.e. $\theta_n(a) = a(n)$. Then $\theta_n[A]' = \mathbb{C}1, \mathcal{M}_2$ or $\mathbb{C}q + \mathbb{C}q^\perp$, for some $q \in \mathcal{P}^1$. But the commutativity of A eliminates the first possibility, so we have some $q \in \ell_\infty(\mathcal{P}^1) \cap A'$, i.e. for all $n \in \mathbb{N}$ and $a \in A$, $q(n) \in \mathcal{P}^1$ and $a(n)q(n) = q(n)a(n)$. By (2.1), replacing $q(n)$ with $q(n)^\perp$ if necessary, we can also assume that $||q - \mathbf{p}||^2 \leq \frac{1}{2}$. Note that q is not necessarily in L or A.

Say $a \in A$ and $\lambda_D^a \neq 0$ for some $D \in \mathcal{L}$. As $aq = qa$, we have $\pi(aq) = \pi(qa)$ and hence $\pi(aqc_D) = \pi(qac_D)$. As $\pi(c_D c_E) = 0$, for all $E \in \mathcal{L} \setminus \{D\}$, this means that $\pi(\lambda_D^a p_D q c_D) = \pi(q\lambda_D^a p_D c_D)$ and hence $\pi(p_D q c_D) = \pi(q p_D c_D)$; i.e. $\pi(p_D c_D)$ and $\pi(qc_D)$ commute. But $||q - \mathbf{p}||^2 \leq \frac{1}{2}$ and $||p_D - \mathbf{p}|| \leq \frac{1}{4}$, and hence $||q - p_D|| < 1$. Thus $||\pi(qc_D) - \pi(p_D c_D)|| \leq ||(q - p_D)c_D|| < 1$, and hence $\pi(qc_D) = \pi(p_D c_D)$, by Lemma 3. So $\lim_{n\in D} ||q(n) - p_D(n)|| = 0$.

As A is non-separable, we must have uncountably many distinct $D \in \mathcal{L}$ for which there is $a \in A$ with $\lambda_D^a \neq 0$. Thus we have uncountable $\mathcal{L}' \subseteq \mathcal{L}$ with $\lim_{n\in D} ||q(n) - p_D(n)|| = 0$, for all $D \in \mathcal{L}'$. As $\mathcal{P}^1$ is a separable metric space, $(p_D)_{D\in\mathcal{L}'}$ must have at least two distinct condensation points r and s, i.e. such that every neighbourhood of r and s contains uncountably many $(p_D)_{D\in\mathcal{L}'}$. Let
$$\mathcal{E} = \{D \in \mathcal{L}' : ||p_D - r|| < \tfrac{1}{2}||r - s||\} \text{ and}$$
$$\mathcal{F} = \{D \in \mathcal{L}' : ||p_D - s|| < \tfrac{1}{2}||r - s||\}.$$
By the triangle inequality, $\mathcal{E}$ and $\mathcal{F}$ are disjoint, as are
$$X = \{n \in \mathbb{N} : ||q(n) - r|| < \tfrac{1}{2}||r - s||\} \text{ and}$$
$$Y = \{n \in \mathbb{N} : ||q(n) - s|| < \tfrac{1}{2}||r - s||\}.$$
As $\lim_{n\in D} ||q(n) - p_D(n)|| = 0$, for all $D \in \mathcal{L}'$, we see that $E \subseteq^* X$, for all $E \in \mathcal{E}$, and $F \subseteq^* Y$, for all $F \in \mathcal{F}$. This contradicts Proposition 2. □

Note that $\pi[L] \cong c_0(\aleph_1)$ has no countable approximate unit, so neither does L. Thus L has no commutative approximate unit by Theorem 5. However, L does have an 'almost idempotent' approximate unit $(h_\lambda) \subseteq A_+^1$ in the sense of [5], Definition II.4.1.1, i.e. satisfying $h_\lambda h_\mu = h_\lambda$ whenever $\lambda < \mu$. Indeed, the projections in L form a lattice and hence an increasing approximate unit for A. As far as we know, it is still an open question whether there is a (necessarily non-separable) C^*-algebra with no almost idempotent approximate unit.

3. The Farah-Wofsey construction

For a separable infinite dimensional Hilbert space H, our goal here is to show the following.

Theorem 6. *There are $\aleph_1$ orthogonal projections in $\mathcal{B}(H)/\mathcal{K}(H)$ containing no uncountable subset that simultaneously lifts to commuting projections in $\mathcal{B}(H)$.*

For the proof, based on [7], Theorem 5.35, we first construct projections in $\mathcal{B}(H)$ with a Luzin-like property with respect to $(K_n)_{n\in\mathbb{N}} \subseteq \mathcal{K}(H)$ as follows.

Lemma 7. *For every $(K_n)_{n\in\mathbb{N}} \subseteq \mathcal{K}(H)$ and every $\epsilon < 1/2$ there are infinite rank projections $(Q_\alpha)_{\alpha\in\aleph_1} \subseteq \mathcal{B}(H)$ such that, for all distinct $\alpha, \beta \in \aleph_1$, $Q_\alpha Q_\beta \in \mathcal{K}(H)$, and for all $\beta \in \aleph_1$, all $n \in \mathbb{N}$, and all but possibly n many $\alpha \in \beta$,*

$$||(Q_\alpha + K_n)(Q_\beta + K_n) - (Q_\beta + K_n)(Q_\alpha + K_n)|| \geq \epsilon.$$

Proof. We construct $(Q_\alpha)_{\alpha\in\aleph_1}$ by recursion as follows. First take any orthogonal non-compact projections $(Q_n)_{n\in\mathbb{N}}$ (which we will delete at the end). Assume $(Q_\alpha)_{\alpha\in\gamma}$ has already been constructed such that, for all distinct $\alpha, \beta \in \gamma$, $Q_\alpha Q_\beta \in \mathcal{K}(H)$. In particular, $\pi(Q_\alpha)$ and $\pi(Q_\beta)$ commute for all $\alpha, \beta \in \gamma$, and hence, by [7], Lemma 5.34, there exist an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of H and $(A_n)_{n\in\mathbb{N}} \subseteq \wp(\mathbb{N})$ such that $\pi(p_{A_n}) = \pi(Q_{\alpha_n})$, for all $n \in \mathbb{N}$, where $n \mapsto \alpha_n$ is any fixed one-to-one mapping of $\mathbb{N}$ onto γ and p_X denotes the projection onto $\overline{\text{span}}(e_n)_{n\in X}$ for $X \subseteq \mathbb{N}$.

Take $\delta > 0$ with $\epsilon \leq \frac{1}{2} - (\frac{1}{\sqrt{2}} + 2)\delta$, and recursively define an increasing sequence $(k_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$ as follows. Let $k_0 (= \{0, \ldots, k_0 - 1\})$ be large enough that

$$||(Q_{\alpha_0} + K_0 - p_{A_0})p^\perp_{k_0}|| < \delta,$$
$$||((Q_{\alpha_0} + K_0)K_0 - K_0(Q_{\alpha_0} + K_0))p^\perp_{k_0}|| < \delta.$$

Once $(k_n)_{n\leq m}$ has been defined, let $k_{m+1} > k_m$ be large enough that there exist distinct $i(m), j(m) \in k_{m+1}\backslash(k_m \cup \bigcup_{n\in m} A_n)$ such that $j(m) \notin A_m$ and $i(m) \in A_m$ and, for all $l \leq m + 1$,

$$||(Q_{\alpha_{m+1}} + K_l - p_{A_{m+1}})p^\perp_{k_{m+1}}|| < \delta,$$
$$||((Q_{\alpha_{m+1}} + K_l)K_l - K_l(Q_{\alpha_{m+1}} + K_l))p^\perp_{k_{m+1}}|| < \delta.$$

Note that, for sufficiently large k_{m+1}, $j(m)$ exists because $A_{m+1}\backslash \bigcup_{n\leq m} A_n$ is infinite.

Now let Q_γ be the projection onto $\overline{\text{span}}\{e_{i(n)} + e_{j(n)} : n \in \mathbb{N}\}$. Note that, for all $m \in \min(\gamma, \omega)$ and $n > m$, we have $i(n), j(n) \notin A_m$ so

$$p_{A_m} Q_\gamma[H] \subseteq \text{span}(\{e_{i(n)} : n \leq m\} \cup \{e_{j(n)} : n \leq m\})$$

so $Q_{\alpha_m} Q_\gamma \in \mathcal{K}(H)$. Also $Q_\gamma e_{i(m)} = \frac{1}{2}(e_{i(m)} + e_{j(m)})$. For any $m \in \mathbb{N}$, $i(m) \in A_m$, and so $p_{A_m}(e_{i(m)} + e_{j(m)}) = e_{i(m)} = p_{A_m} e_{i(m)}$ and hence, for all $l \leq m$,

$$\begin{aligned}
&||(Q_{\alpha_m} + K_l)Q_\gamma e_{i(m)} - \tfrac{1}{2}e_{i(m)}|| \\
&= ||(Q_{\alpha_m} + K_l)(\tfrac{1}{2}(e_{i(m)} + e_{j(m)})) - p_{A_m}(\tfrac{1}{2}(e_{i(m)} + e_{j(m)}))|| \\
&= ||(Q_{\alpha_m} + K_l - p_{A_m})(\tfrac{1}{2}(e_{i(m)} + e_{j(m)}))|| \\
&\leq \delta||e_{i(m)} + e_{j(m)}||/2 \\
&\leq \delta/\sqrt{2}
\end{aligned}$$

and

$$\begin{aligned}
&||Q_\gamma(Q_{\alpha_m}+K_l)e_{i(m)}-\tfrac{1}{2}(e_{i(m)}+e_{j(m)})||\\
&=||Q_\gamma(Q_{\alpha_m}+K_l)e_{i(m)}-Q_\gamma p_{A_m}e_{i(m)}||\\
&\leq ||(Q_{\alpha_m}+K_l-p_{A_m})e_{i(m)}||\\
&\leq \delta,
\end{aligned}$$

so

$$\begin{aligned}
&||(Q_{\alpha_m}+K_l)(Q_\gamma+K_l)e_{i(m)}-(Q_\gamma+K_l)(Q_{\alpha_m}+K_l)e_{i(m)}||\\
&\geq ||\tfrac{1}{2}e_{j(m)}||-||(Q_{\alpha_m}+K_l)Q_\gamma e_{i(m)}-\tfrac{1}{2}e_{i(m)}||\\
&\quad -||Q_\gamma(Q_{\alpha_m}+K_l)e_{i(m)}-\tfrac{1}{2}(e_{i(m)}+e_{j(m)})||\\
&\quad -||((Q_{\alpha_m}+K_l)K_l-K_l(Q_{\alpha_m}+K_l))e_{i(m)}||\\
&\geq 1/2-\delta/\sqrt{2}-\delta-\delta\\
&\geq \epsilon.
\end{aligned}$$

This completes the recursion, and finally we replace each Q_α with $Q_{\omega+\alpha}$ so that $(Q_\alpha)_{\alpha\leq\aleph_1}$ satisfies the required conditions. □

Proof of Theorem 6. Take dense $(K_n)_{n\in\mathbb{N}}\subseteq\mathcal{K}(H)$ and $\epsilon\in(0,1/2)$ and let $(Q_\alpha)_{\alpha\in\aleph_1}$ be obtained from Lemma 7. Assume that we have some uncountable $A\subseteq\aleph_1$ and $(K'_\alpha)_{\alpha\in A}\subseteq\mathcal{K}(H)$ such that $(Q_\alpha+K'_\alpha)_{\alpha\in A}$ commute. By replacing A with an uncountable subset of A if necessary, we may assume that there exists some $M\in\mathbb{R}$ such that $||K'_\alpha||\leq M$ for all $\alpha\in A$. Take $\delta>0$ with $2\delta(1+M)+2(1+M+\delta)\delta<\epsilon$, and pick $n_\alpha\in\mathbb{N}$ such that $||K_{n_\alpha}-K'_\alpha||\leq\delta$, for all $\alpha\in A$. Again replacing A with an uncountable subset of A if necessary, we may assume that there exists some $n\in\mathbb{N}$ such that $K_{n_\alpha}=K_n$ for all $\alpha\in A$. Then, for any $\beta\in A$ and $\alpha\in A\cap\beta$, as we have $(Q_\alpha+K'_\alpha)(Q_\beta+K'_\beta)-(Q_\beta+K'_\beta)(Q_\alpha+K'_\alpha)=0$, we obtain

$$\begin{aligned}
&||(Q_\alpha+K_n)(Q_\beta+K_n)-(Q_\beta+K_n)(Q_\alpha+K_n)||\\
\leq\ &||K'_\alpha-K_n||||Q_\beta+K_n||+||Q_\alpha+K'_\alpha||||K'_\beta-K_n||\\
&+||K'_\beta-K_n||||Q_\alpha+K_n||+||Q_\beta+K'_\beta||||K'_\alpha-K_n||\\
\leq\ &\delta(1+M)+(1+M+\delta)\delta+\delta(1+M)+(1+M+\delta)\delta\\
<\ &\epsilon.
\end{aligned}$$

But for any β such that $A\cap\beta$ contains more than n elements, this contradicts the defining property of the $(Q_\alpha)_{\alpha\in\aleph_1}$.

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Federal University of Bahia, Salvador, Brazil
E-mail address: `Tristan.Bice@gmail.com`

Institute of Mathematics, Polish Academy of Sciences, ul. Śniadeckich 8, 00-656 Warszawa, Poland
E-mail address: `piotr.koszmider@impan.pl`