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TRANSLATION OPERATORS ON WEIGHTED SPACES OF ENTIRE FUNCTIONS

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(Communicated by Thomas Schlumprecht)

To my supervisor, Professor Alexander V. Abanin, on the occasion of his sixtieth birthday

ABSTRACT. We study the dynamical properties of translation operators on both weighted Hilbert and Banach spaces of entire functions. We show that the translation operator on these weighted spaces is always mixing when it is continuous and give necessary and sufficient conditions in terms of weights for the chaos of this operator. We also prove that translation operators can arise as compact perturbations of the identity on weighted Banach spaces.

1. Introduction and notation

Let $H(\mathbb{C})$ denote the space of all entire functions endowed with the topology of uniform convergence on compact subsets of $\mathbb{C}$. The translation operator $T_a f(z) := f(z+a)$ on $H(\mathbb{C})$ was found by Birkhoff [6] since 1929. In this work, the author stated that *for each $a \neq 0$ there exists an entire function f whose orbit $\{T_a^n f\}_{n=0}^\infty$ is dense in $H(\mathbb{C})$*. Later on, Chan and Shapiro [10] investigated the cyclic behavior of the translation operator T_a on weighted Hilbert spaces of entire functions whose growth is defined by a *comparison function* γ.

The aim of this paper is, firstly, to study the other dynamical properties such as the mixing property and the chaos of the translation operator T_a on Hilbert spaces defined in [10]; secondly, to investigate the dynamics of this operator T_a on weighted Banach spaces of entire functions with sup-norm when it is well-defined and continuous; thirdly, to study the perturbation of the identity on these weighted Banach spaces by translation operators. In this view we complete the study on Hilbert spaces of Chan-Shapiro [10] and continue the research in [3, 4, 7–9] for the dynamical behaviour of the differentiation and integration operators on weighted Banach spaces.

A continuous and linear operator T from a Banach space X into itself is called *hypercyclic* if there is a vector $x \in X$ whose orbit under T is dense in X. An operator T on a separable Banach space X is hypercyclic if and only if it is *topologically transitive* in the sense of dynamical systems; i.e., for every pair of non-empty open

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subsets U and V of X there is $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$. A stronger condition is defined as follows: an operator T on X is called *mixing* if for every pair of non-empty open subsets U and V of X there is $N \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$ for all $n \geq N$. According to Devaney [11, page 50] (see also, [13, Section 5] and [14, Definition 2.29]), an operator T on X is said to be *chaotic* if it is hypercyclic and has a dense set of periodic points. We refer the reader to the books by Bayart and Matheron [2] and by Grosse-Erdmann and Peris [14] for more details about linear dynamics.

Throughout the paper, a *weight* v is a continuous increasing function $v : [0, \infty) \to (0, +\infty)$ satisfying $\log r = o(\log v(r))$ as $r \to \infty$. We extend v to $\mathbb{C}$ by $v(z) := v(|z|)$. For such a weight v, we define the following Banach spaces of entire functions:

$$H_v^\infty(\mathbb{C}) := \{f \in H(\mathbb{C}) \mid \|f\|_v := \sup_{z\in\mathbb{C}} |f(z)|v^{-1}(z) < \infty\},$$

$$H_v^0(\mathbb{C}) := \{f \in H(\mathbb{C}) \mid \lim_{z\to\infty} |f(z)|v^{-1}(z) = 0\},$$

endowed with the sup-norm $\|\cdot\|_v$. Clearly, $H_v^0(\mathbb{C})$ is a closed subspace of $H_v^\infty(\mathbb{C})$ and contains the polynomials as a dense subset, and $H_v^0(\mathbb{C})$ is a separable Banach space. In this paper we study the translation operator T_a only on the space $H_v^0(\mathbb{C})$, because if a Banach space X admits a hypercyclic operator, then X must be separable.

Note that one of the most important problems relating to the weighted spaces $H_v^\infty(\mathbb{C})$ and $H_v^0(\mathbb{C})$ is to characterize properties of these spaces and operators between them in terms of the relevant weights. But as is well known, many results in this topic must be formulated in terms of the so-called *associated weights* and not directly in terms of the weight v. Following Bierstedt-Bonet-Taskinen in [5] the associated weight is defined by

$$\widetilde{v}(z) := \sup\{|f(z)| : f \in B_v^\infty(\mathbb{C})\}, z \in \mathbb{C},$$

where $B_v^\infty(\mathbb{C})$ is the unit ball of $H_v^\infty(\mathbb{C})$. By [5, Proposition 1.2], the associated weight $\widetilde{v}$ is continuous, radial (i.e., $\widetilde{v}(z) = \widetilde{v}(|z|)$, $\forall z \in \mathbb{C}$) and $\widetilde{v} \leq v$ on $\mathbb{C}$. It was shown in [5, Observation 1.12] that $H_{\widetilde{v}}^\infty(\mathbb{C})$ coincides isometrically with $H_v^\infty(\mathbb{C})$. Moreover, $\log \widetilde{v}(z)$ is a subharmonic function on $\mathbb{C}$, which is equivalent to $\widetilde{v}$ being *log-convex*; i.e., the associated function $\varphi_{\widetilde{v}}(x) := \log \widetilde{v}(e^x)$ is convex on $\mathbb{R}$.

Following Duyos-Ruiz [12] and Chan-Shapiro [10], let us call an entire function $\gamma(z) = \sum \gamma_n z^n$ a *comparison function* if $\gamma_n > 0$ for each n, and the sequence γ_{n+1}/γ_n decreases to 0 as $n \to \infty$. If, in addition, the sequence $w_n = n\gamma_n/\gamma_{n-1}$ is monotonically decreasing, then we call γ an *admissible* comparison function. For each comparison function γ, we define the following Hilbert space of entire functions:

$$H_\gamma^2(\mathbb{C}) := \{f \in H(\mathbb{C}) \mid f(z) = \sum_{n=0}^{\infty} a_n z^n, \|f\|_{2,\gamma}^2 := \sum_{n=0}^{\infty} |a_n|^2 \gamma_n^{-2} < \infty\}.$$

By [10, Proposition 1.4], the embedding $H_\gamma^2(\mathbb{C}) \to H_\gamma^\infty(\mathbb{C})$ is continuous for every comparison function γ. If the sequence $w_n = n\gamma_n/\gamma_{n-1}$ is bounded, then the embedding $H_\gamma^\infty(\mathbb{C}) \to H_{\gamma_1}^2(\mathbb{C})$ is also continuous for $\gamma_1(z) := z^2\gamma(z),\ z \in \mathbb{C}$. Note that in the definition of the weighted Banach spaces $H_\gamma^\infty(\mathbb{C})$ and $H_\gamma^0(\mathbb{C})$ the symbol γ denotes the weight $\gamma(|z|)$ on $\mathbb{C}$.

In Section 2 we obtain a criterion for the continuity of the translation operator T_a on spaces $H_v^\infty(\mathbb{C})$ and $H_v^0(\mathbb{C})$ in terms of weights.

Section 3 contains the main results concerning the dynamical properties of the operator T_a on both spaces $H_v^0(\mathbb{C})$ and $H_\gamma^2(\mathbb{C})$. In details, we show that the translation operator T_a on these spaces is always mixing, in particular is hypercyclic when it is well-defined and continuous. We also establish complete descriptions of those weights v and comparison functions γ for which the translation operator T_a on the corresponding spaces $H_v^0(\mathbb{C})$ and $H_\gamma^2(\mathbb{C})$ is chaotic.

In Section 4 we prove that the operator $T_a - I$ on weighted Banach space $H_v^0(\mathbb{C})$ can have an arbitrarily small norm and arbitrarily high degree of compactness.

2. Continuity

In this section we investigate when the translation operator T_a on weighted Banach spaces $H_v^\infty(\mathbb{C})$ and $H_v^0(\mathbb{C})$ is continuous.

Theorem 2.1. *Let v be a log-convex weight on $\mathbb{C}$. The following conditions are equivalent:*

(i) *The operator $T_a : H_v^\infty(\mathbb{C}) \to H_v^\infty(\mathbb{C})$ is continuous.*
(ii) *The operator $T_a : H_v^0(\mathbb{C}) \to H_v^0(\mathbb{C})$ is continuous.*
(iii) $\log v(r) = O(r)$ *as* $r \to \infty$.
(iv) $\limsup_{r\to\infty} \dfrac{v'(r)}{v(r)} < \infty$.

Proof. (i) $\Leftrightarrow$ (ii): By [4, Lemma 2.1].

(i) $\Rightarrow$ (iii): (i) means that $T_a : H_{\widetilde{v}}^\infty(\mathbb{C}) \to H_{\widetilde{v}}^\infty(\mathbb{C})$ is continuous. That is, there is a number $C > 1$ such that

$$\|T_a f\|_{\widetilde{v}} \le C\|f\|_{\widetilde{v}} \text{ for all } f \in H_{\widetilde{v}}^\infty(\mathbb{C}).$$

Hence,

$$|f(z+a)| \le C\widetilde{v}(z) \text{ for all } z \in \mathbb{C} \text{ and } f \in B_{\widetilde{v}}^\infty(\mathbb{C}).$$

Taking the supremum over all entire functions $f \in B_{\widetilde{v}}^\infty(\mathbb{C})$, we then obtain

$$\widetilde{v}(z+a) \le C\widetilde{v}(z) \text{ for all } z \in \mathbb{C},$$

i.e.,

$$\varphi_{\widetilde{v}}(\log(r+|a|)) - \varphi_{\widetilde{v}}(\log r) \le \log C, \ \forall r > 0.$$

Thus, in view of the convexity of $\varphi_{\widetilde{v}}$,

$$\varphi'_{\widetilde{v}}(\log r) \le \frac{\log C}{\log(1+|a|r^{-1})}, \ \forall r > 0.$$

Consequently,

$$\limsup_{x\to+\infty} \frac{\varphi'_{\widetilde{v}}(x)}{e^x} < \infty, \text{ i.e., } \limsup_{r\to\infty} \frac{\widetilde{v}'(r)}{\widetilde{v}(r)} < \infty.$$

By the L'Hospital rule, we have

$$\limsup_{r\to\infty} \frac{\log \widetilde{v}(r)}{r} < \infty.$$

Therefore, $\log \widetilde{v}(r) \le M(r+1)$ for some $M > 0$ and all $r > 0$. Moreover, as is known (see, [5, page 157]), $v(r) \le r\widetilde{v}(r)$ for all $r \ge 1$. Thus, $\log v(r) \le M(r+1) + \log r$ for all $r \ge 1$, which implies (iii).

(iii) $\Rightarrow$ (iv): Because φ_v is convex and increasing, we have that, for every $x > 0$,

$$\frac{\varphi'_v(x)}{e^x} \le \frac{\varphi_v(x+1) - \varphi_v(x)}{e^x} \le e\frac{\varphi_v(x+1) - \varphi_v(0)}{e^{x+1}}.$$

From this it follows that

$$\limsup_{x\to+\infty} \frac{\varphi_v'(x)}{e^x} \le e \limsup_{x\to+\infty} \frac{\varphi_v(x)}{e^x}.$$

Hence,

$$\limsup_{r\to\infty} \frac{v'(r)}{v(r)} \le e \limsup_{r\to\infty} \frac{\log v(r)}{r}.$$

Thus, (iii) $\Rightarrow$ (iv).

(iv) $\Rightarrow$ (i): (iv) means that there is a constant $C > 0$ such that

$$(\log v(r))' \le C \text{ for all } r > 0.$$

Using the log-convexity of v, we obtain

$$\log \frac{v(r+|a|)}{v(r)} \le C(r+|a|) \log \frac{r+|a|}{r} \le C(1+|a|)|a| \text{ for all } r \ge 1.$$

Therefore, $v(z+a) \le Mv(z)$ for some $M > 0$ and all $z \in \mathbb{C}$. Consequently, for every $f \in H_v^\infty(\mathbb{C})$,

$$\|T_a f\|_v \le M \sup_{z\in\mathbb{C}} \frac{|f(z+a)|}{v(z+a)} = M\|f\|_v.$$

Thus, (i) holds. □

Remark 2.2. As in the study of the continuity of the differentiation operator on $H_v^\infty(\mathbb{C})$, our assumption that v is log-convex is also essential in Theorem 2.1. To see this, one can use Example 2.11 in [1].

To end this section, we complete the result in [10, Corollary 1.2] concerning the boundedness of the operator T_a on $H_\gamma^2(\mathbb{C})$.

Proposition 2.3. *For each comparison function γ, the operator $T_a : H_\gamma^2(\mathbb{C}) \to H_\gamma^2(\mathbb{C})$ is continuous if and only if the sequence $n\gamma_n/\gamma_{n-1}$ is bounded.*

Proof. In view of [10, Corollary 1.2], it suffices to prove the necessity.

Assume that T_a is continuous on $H_\gamma^2(\mathbb{C})$. That is, there exists a constant $C > 0$ such that

$$\|T_a f\|_{2,\gamma} \le C\|f\|_{2,\gamma} \text{ for all } f \in H_\gamma^2(\mathbb{C}).$$

In particular, for each $n \in \mathbb{N}$ we have that $\|(z+a)^n\|_{2,\gamma} \le C\|z^n\|_{2,\gamma}$, which means that

$$\sum_{k=0}^{n} \gamma_k^{-2}(C_n^k|a|^{n-k})^2 \le C^2\gamma_n^{-2}, \text{ and hence, } \gamma_{n-1}^{-2}(C_n^{n-1}|a|)^2 \le C^2\gamma_n^{-2}.$$

Thus, $n\gamma_n/\gamma_{n-1} \le C/|a|$ for all $n \ge 1$. □

Remark 2.4. The necessary and sufficient conditions for the continuity of the translation operator T_a on Hilbert space $H_\gamma^2(\mathbb{C})$ and Banach spaces $H_v^\infty(\mathbb{C})$ and $H_v^0(\mathbb{C})$ in Proposition 2.3 and Theorem 2.1 do not depend on the number a. This means that if one of the translation operators on these weighted spaces is continuous, then so are all of them.

3. Dynamics

In this section we study the dynamical properties of the translation operator T_a on both spaces $H_v^0(\mathbb{C})$ and $H_\gamma^2(\mathbb{C})$. It should be noted that our results for $H_\gamma^2(\mathbb{C})$ improve and complement the work of Chan-Shapiro [10].

We start with an auxiliary result which contains the hypercyclic comparison principle in [15] and [14, Exercise 2.2.6]. Recall that an operator $T : X \to X$ is called *quasiconjugate* to an operator $S : Y \to Y$ if there exists a continuous map $\phi : Y \to X$ with dense range such that $T \circ \phi = \phi \circ S$ (if ϕ can be chosen to be a homeomorphism, then S and T are called *conjugate*), and a property $\mathcal{C}$ is said to be *preserved under quasiconjugacy* if the following holds: if an operator $S : Y \to Y$ has property $\mathcal{C}$, then every operator $T : X \to X$ that is quasiconjugate to S also has property $\mathcal{C}$. As is well known (see [14, Chapter 2]), the hypercyclicity, the mixing property, the chaos and the property of having a dense set of periodic points are preserved under quasiconjugacy.

Lemma 3.1 ([14, Exercise 2.2.6])**.** *Let $T : X \to X$ be an operator on a Banach space X. Suppose that $Y \subset X$ is a T-invariant dense subspace of X. Furthermore, suppose that Y carries a Banach space topology such that the embedding $Y \to X$ is continuous and $T|_Y : Y \to Y$ is continuous. Then T is quasiconjugate to $T|_Y$. In particular, if $T|_Y$ is hypercyclic (or mixing, chaotic), then so is T; if $T|_Y$ has a dense set of periodic points, then so does T.*

3.1. **Hypercyclicity.** The next result improves Theorem 2.1 in [10] and can be obtained following [14, Exercise 8.1.2].

Proposition 3.2. *For every comparison function γ, the translation operator T_a is mixing on $H_\gamma^2(\mathbb{C})$ when it is continuous.*

From this it follows that

Corollary 3.3. *For every admissible comparison function γ, the translation operator T_a is mixing on $H_\gamma^2(\mathbb{C})$.*

The similar result for weighted Banach spaces $H_v^0(\mathbb{C})$ (see Theorem 3.5 below) can be deduced from [14, Theorem 8.6] (see also [14, Exercises 8.1.6 and 8.1.9]). For the sake of completeness, we include here its proof, which is different from [14] and looks simpler. To do this we need the following lemma.

Lemma 3.4. *For every log-convex weight v, there exists an admissible comparison function γ in $H_v^0(\mathbb{C})$.*

Proof. Take a sequence $(\alpha_n)_{n=0}^\infty$ of positive numbers so that the sequence $n\alpha_n/\alpha_{n-1}$ is decreasing. Clearly, the series $\sum \alpha_n$ converges.

We define $\gamma_n := \alpha_n \exp(-\varphi_v^*(n))$ for each $n \geq 0$, where φ_v^* is the Young conjugate of φ_v. Then

$$\frac{n\gamma_n}{\gamma_{n-1}} = \frac{n\alpha_n}{\alpha_{n-1}} \exp(\varphi_v^*(n-1) - \varphi_v^*(n)), \forall n \in \mathbb{N}.$$

Thus, in view of the convexity of φ_v^*, the sequence γ_n/γ_{n-1} decreases to 0 as $n \to \infty$ and the sequence $n\gamma_n/\gamma_{n-1}$ is decreasing. Therefore, the function $\gamma(z) := \sum \gamma_n z^n$ is an admissible comparison function.

Now we show that $\gamma \in H_v^0(\mathbb{C})$. Fix an arbitrary number $\varepsilon > 0$. Since the series $\sum \alpha_n$ is convergent and $r^n = o(v(r))$ as $r \to \infty$ for all $n \in \mathbb{N}$, there exist numbers

$N \in \mathbb{N}$ and $R > 0$ such that

$$\sum_{n>N} \alpha_n < \frac{\varepsilon}{2} \text{ and } \sum_{n=0}^{N} \gamma_n \frac{r^n}{v(r)} < \frac{\varepsilon}{2} \text{ for all } r \geq R.$$

From this it follows that, for every $r \geq R$,

$$\begin{aligned} \frac{\gamma(r)}{v(r)} &\leq \sum_{n=0}^{N} \gamma_n \frac{r^n}{v(r)} + \sum_{n=N+1}^{\infty} \gamma_n \sup_{r>0} \frac{r^n}{v(r)} \\ &< \frac{\varepsilon}{2} + \sum_{n=N+1}^{\infty} \alpha_n < \varepsilon. \end{aligned}$$

Consequently, $\gamma \in H_v^0(\mathbb{C})$. □

Theorem 3.5. *For every weight v on $\mathbb{C}$, the translation operator T_a is mixing on $H_v^0(\mathbb{C})$ when it is continuous.*

Proof. Since $\widetilde{v}$ is log-convex, by Lemma 3.4, there is an admissible comparison function γ in $H_{\widetilde{v}}^0(\mathbb{C})$ and hence in $H_v^0(\mathbb{C})$. From this it follows that the embedding $H_\gamma^2(\mathbb{C}) \to H_v^0(\mathbb{C})$ is continuous and $H_\gamma^2(\mathbb{C})$ is a dense subspace of $H_v^0(\mathbb{C})$.

By Corollary 3.3, the translation operator T_a is mixing on $H_\gamma^2(\mathbb{C})$. Consequently, by Lemma 3.1, T_a is also mixing on $H_v^0(\mathbb{C})$. □

3.2. **Chaos.** To study the chaos of the translation operator T_a on the space $H_v^0(\mathbb{C})$, we will use the following auxiliary lemma.

Lemma 3.6. *Let v be a weight and $(\lambda_n)_n \subset \mathbb{C}$ a sequence such that $|\lambda_n|$ decreases to zero as $n \to \infty$ and all functions $e^{\lambda_n z}$, $n \in \mathbb{N}$, belong to $H_v^0(\mathbb{C})$. Then the set*

$$\operatorname{span}\{e^{\lambda_n z}; n \in \mathbb{N}\}$$

is dense in $H_v^0(\mathbb{C})$.

Proof. Writing

$$e^{\lambda_n z} - 1 = \lambda_n z \left(1 + \frac{1}{2!}(\lambda_n z) + \frac{1}{3!}(\lambda_n z)^2 + \dots\right), \tag{3.1}$$

we show that $e^{\lambda_n z} \to 1$ in $H_v^0(\mathbb{C})$ as $n \to \infty$. Indeed, fix an arbitrary number $\varepsilon > 0$. By assumption there exist numbers $R > 0$ and $N \geq 2$ such that for all $n \geq N$,

$$\sup_{|z|>R} \frac{|e^{\lambda_n z} - 1|}{v(z)} \leq \sup_{r>R} \frac{|\lambda_2| r e^{|\lambda_2| r}}{v(r)} < \varepsilon \text{ and } \sup_{|z|\leq R} \frac{\left|e^{\lambda_n z} - 1\right|}{v(z)} < \varepsilon.$$

Therefore, $\|e^{\lambda_n z} - 1\|_v < \varepsilon$ for all $n \geq N$. Thus, $e^{\lambda_n z} \to 1$ as $n \to \infty$ in $H_v^0(\mathbb{C})$. This means that the function z^0 belongs to the closure of $\operatorname{span}\{e^{\lambda_n z}; n \in \mathbb{N}\}$ in $H_v^0(\mathbb{C})$.

Next, from (3.1) it follows that

$$\frac{e^{\lambda_n z} - 1}{\lambda_n} - z = z \left(\frac{1}{2!}(\lambda_n z) + \frac{1}{3!}(\lambda_n z)^2 + \dots\right).$$

Arguing as above, we show that

$$\frac{e^{\lambda_n z} - 1}{\lambda_n} \to z \text{ as } n \to \infty \text{ in } H_v^0(\mathbb{C}).$$

This yields that $z^1 \in \overline{\operatorname{span}}\{e^{\lambda_n z}; n \in \mathbb{N}\}$.

Continuing in this way we see that all functions $z^k,\ k \geq 0$, belong to the closure of $\operatorname{span}\{e^{\lambda_n z}; n \in \mathbb{N}\}$ in $H_v^0(\mathbb{C})$.

Thus, the set $\overline{\operatorname{span}}\{e^{\lambda_n z}; n \in \mathbb{N}\}$ contains all polynomials. Hence, $\operatorname{span}\{e^{\lambda_n z}; n \in \mathbb{N}\}$ is dense in $H_v^0(\mathbb{C})$. □

Theorem 3.7. *Suppose that the translation operator $T_a : H_v^0(\mathbb{C}) \to H_v^0(\mathbb{C})$ is continuous. The following assertions are equivalent:*

(i) *T_a is chaotic on $H_v^0(\mathbb{C})$.*
(ii) *T_a has a dense set of periodic points.*
(iii) *T_a has a periodic point different from constant.*
(iv) *$r = O(\log v(r))$ as $r \to \infty$.*

Proof. By Theorem 3.5, T_a is hypercyclic on $H_v^0(\mathbb{C})$. Thus, (i) $\Leftrightarrow$ (ii).

(ii) $\Rightarrow$ (iii): Obvious.

(iii) $\Rightarrow$ (iv): By (iii), there are a non-constant function $h \in H_v^0(\mathbb{C})$ and a number $m \in \mathbb{N}$ such that $h(z + ma) = h(z),\ \forall z \in \mathbb{C}$. This means that h is a non-constant periodic entire function and

$$M_h(r) := \sup\{|h(z)| : |z| = r\} \leq \|h\|_v v(r) \quad \text{for all } r > 0.$$

Then by Jensen's formula it is not difficult to show that there exist numbers $\alpha, c, r_0 > 0$ such that

$$M_h(r) \geq c e^{\alpha r} \quad \text{for all } r \geq r_0.$$

Consequently,

$$\liminf_{r\to\infty} \frac{\log v(r)}{r} = \liminf_{r\to\infty} \frac{\log v(r)}{\log M_h(r)} \frac{\log M_h(r)}{r} \geq \alpha > 0.$$

That is, (iv) is satisfied.

(iv) $\Rightarrow$ (ii): (iv) means that $v(r) \geq e^{\alpha r}$ for some $r_0 > 0$ and all $r \geq r_0$.

Obviously, the functions $e^{\lambda_n z}$ with $\lambda_n = 2\pi i/na,\ n \in \mathbb{N}$, and all functions from their linear span are periodic points of the operator T_a. Since the sequence $|\lambda_n|$ decreases to zero as $n \to \infty$ and $v(r) \geq e^{\alpha r},\ \forall r \geq r_0$, there exists a number $N \in \mathbb{N}$ such that the functions $e^{\lambda_n z}$ belong to $H_v^0(\mathbb{C})$ for all $n \geq N$. Thus, by Lemma 3.6, $\operatorname{span}\{e^{\lambda_n z}; n \geq N\}$ is dense in $H_v^0(\mathbb{C})$.

Consequently, T_a has a dense set of periodic points on $H_v^0(\mathbb{C})$. That is, (ii) holds. □

Using Lemma 3.1 and Theorem 3.7, we obtain the following criteria for the chaos of the operator T_a on $H_\gamma^2(\mathbb{C})$.

Theorem 3.8. *Suppose that the translation operator $T_a : H_\gamma^2(\mathbb{C}) \to H_\gamma^2(\mathbb{C})$ is continuous. The following assertions are equivalent:*

(i) *T_a is chaotic on $H_\gamma^2(\mathbb{C})$.*
(ii) *T_a has a dense set of periodic points on $H_\gamma^2(\mathbb{C})$.*
(iii) *T_a has a periodic point different from constant.*
(iv) *$r = O(\log \gamma(r))$ as $r \to \infty$.*

If, in addition, the comparison function γ is admissble, then $(i)-(iv)$ are equivalent to

(v) *γ is an entire function of order 1 and type $\tau > 0$.*

Proof. As in the proof of Theorem 3.7, (i) $\Leftrightarrow$ (ii) and (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (iv): (iii) means that there is a non-constant periodic entire function $h \in H^2_\gamma(\mathbb{C})$, and hence, by [10, Proposition 1.4], h belongs to $H^\infty_\gamma(\mathbb{C})$. Arguing as above, we show that (iv) holds.

(iv) $\Rightarrow$ (ii): We define

$$\gamma_1(z) := \sum_{n=0}^{\infty} \gamma_{n+2} z^n.$$

Since T_a is continuous on $H^2_\gamma(\mathbb{C})$, by Proposition 2.3, the sequence $n\gamma_n/\gamma_{n-1}$ is bounded. From this it follows that

$$\log \gamma(r) = O(r) \text{ and } \log \gamma_1(r) = O(r),\ r \to \infty.$$

Hence, by Theorem 2.1, T_a is continuous on $H^0_{\gamma_1}(\mathbb{C})$.

Moreover, $r = O(\log \gamma_1(r))$ as $r \to \infty$. Applying Theorem 3.7 to the operator T_a on $H^0_{\gamma_1}(\mathbb{C})$, we have that T_a has a dense set of periodic points on $H^0_{\gamma_1}(\mathbb{C})$.

Clearly, $r^2\gamma_1(r) \le \gamma(r)$, $\forall r > 0$. Then it was shown in [10, Proposition 1.4(b)] that the embedding $H^\infty_{\gamma_1}(\mathbb{C}) \to H^2_\gamma(\mathbb{C})$ is continuous. Consequently, $H^0_{\gamma_1}(\mathbb{C})$ is a dense subspace of $H^2_\gamma(\mathbb{C})$ and the embedding $H^0_{\gamma_1}(\mathbb{C}) \to H^2_\gamma(\mathbb{C})$ is also continuous.

Therefore, by Lemma 3.1, T_a has a dense set of periodic points on $H^2_\gamma(\mathbb{C})$. That is, (ii) is satisfied.

Now suppose that γ is an admissible comparison function on $\mathbb{C}$. Then, by [10, Proposition 1.3], the entire function γ is of order 1 and type $\tau := \lim n\gamma_n/\gamma_{n-1}$. Thus, (iv) $\Rightarrow$ (v).

Conversely, we have that $n\gamma_n/\gamma_{n-1} \ge \tau$ for each $n \in \mathbb{N}$. Hence,

$$\gamma_n \ge \gamma_0 \frac{\tau^n}{n!},\ \forall n \ge 1.$$

Thus, $\gamma(r) \ge \gamma_0 e^{\tau r}$ for every $r \ge 0$. That is, (iv) holds. □

4. Compact perturbation of the identity

In [10, Section 3] Chan and Shapiro proved that the operator $T_a - I$ on weighted Hilbert space $H^2_\gamma(\mathbb{C})$ can be made compact, with approximation numbers decreasing as quickly as desired, choosing a suitable comparison function γ. In this section we obtain a similar result for the operator $T_a - I$ on weighted Banach space $H^0_v(\mathbb{C})$.

Recall that for a bounded linear operator A on Banach space X and a number $n \in \mathbb{N}$, the n^{th} *approximation number of* A, denoted by $\alpha_n(A)$, is defined as follows:

$$\alpha_n(A) := \inf\{\|A - F\| : F \in L(X), \operatorname{rank} F \le n\},$$

where $L(X)$ is the space of all continuous operators $T : X \to X$. According to this definition, the sequence of approximation numbers is decreasing and the operator A is compact when $\alpha_n(A) \to 0$ as $n \to \infty$.

Lemma 4.1. *Suppose that the differentiation operator $D : H^0_v(\mathbb{C}) \to H^0_v(\mathbb{C})$ is continuous. Then, for each $n \in \mathbb{N}$,*

$$\alpha_n(D) \le \sum_{k \ge n} (k+1) \frac{\|z^k\|_v}{\|z^{k+1}\|_v}. \tag{4.1}$$

Proof. Obviously, it suffices to prove (4.1) when the series on its right is convergent.

Given $n \in \mathbb{N}$, consider the finite rank operator

$$P_n : H_v^0(\mathbb{C}) \to H_v^0(\mathbb{C}),$$

$$P_n(f)(z) := \sum_{k=0}^{n-1} a_k z^k \text{ for each } f(z) = \sum_{k=0}^{\infty} a_k z^k \in H_v^0(\mathbb{C}).$$

Clearly, P_n is continuous on $H_v^0(\mathbb{C})$. Hence, by the definition of approximation numbers, $\alpha_n(D) \le \|D - P_n \circ D\|$ for each $n \in \mathbb{N}$.

Fix an arbitrary function $f(z) = \sum_{k=0}^{\infty} a_k z^k \in H_v^0(\mathbb{C})$. Using the classical Cauchy formula, we have that, for all $k \ge 1$ and $\rho > 0$,

$$|a_k| = \frac{|f^{(k)}(0)|}{k!} \le \frac{1}{\rho^k} \max_{|\zeta|=\rho} |f(\zeta)| \le \frac{v(\rho)}{\rho^k} \|f\|_v.$$

Hence,

$$|a_k| \le \|f\|_v \inf_{\rho>0} \frac{v(\rho)}{\rho^k} = \frac{\|f\|_v}{\|z^k\|_v} \quad \text{for all } k \ge 1.$$

Therefore,

$$\begin{aligned} \|Df - P_n \circ Df\|_v &= \|\sum_{k\ge n} (k+1) a_{k+1} z^k\|_v \\ &\le \sum_{k\ge n} (k+1)|a_{k+1}| \|z^k\|_v \\ &\le \sum_{k\ge n} (k+1) \frac{\|z^k\|_v}{\|z^{k+1}\|_v} \|f\|_v. \end{aligned}$$

Consequently, (4.1) holds. □

Theorem 4.2. *Suppose that $(\omega_n)_n$ is a sequence of positive numbers that decreases to zero and $\varepsilon > 0$ is given. Then there exist a weight v and a positive number δ such that the operator T_a is mixing on $H_v^0(\mathbb{C})$ for each $a \neq 0$,*

$$\alpha_n(T_a - I) = o(\omega_n) \text{ as } n \to \infty,$$

and

$$\|T_a - I\| < \varepsilon \text{ for all } |a| < \delta.$$

Proof. We choose the sequence $(\beta_n)_n$ in the following way. Set

$$\beta_0 := -\log \omega_0^2, \ \beta_1 := -\log(\omega_0^2 - \omega_1^2).$$

For each $k \ge 2$ we take a number β_k so that

$$\beta_k \ge \beta_{k-1} - \log(\omega_{k-1}^2 - \omega_k^2) + \log k \ \text{ and } \ \frac{\beta_k + \beta_{k-2}}{2} \ge \beta_{k-1}. \tag{4.2}$$

Obviously, the sequence $(\beta_k - \beta_{k-1})_k$ monotonically increases to ∞ as $k \to \infty$. We construct a function $\varphi : \mathbb{R} \to \mathbb{R}$ as follows:

$$\varphi(x) := \begin{cases} \beta_0, & x \in (-\infty, 0], \\ (\beta_{k+1} - \beta_k)(x-k) + \beta_k, & x \in (k, k+1], \ k \ge 0. \end{cases}$$

Evidently, φ is convex on $\mathbb{R}$ and $x = o(\varphi(x))$ as $x \to +\infty$. Then the function

$$v(r) := \exp \varphi^*(\log r), \ r > 0,$$

is a log-convex weight on $\mathbb{C}$, where, as before, φ^* is the Young conjugate of φ. A simple calculation gives that

$$\|z^k\|_v = \exp\varphi(k) = \exp\beta_k \text{ for all } k \in \mathbb{N}.$$

Consider the differentiation operator $D : H_v^0(\mathbb{C}) \to H_v^0(\mathbb{C})$. Arguing as in the proof of Lemma 4.1, we show that, for each function $f(z) = \sum_{k\geq 0} a_k z^k \in H_v^0(\mathbb{C})$,

$$\|Df\|_v \leq \|f\|_v \sum_{k\geq 0}(k+1)\frac{\|z^k\|_v}{\|z^{k+1}\|_v} = \|f\|_v \sum_{k\geq 0}(k+1)\exp(\beta_k - \beta_{k+1}).$$

From this and (4.2), it is easy to see that $\|Df\|_v \leq C\|f\|_v$ for some $C > 0$ and all $f \in H_v^0(\mathbb{C})$; i.e., the operator D is continuous on $H_v^0(\mathbb{C})$. Hence, the operator T_a is continuous on $H_v^0(\mathbb{C})$ and, by Theorem 3.5, T_a is mixing on $H_v^0(\mathbb{C})$ for each $a \neq 0$. Moreover, by Lemma 4.1 and (4.2), we obtain, for each $n \in \mathbb{N}$,

$$\begin{aligned}\alpha_n(D) &\leq \sum_{k\geq n}(k+1)\frac{\|z^k\|_v}{\|z^{k+1}\|_v}\\ &= \sum_{k\geq n}\exp(\beta_k - \beta_{k+1} + \log(k+1))\\ &\leq \sum_{k\geq n}(\omega_k^2 - \omega_{k+1}^2) = \omega_n^2.\end{aligned}$$

Consequently, $\alpha_n(D) = o(\omega_n),\ n \to \infty$.

Similarly to the proof of [10, Theorem 3.2], we consider the entire function

$$\Phi(z) := \frac{e^{az}-1}{az} = \sum_{k=0}^{\infty}\frac{(az)^k}{(k+1)!}.$$

Then $\Phi(D) := \sum_{k=0}^{\infty}\frac{(aD)^k}{(k+1)!}$ defines a bounded operator on $H_v^0(\mathbb{C})$. Moreover,

$$T_a - I = e^{aD} - I = aD\Phi(D).$$

Therefore,

$$\alpha_n(T_a - I) \leq |a|\|\Phi(D)\|\alpha_n(D) = o(\omega_n) \text{ as } n \to \infty,$$

and

$$\|T_a - I\| \leq |a|\|D\Phi(D)\| < \varepsilon, \text{ if } |a| < \delta := \frac{\varepsilon}{\|D\Phi(D)\|}.$$

□

Remark 4.3. In this section we actually extended the study of compact perturbations by translation operators of the identity on weighted Hilbert spaces $H_\gamma^2(\mathbb{C})$ to weighted Banach spaces $H_v^0(\mathbb{C})$.

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