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ENTROPY FLUX - ELECTROSTATIC CAPACITY - GRAPHICAL MASS

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Abstract. This note shows that the optimal inequality

$$\mathsf{F}(K,\kappa) \le \mathsf{C}(K) \le 2(n-2)\sigma_{n-1}\mathsf{M}(\mathbb{R}^n \setminus K^\circ, \delta + df \otimes df)$$

holds for the entropy flux $\mathsf{F}(K,\kappa)$, the electrostatic capacity $\mathsf{C}(K) = \mathsf{C}(\partial K)$ and the graphical mass $\mathsf{M}(\mathbb{R}^n \setminus K^\circ, \delta + df \otimes df)$ generated by a compact $K \subset \mathbb{R}^{n\ge 3}$ with nonempty interior K° and smooth boundary ∂K.

This note stems from an optimal imbedding of the electrostatic capacity between the entropy flux and the graphical mass. The details for such an embedding process are presented in §2 (the entropy flux existing as a lower bound of the electrostatic capacity) and §3 (the graphical mass existing as an upper bound of the electrostatic capacity) right after §1 (the electrostatic capacity serving as an introduction to the current note).

1. Electrostatic capacity being reviewed

The electrostatic capacity of a compact set K in $\mathbb{R}^{n\ge 3}$ in electrostatics is the maximal charge which can be placed on K when the electric potential of the vector field created by this charge is controlled by 1. To be more precise, if the charge distribution is determined by a Radon measure μ with compact support in $\mathbb{R}^n$ and σ_{n-1} is the surface area of the unit sphere of $\mathbb{R}^n$, then the Newtonian potential generated by μ (with compact support $\operatorname{supp}(\mu) \subset \mathbb{R}^n$) is

$$P_\mu(x) = \big((n-2)\sigma_{n-1}\big)^{-1} \int_{\mathbb{R}^n} |x-y|^{2-n}\, d\mu(y) \quad \forall x \in \mathbb{R}^n,$$

and hence one has the Laplace equation

$$\Delta P_\mu(x) = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2} P_\mu(x) = 0 \quad \forall x \in \mathbb{R}^n \setminus \operatorname{supp}(\mu).$$

When $d\mu = \rho\, d\nu$ (where $d\nu$ is the n-dimensional Lebesgue measure on $\mathbb{R}^n$) and $\rho \in C_c^1(\mathbb{R}^n)$ (where $C_c^1(\mathbb{R}^n)$ comprises all C^1-functions with compact support in $\mathbb{R}^n$), then $-\Delta P_\mu = \rho$ weakly. The desired definition of an electrostatic capacity of K, due to N. Wiener, is given by:

(1.1)

$$\mathsf{C}(K) = \sup\Big\{\mu(K) :\ \text{nonnegative Radon measures } \mu \text{ with } P_\mu(x) \le 1\, \forall\, x \in \mathbb{R}^n \setminus K\Big\}.$$

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As a matter of fact, the supremum in (1.1) is achievable; namely, there is a unique nonnegative Radon measure μ_K supported on the boundary ∂K of K such that $\mathsf{C}(K) = \mu_K(K)$. This maximizing measure is called the equilibrium distribution of the charge. Consequently, the electrostatic capacity has the following alternatives:

- Kelvin's principle (cf. [17, p. 42, Theorem 1.59]):

$$\big(\mathsf{C}(K)\big)^{-1} = \inf\left\{\int_{\mathbb{R}^n} P_\mu\, d\mu : \text{ nonnegative Radon measures } \mu \text{ with } \operatorname{supp}(\mu) \subseteq K \text{ and } \mu(K) = 1\right\} \tag{1.2}$$

  with the convention for (1.2) that $\big(\mathsf{C}(K)\big)^{-1} = \infty$ as $\mathsf{C}(K) = 0$.
- Wiener's principle (cf. [23, p. 4, Definition 1.2]):

$$\mathsf{C}(K) = \inf\left\{\int_{\mathbb{R}^n} |\nabla u|^2\, d\nu = \int_{\mathbb{R}^n} \sum_{j=1}^n \Big(\frac{\partial u}{\partial x_j}\Big)^2 d\nu : \ u \in C^\infty(\mathbb{R}^n),\ u|_K = 1,\ \lim_{|x|\to\infty} u(x) = 0\right\}. \tag{1.3}$$

  Here it is perhaps appropriate to point out that the infimum in (1.3) can be taken over either all $u \in C_c^\infty(\mathbb{R}^n)$ (all C^∞-functions with compact support in $\mathbb{R}^n$), with $u = 1$ in an open set containing K, or all Lipschitz functions u on $\mathbb{R}^n$, with $u = 1$ in a neighbourhood of K.
- Riesz's principle (cf. [14, p. 293, Theorem 11.16]): there are a unique $h \in L^2(\mathbb{R}^n)$ and a dimensional constant c_n such that the convolution $|x|^{1-n} * h \ge 1$ on K and

$$\mathsf{C}(K) = \int_{\mathbb{R}^n} |\nabla(|x|^{1-n} * h)|^2\, d\nu = c_n \int_{\mathbb{R}^n} h^2\, d\nu.$$

As a set function on compact subsets of $\mathbb{R}^n$, $\mathsf{C}(\cdot)$ enjoys the following fundamental properties (cf. [8, p. 28 and p. 32] and [16, Section 2.2]):

- Boundarization: $\mathsf{C}(\partial K) = \mathsf{C}(K)$ for any compact $K \subset \mathbb{R}^n$;
- Monotonicity: $\mathsf{C}(K_1) \le \mathsf{C}(K_2)$ for any compact $K_1, K_2 \subset \mathbb{R}^n$ with $K_1 \subseteq K_2$;
- Strong sub-additivity: $\mathsf{C}(K_1 \cup K_2) + \mathsf{C}(K_1 \cap K_2) \le \mathsf{C}(K_1) + \mathsf{C}(K_2)$ for any compact $K_1, K_2 \subset \mathbb{R}^n$;
- Downward monotone-convergence: $\mathsf{C}(\bigcap_{j=1}^\infty K_j) = \lim_{j\to\infty} \mathsf{C}(K_j)$ for any sequence $(K_j)_{j=1}^\infty$ of compact $K_j \subset \mathbb{R}^n$ with $K_1 \supseteq K_2 \supseteq \cdots \supseteq \bigcap_{j=1}^\infty K_j$;
- Ball capacity: $\mathsf{C}(B(x,r)) = (n-2)\sigma_{n-1} r^{n-2}$ for the Euclidean closed ball $B(x,r)$ with centre x and radius r, while

$$\mathsf{C}(B(0,r) \cap \mathbb{R}^{n-1}) = \Big(\int_0^\infty (\cosh t)^{2-n}\, dt\Big)^{-1} \sigma_{n-1} r^{n-2}.$$

2. Entropy flux bounding electrostatic capacity from below

Assume that K is a compact subset of $\mathbb{R}^{n\geq 3}$ with its interior K° being a nonempty open set and its boundary ∂K being a smooth $(n-1)$-dimensional hypersurface. Given a nonnegative constant κ and a nonnegative Hölder continuous function ϕ which is not identical with zero in K but equals zero in $K^c = \mathbb{R}^n \setminus K$. Consider the Dirichlet problem in $\mathbb{R}^n$:

$$\begin{cases} -\Delta u = \phi \quad \text{in} \quad \mathbb{R}^n; \\ \lim_{|x|\to\infty} u(x) = \kappa. \end{cases}$$

It is well-known that the solution to this problem is

$$u(x) = \kappa + \big((n-2)\sigma_{n-1}\big)^{-1} \int_{\mathbb{R}^n} |x-y|^{2-n}\phi(y)\, d\nu(y) > 0 \quad \forall x \in \mathbb{R}^n.$$

Borrowing the concepts from [4, pp. 18-21] (cf. [5, Chapter 4]), we call

$$\begin{cases} \mathsf{S}(K,\kappa) = \int_K u^{-1}\phi\, d\nu; \\ \mathsf{R}(K,\kappa) = \int_K (u^{-1}|\nabla u|)^2\, d\nu; \\ \mathsf{F}(K,\kappa) = -\int_{\partial K} u^{-1}\nabla u \cdot \mathbf{n}\, d\sigma \end{cases}$$

the entropy supply to K; the rate of generation of entropy in K; the entropy flux through ∂K oriented by its outer unit normal vector $\mathbf{n}$, where $d\sigma$ is the $(n-1)$-dimensional Hausdorff measure.

To proceed, let us make a twofold observation:

- The entropy balance law

$$\mathsf{S}(K,\kappa) + \mathsf{R}(K,\kappa) = \mathsf{F}(K,\kappa)$$

  follows from the well-known Green's theorem. Also, the entropy is actually generated by the mechanism of internal heat conduction.
- In accordance with the Pólya description in [18], up to a constant factor $\mathsf{C}(K)$ can be interpreted as the heat lost by K (treated as a body at constant temperature in a uniform medium with temperature at infinity being zero) in unit time in steady state, and this heat simply propagates through space and can be evaluated on ∂K (serving as the boundary of the body).

Thus, it is natural to discover the following result (2.2), which improves essentially Day's entropy supply inequality in [4, p. 21]: $\mathsf{S}(K,\kappa) \leq (n-2)\sigma_{n-1}\big(\mathrm{diam}(K)\big)^{n-2}$, where $\mathrm{diam}(K)$ is the diameter of K.

Theorem 2.1. *For a compact $K \subset \mathbb{R}^{n\geq 3}$ with smooth boundary ∂K, one has:*

(i)

$$\mathsf{F}(K,\kappa) \leq \mathsf{C}(K) \tag{2.1}$$

with equality if and only if $\kappa = 0 = u|_{\partial K} - \gamma$ for some constant γ;

(ii)

$$\mathsf{F}(K,\kappa) \leq (n-2)\sigma_{n-1}\big(2^{-1}\mathit{diam}(K)\big)^{n-2} \tag{2.2}$$

with equality if and only if $\kappa = 0 = u|_{\partial K} - \gamma$ for some constant γ and $\left(\frac{\mathsf{C}(K)}{(n-2)\sigma_{n-1}}\right)^{\frac{1}{n-2}} = 2^{-1}\mathit{diam}(K)$.

Proof. (i) It is well-known (cf. [3, 13]) that there is a unique solution w to

$$\begin{cases} \Delta w = 0 \quad \text{in} \quad K^c; \\ w|_{\partial K} = 1; \\ \lim_{|x|\to\infty} w(x) = 0, \end{cases}$$

such that

$$\begin{cases} w \in C^\infty(K^c) \cap C(\mathbb{R}^n \setminus K^\circ); \\ 0 < w < 1; \\ |\nabla w| \neq 0 \text{ in } K^c \end{cases}$$

and

$$\mathsf{C}(K) = \int_{K^c} |\nabla w|^2 \, d\nu = -\int_{\partial K} \nabla w \cdot \mathbf{n} \, d\sigma. \tag{2.3}$$

Next, let B_r be the origin-centred Euclidean open ball with the radius r being sufficiently large so that $K \subset B_r$. Then, a direct computation with u and ϕ shows that

$$\sum_{j=1}^n \frac{\partial}{\partial x_j}\Big(w^2 u^{-1} \frac{\partial u}{\partial x_j}\Big) = |\nabla w|^2 - w^2 \phi u^{-1} - \sum_{j=1}^n w^2 \Big(u^{-1}\frac{\partial u}{\partial x_j} - w^{-1}\frac{\partial w}{\partial x_j}\Big)^2.$$

An application of the Green theorem gives

$$\begin{aligned} \int_{\partial B_r} w^2 u^{-1} \nabla u \cdot \frac{x}{|x|} \, d\sigma - \int_{\partial K} w^2 u^{-1} \nabla u \cdot \mathbf{n} \, d\sigma \\ &= \int_{B_r \setminus K} \sum_{j=1}^n \frac{\partial}{\partial x_j}\Big(w^2 u^{-1}\frac{\partial u}{\partial x_j}\Big) \, d\nu \\ &= \int_{B_r\setminus K} |\nabla w|^2 \, d\nu - \int_{B_r \setminus K} w^2 \phi u^{-1} \, d\nu \\ &\quad - \sum_{j=1}^n \int_{B_r\setminus K} w^2 \Big(u^{-1}\frac{\partial u}{\partial x_j} - w^{-1}\frac{\partial w}{\partial x_j}\Big)^2 \, d\nu. \end{aligned} \tag{2.4}$$

Note that

$$\begin{cases} \phi|_{K^c} = 0; \\ w|_{\partial K} - 1 = 0 = \lim_{|x|\to\infty} w(x); \\ \lim_{|x|\to\infty} u(x) = \kappa. \end{cases} \tag{2.5}$$

So, letting $r \to \infty$ in (2.4) implies that

$$-\int_{\partial K} u^{-1} \nabla u \cdot \mathbf{n} \, d\sigma = \int_{K^c} |\nabla w|^2 \, d\nu - \sum_{j=1}^n \int_{K^c} w^2 \Big(u^{-1}\frac{\partial u}{\partial x_j} - w^{-1}\frac{\partial w}{\partial x_j}\Big)^2 \, d\nu \tag{2.6}$$

$$\leq \int_{K^c} |\nabla w|^2 \, d\nu. \tag{2.7}$$

Clearly, (2.1) follows from the definition of entropy flux, (2.3) and (2.6).

Moreover, if equality of (2.1) is valid, then (2.6) is utilized to deduce that

$$\sum_{j=1}^n \int_{K^c} w^2 \Big(u^{-1}\frac{\partial u}{\partial x_j} - w^{-1}\frac{\partial w}{\partial x_j}\Big)^2 \, d\nu = 0,$$

and hence

$$u^{-1}\frac{\partial u}{\partial x_j} = w^{-1}\frac{\partial w}{\partial x_j} \quad \text{in} \quad K^c.$$

Also $\phi = 0$ in K^c, so u is harmonic in K^c. This in turn implies that $u = \gamma w$ for some positive constant γ; however, one has (2.5), thereby getting $\kappa = 0$ and $u|_{\partial K} = \gamma$. Conversely, if $\kappa = 0$ and $u|_{\partial K} = \gamma$ (a positive constant), then an application of the maximum principle for the harmonic function $u - \gamma w$ in K^c yields $u = \gamma w$ in K^c, and thus equality of (2.1) holds.

(ii) According to [25, Lemma 2.1] with $p = 2 < n$ (extending Pólya's case $n = 3$ in [18, (5)] as K is convex), one has the inequality

$$\mathsf{C}(K) \le (n-2)\sigma_{n-1}\big(2^{-1}\mathrm{diam}(K)\big)^{n-2} \tag{2.8}$$

with equality if K is a Euclidean ball. Now, a combination of (2.1) and (2.8) derives (2.2).

If $\kappa = 0 = u|_{\partial K} - \gamma$ for some constant $\gamma > 0$ and

$$\mathsf{C}(K) = (n-2)\sigma_{n-1}\big(2^{-1}\mathrm{diam}(K)\big)^{n-2},$$

then (2.1) and (2.8) take their equalities, and hence equality of (2.2) occurs. Conversely, if (2.2) takes its equality, then (2.1) takes its equality and so does (2.8) due to the fact that $\mathsf{C}(K)$ is between $\mathsf{F}(K,\kappa)$ and $(n-2)\sigma_{n-1}\big(2^{-1}\mathrm{diam}(K)\big)^{n-2}$. Consequently, one has not only $\kappa = 0 = u|_{\partial K} - \gamma$ for some constant γ but also $\mathsf{C}(K) = (n-2)\sigma_{n-1}\big(2^{-1}\mathrm{diam}(K)\big)^{n-2}$. □

3. Graphical mass bounding electrostatic capacity from above

Following [12] we consider the so-called graphical mass which is regarded as a test case for some important and open questions in classical or general relativity. For $f(x) = f(x_1, \dots, x_n)$ and $i, j, k = 1, 2, \dots, n$ we write

$$\begin{cases} f_i = \frac{\partial f}{\partial x_i}; \\ f_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}; \\ f_{ijk} = \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}; \\ \delta_{ij} = \begin{cases} 0 \text{ as } i \ne j \\ 1 \text{ as } i = j. \end{cases} \end{cases}$$

Assume that O is a bounded open set in $\mathbb{R}^{n\ge 3}$ with boundary ∂O. We say that a smooth function $f : \mathbb{R}^n \setminus O \mapsto \mathbb{R}$ is asymptotically flat provided that there is a constant $\gamma > \frac{n}{2} - 1$ such that

$$|f_i(x)| + |x||f_{ij}(x)| + |x|^2|f_{ijk}(x)| = \mathcal{O}(|x|^{-\frac{\gamma}{2}}) \quad \text{as} \quad |x| \to \infty.$$

Now, the graph attached to f,

$$\big(\mathbb{R}^n \setminus O, \delta + df \otimes df\big) = \big(\mathbb{R}^n \setminus O, (\delta_{ij} + f_i f_j)\big),$$

is a complete Riemannian manifold. Then, the ADM (named after three physicists: Arnowitt, Deser, and Misner) mass of this graph, simply, the graphical mass, is determined by

$$\mathsf{M}(\mathbb{R}^n \setminus O, \delta + df \otimes df) = \lim_{r\to\infty} \int_{S_r} \sum_{i,j=1}^{n} \frac{(f_{ii}f_j - f_{ij}f_i)x_j|x|^{-1}}{2(n-1)\sigma_{n-1}(1+|\nabla f|^2)}\, d\sigma,$$

where S_r is the coordinate sphere of radius r. It is perhaps appropriate to mention that this definition of the graphical mass coincides with the definition of the original mass of an asymptotically flat manifold; see [9,10] and [12] containing a brief review on the Riemannian positive mass theorem (cf. Schoen-Yau ([20,21]) and Witten [24]) and its variations (cf. Huisken-Illmanen [11], Bray [1] and Bray-Miao [2]).

Quite interestingly, the next assertion, whose equation (3.2) corresponds nicely to Schwarz's volumetric Penrose inequality in [22, Theorem 1], indicates up to a constant factor that the electrostatic capacity is an optimal lower bound of the graphical mass.

Theorem 3.1. *For a compact $K \subset \mathbb{R}^{n\geq 3}$ let its interior $K^\circ \neq \emptyset$ be smooth, its volume be $V(K)$, and its boundary be ∂K (not necessarily connected); and let it have positive mean curvature $H(\partial K, \cdot)$ (the average of the $n-1$ principal curvatures of ∂K) and satisfy one of the following three conditions:*

- $n - 1 < 7$;
- *∂K is outer-minimizing; namely, if K is contained in L (a smoothly bounded domain), then $\sigma(\partial K) \leq \sigma(\partial L)$;*
- *the solution of inverse mean curvature flow in K^c with initial hypersurface ∂K is always smooth.*

Suppose that $f : \mathbb{R}^n \setminus K^\circ \mapsto \mathbb{R}$ is a smooth asymptotically flat function with:

- *each connected component of ∂K being in a level set of f;*
- $\lim_{x\to\partial K} |\nabla f(x)| = \infty$;
- $R_f = \sum_{j=1}^n \frac{\partial}{\partial x_j} \sum_{i=1}^n \left(\frac{f_{ii}f_j - f_{ij}f_i}{1+|\nabla f|^2}\right) \geq 0$ *and* $\int_{\mathbb{R}^n\setminus K^\circ} R_f\, d\nu < \infty$.

Then:

(i)

$$\big((n-2)\sigma_{n-1}\big)^{-1}\mathsf{C}(K) \leq 2\mathsf{M}(\mathbb{R}^n \setminus K^\circ, \delta + df \otimes df) \tag{3.1}$$

with equality when and only when K is a Euclidean ball and $R_f = 0$ in $\mathbb{R}^n \setminus K^\circ$;

(ii)

$$\Big(\big(n\sigma_{n-1}\big)^{-1}\mathsf{V}(K)\Big)^{\frac{n-2}{n}} \leq 2\mathsf{M}(\mathbb{R}^n \setminus K^\circ, \delta + df \otimes df) \tag{3.2}$$

with equality when and only when K is a Euclidean ball and $R_f = 0$ in $\mathbb{R}^n \setminus K^\circ$.

Proof. (i) Note that Lam's [12, Theorem 6] reveals that

$$\begin{aligned} &2\mathsf{M}(\mathbb{R}^n \setminus K^\circ, \delta + df \otimes df) \\ &\quad= {\sigma_{n-1}}^{-1}\int_{\partial K} H(\partial K, \cdot)\, d\sigma + \big((n-1)\sigma_{n-1}\big)^{-1}\int_{\mathbb{R}^n\setminus K^\circ} R_f\, d\nu. \end{aligned} \tag{3.3}$$

Since R_f is actually the scalar curvature of the graph $(\mathbb{R}^n \setminus K^\circ, \delta + df \otimes df)$ (cf. [12, Lemma 10]), (3.3) may be regarded as the Gauss-Bonnet-like formula for the graphical mass, and then it, along with the hypothesis on K and $R_f \geq 0$ and $\int_{\mathbb{R}^n\setminus K^\circ} R_f\, d\nu < \infty$ as well as [7, Theorem 5 (II)], implies that

$$\big((n-2)\sigma_{n-1}\big)^{-1}\mathsf{C}(K) \leq {\sigma_{n-1}}^{-1}\int_{\partial K} H(\partial K, \cdot)\, d\sigma \leq 2\mathsf{M}(\mathbb{R}^n\setminus K^\circ, \delta+df\otimes df). \tag{3.4}$$

Consequently, (3.1) holds.

Furthermore, if equality of (3.1) is valid, then (3.4) forces

$$\big((n-2)\sigma_{n-1}\big)^{-1}\mathsf{C}(K)={\sigma_{n-1}}^{-1}\int_{\partial K}H(\partial K,\cdot)\,d\sigma \tag{3.5}$$

and

$$ {\sigma_{n-1}}^{-1}\int_{\partial K}H(\partial K,\cdot)\,d\sigma=2\mathsf{M}(\mathbb{R}^n\setminus K^\circ,\delta+df\otimes df). \tag{3.6}$$

Note that (3.5) makes the equality case of [7, Theorem 5 (II)], so, K must be a Euclidean ball. Meanwhile, combining (3.6) and (3.3) implies that $\int_{\mathbb{R}^n\setminus K^\circ}R_f\,d\nu=0$, whence $R_f=0$ in $\mathbb{R}^n\setminus K^\circ$ through utilizing $R_f\ge 0$. Conversely, if K is a Euclidean ball and $R_f=0$ in $\mathbb{R}^n\setminus K^\circ$, then a direct computation with (3.3) shows that (3.1) takes its equality.

(ii) Recalling the well-known isocapacitary inequality for volume (cf. [15] as well as [18, (5)] for $n=3$ whenever K is convex):

$$\Big(\big(n\sigma_{n-1}\big)^{-1}\mathsf{V}(K)\Big)^{\frac{n-2}{n}}\le\big((n-2)\sigma_{n-1}\big)^{-1}\mathsf{C}(K) \tag{3.7}$$

with equality if K is a Euclidean ball, one utilizes (3.1) and (3.7) to gain (3.2).

Moreover, if equality of (3.2) is valid, then an application of (3.7) and (3.1) produces equality of (3.1), and hence K is a Euclidean ball and $R_f=0$ in $\mathbb{R}^n\setminus K^\circ$. Conversely, if this last statement is true, then both (3.1) and (3.7) take their equalities, and hence equality of (3.2) occurs. □

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