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INEQUALITY ON $t_\nu(K)$ DEFINED BY LIVINGSTON AND NAIK AND ITS APPLICATIONS

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ABSTRACT. Let $D_+(K,t)$ denote the positive t-twisted double of K. For a fixed integer-valued additive concordance invariant ν that bounds the smooth four genus of a knot and determines the smooth four genus of positive torus knots, Livingston and Naik defined $t_\nu(K)$ to be the greatest integer t such that $\nu(D_+(K,t)) = 1$. Let K_1 and K_2 be any knots; then we prove the following inequality: $t_\nu(K_1) + t_\nu(K_2) \leq t_\nu(K_1 \# K_2) \leq min(t_\nu(K_1) - t_\nu(-K_2), t_\nu(K_2) - t_\nu(-K_1))$. As an application we show that $t_\tau(K) \neq t_s(K)$ for infinitely many knots and that their difference can be arbitrarily large, where $t_\tau(K)$ (respectively $t_s(K)$) is $t_\nu(K)$ when ν is an Ozváth-Szabó invariant τ (respectively when ν is a normalized Rasmussen s invariant).

1. Introduction

Let ν be any integer-valued concordance invariant with the following properties:

(1) additive under connected sum,
(2) $|\nu(K)| \leq g_4(K)$,
(3) $\nu(T_{p,q}) = (p-1)(q-1)/2$ for $p, q > 0$.

Notice that the Ozváth-Szabó invariant τ satisfies the above properties [OS03], as does the Rasmussen s invariant when suitably normalized (i.e. when $\nu = -s/2$) [Ras10]. Let $D_\pm(K,t)$ denote the positive or negative t-twisted double of K. Then for a fixed concordance invariant ν, Livingston and Naik [LN06] show that $\nu(D_+(K,t))$ is always 1 or 0 (see Theorem 2.1) and define $t_\nu(K)$ to be the greatest integer t such that $\nu(D_+(K,t)) = 1$. Specializing to τ and s, we have the two concordance invariants $t_\tau(K)$ (respectively $t_s(K)$), which is the greatest integer t where $\tau(D_+(K,t)) = 1$ (respectively $-s(D_+(K,t))/2 = 1$). Hedden and Ording [HO08] show that there exist K for which $t_\tau(K) \neq t_s(K)$. In particular they show that $t_\tau(T_{2,2n+1}) = 2n-1$ whereas $t_s(T_{2,3}) \geq 2$, $t_s(T_{2,5}) \geq 5$, and $t_s(T_{2,7}) \geq 8$. (In fact, it is easy to verify that $t_s(T_{2,3}) = 2$ and $t_s(T_{2,5}) = 5$ using Bar-Natan's program [BN]). This was the first example known of a knot K for which $\tau(K) \neq -s(K)/2$. (Note that it is proven that $\tau \neq -s/2$ even for topologically slice knots by Livingston [Liv08].) Further they make a remark that it would be reasonable to guess that $t_s(T_{2,2n+1}) = 3n-1$, which would imply that $t_\tau(K) \neq t_s(K)$ for infinitely many different knots.

Hedden [Hed07] showed that $t_\tau(K)$ does not give more information than $\tau(K)$:

Theorem 1.1 ([Hed07, Theorem 1.5]). $t_\tau(K) = 2\tau(K) - 1$.

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However, $t_s(K)$ is not well understood. In this paper we show the following inequality:

Theorem 1.2. *Let K_1 and K_2 be any knots and ν any integer-valued concordance invariant with properties* (1), (2), *and* (3) *as above. Then the following inequality holds:*

$$t_\nu(K_1) + t_\nu(K_2) \le t_\nu(K_1 \# K_2) \le min(t_\nu(K_1) - t_\nu(-K_2), t_\nu(K_2) - t_\nu(-K_1)).$$

We have the following as the immediate corollary:

Corollary 1.3. *For any positive integer n, there exists a knot K_n such that*

$$|t_\tau(K_n) - t_s(K_n)| > n.$$

Proof. Let K_n be an n connected sum of $T_{2,5}$. Then by Theorem 1.2 and the fact that $t_\tau(K_n) = 2n \cdot \tau(T_{2,5}) - 1 = 4n - 1$ and $t_s(T_{2,5}) \ge 5$ by Theorem 1.1 and [HO08], the result follows. □

We end this section with the following remark:

Remark 1.4. If we assume that $t_s(K)$ is a polynomial of $-s(K)/2$ with integer coefficients, it is easy to verify that $t_s(K) = 3 \cdot (-s(K)/2) - 1$ using Theorem 1.2.

2. Proof of Theorem 1.2

We will denote by $TB(K)$ the maximum value of the Thurston-Bennequin number, taken over all possible Legendrian representatives of K. Recall the following theorem from [LN06]:

Theorem 2.1 ([LN06, Theorem 2])**.** *For each knot K there is an integer $t_\nu(K)$ such that*

$$\nu(D_+(K,t)) = \begin{cases} 0 & \text{for } t > t_\nu(K), \\ 1 & \text{for } t \le t_\nu(K), \end{cases}$$

where $t_\nu(K)$ satisfies $TB(K) \le t_\nu(K) < -TB(-K)$.

A similar result holds for $D_-(K,t)$ using $t_\nu(-K)$:

$$\nu(D_-(K,t)) = \begin{cases} -1 & \text{for } t \ge -t_\nu(-K), \\ 0 & \text{for } t < -t_\nu(-K), \end{cases}$$

where $t_\nu(-K)$ satisfies $TB(-K) \le t_\nu(-K) < -TB(K)$.

Now, we are ready to prove Theorem 1.2. The proof completely relies on Theorem 2.1.

Proof of Theorem 1.2. Let t_1 and t_2 be integers and consider $D_+(K_1,t_1)\# D_+(K_2,t_2)$ and $D_+(K_1\#K_2, t_1+t_2)$. Then there is a genus one cobordism from $D_+(K_1,t_1)\#D_+(K_2,t_2)$ to $D_+(K_1\#K_2,t_1+t_2)$ (see Figure 1). Hence if $\nu(D_+(K_1,t_1)) = \nu(D_+(K_2,t_2)) = 1$, then $\nu(D_+(K_1\#K_2,t_1+t_2)) = 1$. Letting $t_1 = t_\nu(K_1)$ and $t_2 = t_\nu(K_2)$, we have $\nu(D_+(K_1,t_1)) = \nu(D_+(K_2,t_2)) = 1$ by Theorem 2.1. Using Theorem 2.1 again we have $t_\nu(K_1) + t_\nu(K_2) \le t_\nu(K_1\#K_2)$.

Using a similar argument, notice that there is a genus one cobordism from $D_+(K_1,t_1)\# D_-(K_2,t_2)$ to $D_+(K_1\#K_2,t_1+t_2)$ by simply changing the sign of the clasp in Figure 1. Therefore if $\nu(D_+(K_1,t_1)) = 0$ and $\nu(D_-(K_2,t_2)) = -1$, then $\nu(D_+(K_1\#K_2,t_1+t_2)) = 0$ by Theorem 2.1. Letting $t_1 = t_\nu(K_1)+1$ and $t_2 = -t_\nu(-K_2)$, we have $\nu(D_+(K_1,t_1)) = 0$ and $\nu(D_-(K_2,t_2)) = -1$ by Theorem 2.1. Using Theorem 2.1 again we have $t_\nu(K_1) + 1 - t_\nu(-K_2) \ge t_\nu(K_1\#K_2) + 1$,

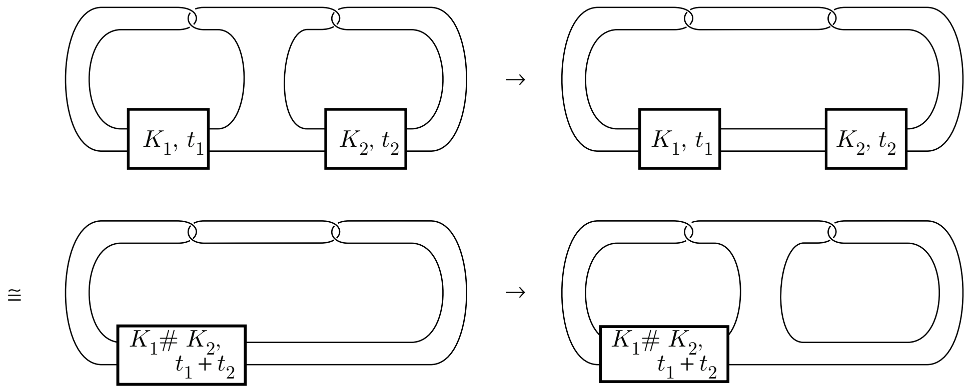

FIGURE 1. A genus one cobordism from $D_+(K_1, t_1)\#D_+(K_2, t_2)$ to $D_+(K_1\#K_2, t_1 + t_2)$. The top left figure is $D_+(K_1, t_1)\#D_+(K_2, t_2)$; the top right figure is obtained from the top left figure after one band sum; the bottom left figure is obtained from the top right figure after an isotopy; and the bottom right figure is obtained from the bottom left figure after one band sum, and it is isotopic to $D_+(K_1\#K_2, t_1 + t_2)$.

hence $t_\nu(K_1\#K_2) \leq t_\nu(K_1) - t_\nu(-K_2)$. Finally, by switching the roles of K_1 and K_2 we also get $t_\nu(K_1\#K_2) \leq t_\nu(K_2) - t_\nu(-K_1)$, which completes the proof. □

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