BIRCH’S LEMMA OVER GLOBAL FUNCTION FIELDS

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ABSTRACT. We obtain a function field version of Birch’s Lemma, which reveals non-torsion points in quadratic twists of an elliptic curve over a global function field, where the quadratic twists have many prime factors. The proof uses Brown’s Euler system of Heegner points over function fields and a result of Vigni on the ring class eigenspaces of Mordell-Weil groups in positive characteristic.

1. INTRODUCTION AND MAIN RESULTS

In this note, we shall give a function field version of Coates-Li-Tian-Zhai’s generalization of Birch’s Lemma.

1.1. Birch’s Lemma. Let $E$ be an elliptic curve over $\mathbb{Q}$ of conductor $N$, and let $f : X_0(N) \to E$ be a modular parametrization of $E$ such that the cusp $[\infty] \in f^{-1}(O)$. Assume $f([0]) \not\in 2E(\mathbb{Q})$. Assume $l > 3$ is a prime number such that $l \equiv 3 \pmod{4}$ and every prime factor of $N$ splits in $\mathbb{Q}(\sqrt{-l})$, i.e., the Heegner Hypothesis is satisfied for $(\mathbb{Q}(\sqrt{-l}), N)$. Birch showed that if $E^{(-l)}(\mathbb{Q})$ is the quadratic twist of $E$ by $-l$, then the Mordell-Weil group $E^{(-l)}(\mathbb{Q})$ has rank 1.

Recently Birch’s Lemma was generalized by Coates, Li, Tian and Zhai in [CLTZ, §2]. If there is a good supersingular prime $q_1$ for $E$ such that $q_1 \equiv 1 \pmod{4}$ and $N$ is a square module $q_1$, they showed that for any fixed integer $k \geq 1$, there are infinitely many square free integers $M$ with exactly $k$ prime factors, such that the Mordell-Weil rank of the quadratic twist $E^{(M)}$ is 1. In particular, $E = X_0(14)$ with $q_1 = 5$ and $E = X_0(49)$ with $q_1 = 5$ are two examples satisfying the assumptions.

1.2. Heegner points over function fields and Vigni’s result. Let $\mathcal{C}$ be a geometrically connected, smooth, projective algebraic curve over a finite field $\mathbb{F}$ of characteristic $p > 2$. Denote by $F := \mathbb{F}(\mathcal{C})$ the function field of $\mathcal{C}$. Let $\infty$ be a fixed closed point of $\mathcal{C}$ and denote by $\mathcal{O}_F$ the Dedekind domain of elements of $F$ that are regular outside $\infty$. Let $F_\infty$ be the completion of $F$ at $\infty$ and let $\mathcal{C}$ be the completion of a fixed algebraic closure of $F_\infty$.

Suppose $E/F$ is a non-isotrivial (i.e., $j(E) \not\in \mathbb{F}$) elliptic curve defined over $F$. We assume that $E$ has split multiplicative reduction at $\infty$. This assumption is not essential since we can replace $F$ by a suitable finite separable extension and $\infty$ by another closed point. Then the conductor of $E$ can be written as $n\infty$ with $n$ an ideal of $\mathcal{O}_F$. As explained in [GR], there is a non-constant morphism

$$f : X_0(n) \to E$$

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defined over $F$, where $X_0(n)$ is the compactified Drinfeld’s modular curve of level $n$. Fix a cusp $P_0$ defined over $F$; then we can translate the modular parametrization $f$ to ensure that $f^{-1}(O)$ contains $P_0$.

Let $K = F(\sqrt{I})$ ($I \in \mathcal{O}_F$) be a quadratic extension of $F$, and $\mathcal{O}_K$ be the integral closure of $\mathcal{O}_F$ in $K$. Write $\text{Gal}(K/F) = \{1, \tau\}$.

**Assumption I.** The place $\infty$ is ramified in $K$ and the class number $h$ of $\mathcal{O}_K$ is odd.

*Note.* Assumption I means that the class number of $K$ and the degree of $\infty$ are both odd, and the constant field of $K$ is still $F$. By abuse of notation, we denote by $\infty$ the only place of $K$ above $\infty$ and identify $\text{Gal}(K_\infty/F_\infty) = \text{Gal}(K/F)$.

**Assumption II.** The pair $(K, n)$ satisfies the Heegner Hypothesis, i.e., every prime dividing $n$ splits in $K$.

*Note.* By Assumption II, $n\mathcal{O}_K = \mathfrak{M}_n^\tau$ with $\mathfrak{M}$ an ideal of $\mathcal{O}_K$.

Fix a non-zero ideal $\mathfrak{M}$ of $\mathcal{O}_F$ which is prime to $n$. Then we can construct a Drinfeld’Heegner point as follows. Let $\mathcal{O}_{\mathfrak{M}} = \mathcal{O}_F + n\mathcal{O}_K$ be the order of conductor $\mathfrak{M}$ in $\mathcal{O}_K$. The proper ideal $\mathfrak{M}_\mathfrak{M} = \mathfrak{M} \cap \mathcal{O}_K$ of $\mathcal{O}_{\mathfrak{M}}$ satisfies

$$\mathfrak{M}_\mathfrak{M} \cap \mathfrak{M}_\mathfrak{M} \cong \mathcal{O}_K/\mathfrak{M} \cong \mathcal{O}_F/n.$$ 

Thus the two lattices $\mathcal{O}_{\mathfrak{M}}$ and $\mathcal{O}_{\mathfrak{M}}^{-1}$ of $C$ give a pair $(\Phi_{\mathfrak{M}}, \Phi_{\mathfrak{M}}')$ of Drinfeld’s modules of rank 2 with a cyclic $n$-isogeny, hence define a point $P_{\mathfrak{M}}$ on $X_0(n)$. Furthermore, $P_{\mathfrak{M}}$ is defined over the ring class field $H_{\mathfrak{M}}$ of conductor $\mathfrak{M}$ of $K$. As described in [B2] Chapter 2], this field is an abelian extension of $K$ which is unramified outside $\mathfrak{M}$. Moreover, $\infty$ splits completely in $H_{\mathfrak{M}}$, thus we can embed $H_{\mathfrak{M}}$ into $K_\infty$ and we regard $H_{\mathfrak{M}}$ as a subfield of $K_\infty$ from now on.

Denote

$$x_{\mathfrak{M}} = f(P_{\mathfrak{M}}).$$

For a complex character $\chi$ of $G = \text{Gal}(H_{\mathfrak{M}}/K)$, let

$$E(H_{\mathfrak{M}})_C := \{ x \in E(H_{\mathfrak{M}}) \otimes \mathbb{C} : x^\sigma = \chi(\sigma)x \text{ for all } \sigma \in G \}$$

be the $\chi$-eigenspace of $E(H_{\mathfrak{M}}) \otimes \mathbb{C}$. Denote

$$\chi^{-1} \cdot \text{Tr}_{H_{\mathfrak{M}}/K}(x_{\mathfrak{M}}) = \sum_{\sigma \in G} \chi^{-1}(\sigma)\sigma.$$

Vigni in [V] Theorem 1.1] shows that

$$\chi^{-1} \cdot \text{Tr}_{H_{\mathfrak{M}}/K}(x_{\mathfrak{M}}) \neq 0 \text{ in } E(H_{\mathfrak{M}})_C \Rightarrow \dim_{\mathbb{C}} E(H_{\mathfrak{M}})_C = 1.$$ 

For a quadratic extension $K(\sqrt{M})$ of $K$ in $H_{\mathfrak{M}}$ with $M \in \mathcal{O}_F$, let $\chi_M$ be the associated quadratic character. Under certain assumptions, we will show that $\chi_M \cdot \text{Tr}(x_{\mathfrak{M}})$ is non-torsion for some $M$.

**1.3. Main results.** For a finite prime $q$ of $\mathcal{O}_F$, i.e., $q \neq \infty$, denote

$$a_q = #\kappa(q) + 1 - \hat{E}(\kappa(q)),$$

where $\hat{E}$ is the reduced curve of $E$ and $\kappa(q)$ is the residue field of $\mathcal{O}_F$ at $q$. Let $d_q$ be the order of the class of $q$ in the class group of $\mathcal{O}_F$. Let $q^* \in \mathcal{O}_F$ be a generator of $q^{d_q}$ such that $q^*$ is a square in $K_\infty$. This is reachable since $\infty$ is ramified in $K/F$ and any generator of $q^{d_q}$ is of even valuation at $\infty$ in $K_\infty$. Adjusting it by a
suitable root of unity we can make it a square in $K_{\infty}$. Let $q = q^* \text{ or } lq^*$ such that $\tau(\sqrt{q}) = \sqrt{q}$. Denote this by $q_i$ for $q = q_i$.

**Definition.** A finite prime $q$ is called *sensitive* for $E$ if it satisfies (i) $a_q = 0$, (ii) $\#\kappa(q) \equiv 1 \pmod{4}$, and (iii) the Artin symbol $[n, F(\sqrt{q^*})/F] = 1$.

**Assumption III.** The curve $E$ possesses a sensitive prime $q_1$ of $\mathcal{O}_F$ inert in $K$.

**Definition.** For each integer $k \geq 2$, $\Sigma_k$ is the set of finite primes $q \neq q_1$ of $\mathcal{O}_F$ satisfying (i) $a_q \equiv 0 \pmod{2^k}$, (ii) $\#\kappa(q) \equiv 1 \pmod{4}$, (iii) $[n, F(\sqrt{q^*})/F] = 1$, (iv) $q$ is inert in $K$.

Let

$$d_n := \text{ the order of } n \text{ in } \text{Pic}(\mathcal{O}_F)$$

and $n^\dagger$ be a generator of $n^{d_n}$ such that $(-1)^{\text{deg}(n)}n^\dagger$ is a square in $K_\infty$. Then by Hasse’s reciprocity law and the fact that the Hilbert symbol $(q^*, n^\dagger)_\infty = 1$,

$$[q, F(\sqrt{n^\dagger})/F] = [n, F(\sqrt{q^*})/F].$$

**Note.** We will see in Lemma 2.5 that $\Sigma_k$ is infinite if Assumption III is satisfied.

The Atkin-Lehner operator $w_n$ acts on a pair $(D, Z) \in X_0(n)$ of Drinfel’d modules as follows:

$$w_n = \prod_p w_p,$$

$$w_p(D, Z) = (D/Z_{p^k}, (D_{p^k} + Z)/Z_{p^k}),$$

where $p^k || n$ and $D_{p^k}$ (resp. $Z_{p^k}$) is the subgroup scheme of $D$ (resp. $Z$) annihilated by $p^k$. Let

$$w := w_n^{d_n}.$$  

If we compose $f$ with multiplication by a suitable odd integer, we may assume $f(P_0^w)$ is of order a power of 2.

**Assumption IV.** $f(P_0^w) \notin 2E(F)$.

**Theorem A.** Assume Assumptions I-IV. For each integer $k \geq 0$, let $q_2, \ldots, q_k$ be distinct primes in the set $\Sigma_k$ and $M = q_1 \cdots q_k$. Then $E(\sqrt{\sqrt{M}})$, the $\tau = -1$ part of $E(F(\sqrt{\sqrt{M}}))$, is infinite. Moreover, $E^{(IM)}(F)$ has Mordell-Weil rank 1 and the BSD conjecture holds for $E^{(IM)}/F$.

**Theorem B.** Under Assumptions I-IV, if the degree of $q_1$ is even, then for each integer $k \geq 1$ there are infinitely many square-free $M$ having exactly $k$ prime factors, such that $E^{(IM)}(F)$ has Mordell-Weil rank 1 and the BSD conjecture holds for $E^{(IM)}/F$.

2. Proofs of Theorems A and B

2.1. Quadratic subfields.

**Lemma 2.1.** Let $q$ be a finite prime of $\mathcal{O}_F$ unramified in $K$.

i) The order of $q$ in the ideal class group of $\mathcal{O}_F$ divides $h = h(\mathcal{O}_{K})$.

ii) If the size of its residue field $\kappa(q)$ is $\equiv 1 \pmod{4}$, then $H_q$ contains a unique quadratic extension of $K$, which is $K(\sqrt{q})$. 

Proof. i) Let $a$ be a generator of $q^d$ where $d$ is the order of $q$ in $\text{Pic}(\mathcal{O}_F)$. We claim that $d$ is odd. If not, $q^{d/2}\mathcal{O}_K$ is principal since $h$ is odd by Assumption II. Let $b$ be a generator of $q^{d/2}\mathcal{O}_K$; then $b^2 = a\epsilon$ for some $\epsilon \in \mathbb{F}^\times$, and $K = F(\sqrt{a\epsilon})$. Since the degree of $\infty$ is odd, this implies that the valuation of $a\epsilon$ at $\infty$ in $\mathcal{O}_F$ is even, which contradicts the fact that $\infty$ is ramified in $K$.

The order of $q\mathcal{O}_K$ in $\text{Pic}(\mathcal{O}_K)$ divides the greatest common divisor $(d, h)$, the ideal $(q\mathcal{O}_K)^{(d, h)}$ is principal and generated by some $c \in \mathcal{O}_K$. If $d \nmid h$, let $a = d/(d, h)$; then $c \in a^{1/\alpha}\mathbb{F}^\times$. But $\alpha > 2$ is impossible! Hence $d \mid h$.

ii) By class field theory, there is a canonical isomorphism

$$\text{Gal}(H_q/H_K) \cong \frac{(\mathcal{O}_K/q\mathcal{O}_K)^\times}{(\mathcal{O}_F/q)^\times},$$

so $\text{Gal}(H_q/H_K)$ has cardinality $\#\kappa(q) + 1$ (see [B2, (2.3.8)]). By Assumption II, the degree of the extension $H_K/K$ is odd, thus $[H_q : K] = 2 \mod 4$ and there exists a unique quadratic sub-extension, denoted by $K(\sqrt{a'})$, of $H_q/K$.

We see that $q$ is the only prime ramified in $K(\sqrt{a})/K$ and $K(\sqrt{a'})/K$. Then $a'/a$ has even valuations at every finite place, and $(a'/a)\mathcal{O}_K = I^2$ for a fractional ideal $I$ of $\mathcal{O}_K$. Since $h$ is odd, $I$ must be principal, and $K(\sqrt{a'}) = K(\sqrt{\epsilon a})$ with $\epsilon \in \mathbb{F}^\times$. Hence we may assume $a' = \epsilon a$.

Notice that $\infty$ is ramified and $K_\infty$ and $F_\infty$ have the same residue fields. Since $a'$ is a square in $K_\infty$, it follows that $K(\sqrt{a'}) = K(\sqrt{a})$. \qed

### 2.2. Heegner points and the Atkin-Lehner operator

Let $\Lambda, \Lambda'$ be two $\mathcal{O}_F$-lattices of rank 2 in $C$ with $\Lambda'/\Lambda \cong \mathcal{O}_F/n$. They define a pair of Drinfel’d modules with an $n$-isogeny, thus a point on $X_0(n)$, which we denote by $P(\Lambda, \Lambda')$.

For a non-zero ideal $\alpha$ of $\mathcal{O}_F$, the Galois group acts on the set of the Heegner points by

$$P(\alpha, \alpha \mathfrak{n}_{2\mathfrak{m}}^{-1})|_{\alpha, H_\mathfrak{m}/K} = P(\alpha \alpha^{-1}, \alpha \mathfrak{n}_{2\mathfrak{m}}^{-1})$$

where $\alpha$ is a non-zero fractional ideal prime to $\mathfrak{m}\mathfrak{r}$ and $[-, H_\mathfrak{m}/K]$ is the Artin symbol; see [B2, §4.5]. The Atkin-Lehner operator $w_\mathfrak{n}$ acts on the Heegner points by

$$w_\mathfrak{n} P(\alpha, \alpha \mathfrak{n}_{2\mathfrak{m}}^{-1}) = P(\alpha \mathfrak{n}_{2\mathfrak{m}}^{-1}, \alpha \mathfrak{n}^{-1}).$$

Let $w$ be as in [S]. By Lemma 2.1 the order $d_\mathfrak{n}$ of $\mathfrak{n}$ in the ideal class group of $F$ is odd, thus

$$w P(\alpha, \alpha \mathfrak{n}_{2\mathfrak{m}}^{-1}) = P(\alpha \mathfrak{n}_{2\mathfrak{m}}^{-1}, \alpha \mathfrak{n}^{-1/d_\mathfrak{n}}).$$

Let

$$P_{2\mathfrak{m}} := P(\mathcal{O}_{2\mathfrak{m}}, \mathfrak{n}_{2\mathfrak{m}}^{-1}),$$

then (see [B2, 4.6.17])

$$\tau P_{2\mathfrak{m}}^{\mathfrak{n}^{-d_\mathfrak{n}}/2} P_{2\mathfrak{m}}^{\mathfrak{n}^{-d_\mathfrak{n}}/2, H_{2\mathfrak{m}}/K} = w_\mathfrak{n}(P_{2\mathfrak{m}}).$$

Let $H_0 = K(\sqrt{q_1}, \ldots, \sqrt{q_6})$. This is a subfield of $H_{2\mathfrak{m}}$ and $[H_{2\mathfrak{m}} : H_0]$ is odd.

**Lemma 2.2.** Let $S$ be the orbit of $P_{2\mathfrak{m}}$ under the action of $\text{Gal}(H_{2\mathfrak{m}}/H_0)$. Then $wS = \tau S$ set-theoretically.

**Proof.** This is because the restriction of $[\mathfrak{n}^\tau n^d, H_{2\mathfrak{m}}/K]$ to $F(\sqrt{q_i})$ is

$$[n, F(\sqrt{q_i})/F]^{d_\mathfrak{n}} = [n, F(\sqrt{q_i^*})/F]^{d_\mathfrak{n}} = 1.$$  \qed
Lemma 2.3. The operator $w$ has a fixed point on $X_0(n)$.

Proof. Since the degree of $\infty$ in $F$ is odd, we may choose $c \in C - F_\infty$ such that $c^2$ generates $n^2$. Note that $d_n$ is odd by Lemma 2.1 and write $d_n = 2t + 1$. Let $\Lambda = n + n^{-t}c$ and $\Lambda' = \mathcal{O}_F + n^{-t}c$ be two lattices in $C$; then $\Lambda'/\Lambda \cong \mathcal{O}_F/n$ and

$$w\mathcal{P}(\Lambda, \Lambda') = \mathcal{P}(n^{-t}\Lambda', n^{-t-1}\Lambda)$$

$$= \mathcal{P}(n^{-t} + n^{-2t}c^{-1}, n^{-t} + \mathcal{O}_F c)$$

$$= \mathcal{P}(n^{-t}c^{-1} + n^{-2t}c^{-2}, n^{-t}c^{-1} + \mathcal{O}_F)$$

$$= \mathcal{P}(\Lambda, \Lambda') \in X_0(n).$$

That is to say, $P(\Lambda, \Lambda')$ is a fixed point of $w$. □

Lemma 2.4. The morphism $f + f \circ w : X_0(n) \to E$ is constant.

Proof. We can write $f$ as the composite of

$$X_0(n) \to J_0(n) = \text{Jac}(X_0(n)) \xrightarrow{g} A = J_0(n)/(T_p - a_p; p \nmid n) \xrightarrow{h} E.$$ 

Here $T_p$ is the $p$-th Hecke operator and $h$ is an isogeny. Let $f_A : P \mapsto g([P] - [P_0])$ be the composite of the first two maps.

By definition, $w$ is a linear involution on $J_0(n)$ as

$$w([P] - [P_0]) = [P^w] - [P_0^w].$$

It induces a linear involution $w = \pm 1$ on $A$ since $wT_n = T_n \circ w$.

If $w = +1$, then

$$(f_A - f_A \circ w)(P) = w(f_A - f_A \circ w)(P)$$

$$= w \circ g(([P] - [P_0]) - ([P^w] - [P_0^w])) = w \circ g([P] - [P^w])$$

$$= g([P^w] - [P]) = (f_A \circ w - f_A)(P).$$

The image of $f_A - f_A \circ w$ lies in $A[2]$, which is finite. Thus $f_A - f_A \circ w$ is a constant. Let $Q$ be a fixed point of $w$; then

$$f_A(P_0^w) = f_A(P_0^w) - f_A(P_0) = f_A(Q^w) - f_A(Q) = O$$

and $f(P_0^w) = O$, which contradicts Assumption. Hence $w = -1$.

On one hand,

$$2g([P] + [P^w] - [P_0] - [P_0^w]) = f_A(P) + f_A(P^w) + wf_A(P) + w f_A(P^w) = 0.$$ 

On the other hand,

$$g([P] + [P^w] - [P_0] - [P_0^w])$$

$$= (f_A + f_A \circ w)(P) - g([P_0^w] - [P_0])$$

$$= (f_A + f_A \circ w)(P) - f_A(P_0^w).$$

The image of $f_A + f_A \circ w$ lies in $f_A(P_0^w) + A[2]$, which is finite. Thus $f_A + f_A \circ w$ is constant, and so is $f + f \circ w = f(P_0^w)$. □

Lemma 2.5. Assume $E$ possesses a sensitive prime $q_1$ of $\mathcal{O}_F$ inert in $K$. Then for each integer $k \geq 2$, $\Sigma_k$ is infinite of positive density in the set of primes.
Proof. Set $J = F(\sqrt{n}, E[2^k])$; then $K \cap J = F$ and $q_1$ is unramified in $J$. There is a unique element $\sigma$ in $\Delta = \text{Gal}(JK/F)$ whose restriction to $K$ is $\tau$ and whose restriction to $J$ is the Frobenius automorphism of some prime of $J$ above $q_1$.

Assume $q$ is a finite prime not dividing $lq_1n$, whose Frobenius automorphisms in $\Delta$ lie in the conjugate class of $\sigma$. The characteristic polynomials of the Frobenius automorphisms of $q_1$ and $q$ acting on the 2-adic Tate module $T_2(E)$ are $X^2 + \#(\kappa(q_1))$ and $X^2 + a_qX + \#(\kappa(q))$, respectively. Since $E[2^k] = T_2(E)/2^kT_2(E)$, we have $a_q \equiv 0 \mod 2^k$ and $\#(\kappa(q)) \equiv \#(\kappa(q_1)) \mod 2^k$. Also $q$ is inert in $K$ since $q_1$ is inert in $K$, and $q$ splits in $F(\sqrt{n})$ since $q_1$ splits in this field. Hence $\Sigma_k$ contains all such primes and it follows that $\Sigma_k$ is infinite of positive density in the set of all primes by the Chebotarev density theorem. 

**Lemma 2.6.** We have $E(H_0)[2^\infty] = E(F)[2]$.

**Proof.** Since in every subfield of $H_0$ which is strictly larger than $F$, at least one prime dividing $lq_1 \cdots q_k$ ramifies, but only the primes dividing $2a\infty$ may ramify in the field $F(E[2^\infty])$, we have

$$E(H_0)[2^\infty] = E(F)[2^\infty] = E(F)[2].$$

Note that $q_1$ is a sensitive prime for $E$, reduction modulo $q_1$ is injective on $E(F)[2^\infty]$, and there are $\#(\kappa(q_1)) + 1$ points with coordinates in $\kappa(q_1)$ on the reduced curve $E$. It follows that $E(F)[2^\infty]$ has order at most 2.

**2.3. Euler system.** For a factor $d$ of $\mathfrak{M}$, let $d = \prod_{q_i \mid d} q_i$. The following result ensures that Heegner points form an Euler system, as in the classical case (see [12], (4.6.8), (4.8.3)): 

**Proposition 2.7.** For $q \mid \frac{\mathfrak{M}}{d}$, we have $\text{Tr}_{H_q/H_0} x_q = a_q x_d$.

Let $\psi_M = \text{Tr}_{H_0/H_0} x_{2\mathfrak{M}}$. Define $K((\sqrt{d}))$-points $y_d, z_d$ of $E$ by

$$z_d := \chi_d^{-1} \text{Tr}_{H_{2\mathfrak{M}}/K}(x_{d}) = \chi_d^{-1} \text{Tr}_{H_0/K}(\psi_M),$$

$$y_d := \chi_d^{-1} \text{Tr}_{H_{d}/K}(x_d).$$

Then $z_M = y_M$ and $z_d = b_d y_d$ where $b_d = \prod_{q \mid \frac{\mathfrak{M}}{d}} a_q = 2^k e_d$ for $d \neq \mathfrak{M}$.

**2.4. End of proof.**

**Proof of Theorem A.** If $k = 0$, $y_1 = \text{Tr}_{H/K}(x_1)$, $y_1 + \tau(y_1) = h(\mathcal{O}_K) f(P_0^w) = f(P_0^w)$. If $y_1$ is torsion, then there is an odd number $m$ such that $my_1 \in E(K)[2^\infty] = E(F)[2]$. It follows that $f(P_0^w) = m(y_1 + \tau(y_1)) = 2my_1$, which contradicts Assumption [IV]. Hence $y_1$ is non-torsion, and so is $2y_1 \in E(K)^\times$.

Now assume $k \geq 1$. Let $\sigma \in \text{Gal}(H_0/K)$ which maps $\sqrt{q_1}$ to $-\sqrt{q_1}$. Then $\sqrt{q_1}$ for $i > 1$. Then

$$\sigma(\psi_M) + \psi_M = \text{Tr}_{H_{\mathfrak{M}}/K(\sqrt{q_1}, i > 1)}(x_{\mathfrak{M}}) = a_{q_1} \text{Tr}_{H_{2\mathfrak{M}}/K(\sqrt{q_1}, i > 1)}(x_{\mathfrak{M}}) = 0.$$ 

Since $a_{q_1} = 0$,

$$\sigma(v_M) + v_M = \text{Tr}_{H_{\mathfrak{M}}/K(x_{\mathfrak{M}})} = a_{q_1} \text{Tr}_{H_{2\mathfrak{M}}/K(x_{\mathfrak{M}})} = 0,$$

where

$$v_M = \text{Tr}_{H_{\mathfrak{M}}/K(\sqrt{M})}(x_{\mathfrak{M}}) = \text{Tr}_{H_0/K(\sqrt{M})}(\psi_{2\mathfrak{M}}).$$
Then $y_M = v_M - \sigma(v_M) = 2v_M, \sigma(y_M) + y_M = 0$.
By Lemma 2.2 and Lemma 2.4, we have
\[ \psi_M + \tau(\psi_M) = [H_\infty : H_0] f(P_w^\infty) = f(P_0^w). \]
Thus $y_M + \tau(y_M) = 2(v_M + \tau(v_M)) = 0$. Hence $y_M \in E(F(\sqrt{M}))^{-}$. Similarly, we have $y_d + \tau(y_d) = 0$ if $q_1 \nmid d$.
By the definition of $y_d$, we have
\[ y_M + \sum_{d \mid M, d \neq M} z_d = 2^k \psi_M. \]
Let
\[ u_M = \psi_M - \sum_{d \mid M, d \neq M} e_d y_d; \]
then $y_M = 2^k u_M$. Since $e_d = 0$ if $q_1 \nmid d$, it follows that $u_M + \tau(u_M) = f(P_0^w)$.
If $u_M$ is torsion, then there is an odd number $m$ such that $mu_M \in E(H_0)[2^\infty] = E(F)[2]$. It follows that $f(P_0^w) = m(u_M + \tau(u_M)) = 2mu_M$, which contradicts Assumption IV. Hence $u_M$ is non-torsion, and so is $y_M$.
The rest of the proof is similar to [V] Theorem 7.1. By [V] Theorem 6.1, we can take a suitable rational prime $t$ such that the $F_t$-vector space $\text{Sel}_t(E/H_\infty)^{\chi_M}$ is one-dimensional and $E[t](H_\infty) = 0$. Since the Selmer groups can be controlled via the injections
\[ \text{Sel}_t(E/F(\sqrt{M}))^{\chi_M} \hookrightarrow \text{Sel}_t(E/K(\sqrt{M}))^{\chi_M} \hookrightarrow \text{Sel}_t(E/H_\infty)^{\chi_M}, \]
they must be all one-dimensional $F_t$-vector spaces.
We know that $E(\iota_M)(F) \cong E(F(\sqrt{M}))^{-}$. By injectivity of the restriction map, $\dim_{F_t} \text{Sel}_t(E(\iota_M)/F) = 1$ and $\text{III}(E(\iota_M)/F)[t] = 0$. By the result of Tate, Milne, Kato and Trihan ([V] Theorem 7.2), the conjecture of BSD holds for $E(\iota_M)/F$.

**Proof of Theorem B.** If the degree of $q_1$ is even, the 2-adic valuation of $\# \kappa(q_1)$ is $r \geq 2$; then the 2-adic valuation of $\# F - 1$ is less than $r$. Take $q_2, \ldots, q_k$ in $\Sigma_{k+r}$. We have $\# \kappa(q) \equiv \# \kappa(q_1) \mod 2^r$ as in Lemma 2.5. Thus the degree of $q_i$ is even and then $q_i = q_i^e$. Therefore, $M$ has exactly $k$ prime factors and the result follows from Lemma 2.5.

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**References**


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