PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 145, Number 2, February 2017, Pages 617–626
http://dx.doi.org/10.1090/proc/13414
Article electronically published on October 13, 2016

ON THE RADON-NIKODYM PROPERTY IN FUNCTION SPACES

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(Communicated by Thomas Schlumprecht)

ABSTRACT. We exhibit a large class of Banach function spaces which fail to have the Radon-Nikodym property.

The identification of Banach spaces which possess the Radon-Nikodym property (RNP), or do not possess it, began with a penetrating paper of Dunford and Pettis, [6], and has continued ever since. Large classes of Banach spaces for which the answer is known can be found in [5], [10], [12], for example, and the references therein.

This note is motivated by the recent papers [1], [7]. In [1] Astashkin and Maligranda proved that the separable Banach function space (B.f.s.)

$$Ces_p(I) := \left\{ f : x \mapsto \frac{1}{x} \int_0^x |f(y)|\, dy \in L^p(I) \right\}, \quad 1 \le p < \infty,$$

is not a dual Banach space (with $I = [0,1]$ or $I = [0,\infty)$), from which they deduce that $Ces_p(I)$ fails RNP. In [7] Kamińska and Kubiak consider the weighted (separable) Cesàro spaces

$$Ces_{p,w}(I) := \left\{ f : x \mapsto w(x) \int_0^x |f(y)|\, dy \in L^p(I) \right\}$$

and prove that they fail RNP, from which they deduce that $Ces_{p,w}$ cannot be a dual Banach space. It will be shown, via different methods, that the B.f.s.'s $Ces_{p,w}(I)$, which include $Ces_p(I)$, are particular examples from a significantly larger class of B.f.s.'s, all of which fail RNP.

A Banach space X has RNP if, for every finite measure space (Ω, Σ, μ) and every μ-continuous vector measure of finite variation $m\colon \Sigma \to X$ there exists a Bochner μ-integrable function $F \in L^1(\mu, X)$ such that $m(A) = \int_A F\, d\mu$ for all $A \in \Sigma$. If a Banach space X has RNP, then so does each of its closed subspaces [5, III Theorem 3.2].

Recall that a *Banach function space* E over a σ-finite measure space (Ω, Σ, μ) is a Banach space of classes of measurable functions on Ω satisfying the ideal property; that is, $g \in E$ and $\|g\|_E \le \|f\|_E$ whenever $f \in E$ and $|g| \le |f|$ μ–a.e. We denote

Received by the editors January 11, 2016.
2010 *Mathematics Subject Classification.* Primary 46G05, 46E30; Secondary 28B05.
Key words and phrases. Radon-Nikodym property, Banach function space, vector measure, kernel operator.
The first author acknowledges the support of the "International Visiting Professor Program 2015" from the Ministry of Education, Science and Art, Bavaria (Germany), and of MTM2012-36732-C03-03, MINECO (Spain).

by E^+ the cone in E consisting of all $f \in E$ satisfying $f \geq 0$ μ-a.e. For further details concerning B.f.s.'s we refer to [9], [13].

Theorem 1. *Let E be a B.f.s. over a measure space (Ω, Σ, μ). The following conditions are assumed to hold.*

(a) *There exists a non-null set $\Delta \in \Sigma$ such that the restricted measure space $(\Delta, \Sigma_\Delta, \mu)$ is finite and non-atomic.*

(b) *There exists $\psi \in E$, with support Δ and a constant $M > 0$, such that*

$$\sum_{n=1}^{\infty} \left\|\psi\chi_{A_n}\right\|_E \leq M, \quad \forall\ \Sigma\text{-partitions } (A_n)_1^\infty \text{ of } \Delta.$$

Then E does not have RNP.

Proof. The proof is via a series of steps.

Step 1. We first prove the theorem in the case when $\psi = \chi_\Omega$. Then $\Delta = \Omega$, and hence (Ω, Σ, μ) is a finite, non-atomic measure space. Define the finitely additive, E-valued measure m by

$$m \colon A \mapsto m(A) := \chi_A \in E, \quad A \in \Sigma.$$

Condition (b) implies, for any sequence of pairwise disjoint sets $(A_n)_1^\infty \subseteq \Sigma$, that the series $\sum_1^\infty m(A_n)$ in E is absolutely convergent. Since the sequence $m(\bigcup_1^m A_n) = \chi_{\bigcup_1^m A_n}$, for $m \in \mathbb{N}$, converges pointwise a.e. to $m(\bigcup_1^\infty A_n) = \chi_{\bigcup_1^\infty A_n}$, it follows that $\sum_1^\infty m(A_n) = m(\bigcup_1^\infty A_n)$. Accordingly, m is σ-additive; i.e., it is a vector measure. Moreover, m is absolutely continuous with respect to μ as they have the same null sets. From (b), m has finite variation.

Let η be a *Rybakov control measure* for m; that is, $\eta = |x^*m|$ for a suitable element $x^* \in E^*$ (with $\|x^*\| = 1$) and such that η and m have the same null sets [5, IX Theorem 2.2]. Here E^* is the Banach space dual of E and $|x^*m|$ is the variation measure of the $\mathbb{R}$-valued measure $x^*m : A \mapsto \langle x^*, m(A)\rangle$ for $A \in \Sigma$. Then E is also a B.f.s. over (Ω, Σ, η). For a simple function $s = \sum_1^n a_i\chi_{A_i}$ with $(A_i)_1^n \subseteq \Sigma$ pairwise disjoint, it follows from the Hahn-decomposition of x^*m that

$$\begin{aligned}
\|s\|_{L^1(\eta)} &= \Big\|\sum_1^n a_i\chi_{A_i}\Big\|_{L^1(\eta)} = \sum_1^n |a_i|\eta(A_i) = \sum_1^n |a_i| \cdot |x^*m|(A_i) \\
&= \sum_1^n |a_i|\big((x^*m)(A_i^+) - (x^*m)(A_i^-)\big) \\
&= x^*\Big(\sum_1^n |a_i|(m(A_i^+) - m(A_i^-))\Big) \\
&\leq \|x^*\|_{E^*}\Big\|\sum_1^n |a_i|(m(A_i^+) - m(A_i^-))\Big\|_E \\
&= \Big\|\sum_1^n |a_i|(\chi_{A_i^+} - \chi_{A_i^-})\Big\|_E = \Big\|\sum_1^n a_i\chi_{A_i}\Big\|_E = \|s\|_E.
\end{aligned}$$

Let $f \in E$ and $\{s_n\}$ be simple functions such that $0 \leq s_n \leq s_{n+1} \uparrow |f|$ μ-a.e. (hence, η-a.e.). Then $\|s_n\|_{L^1(\eta)} \leq \|s_n\|_E \leq \|f\|_E$ and so $\sup_n \|s_n\|_{L^1(\eta)} \leq \|f\|_E$.

Hence, $f \in L^1(\eta)$ and $\|f\|_{L^1(\eta)} \le \|f\|_E$ for all $f \in E$. For each $g \in L^\infty(\eta)$ it follows that

$$\Big| \int_\Omega fg\, d\eta \Big| \le \|g\|_\infty \|f\|_{L^1(\eta)} \le \|g\|_\infty \|f\|_E, \quad f \in E.$$

Hence, the linear functional $\varphi_g \colon f \to \int_\Omega fg\, d\eta$ is continuous on E with $\|\varphi_g\|_{E^*} \le \|g\|_\infty$. Since $\varphi_g = 0$ if and only if $g = 0$, this provides an embedding of $L^\infty(\eta)$ into E^*.

Suppose that E has RNP. Then there exists an E-valued Bochner η-integrable function $F \colon w \mapsto F_w \in E$, for $w \in \Omega$, such that $m(A) = \int_A F_w\, d\eta(w)$, for $A \in \Sigma$; that is, from the definition of m,

$$\text{(1)} \qquad \chi_A = \int_A F_w\, d\eta(w), \quad A \in \Sigma.$$

It will be shown that (1) leads to a contradiction.

Since $\chi_A \in E^+$, for $A \in \Sigma$, we can conclude from (1) that $\int_A F_w\, d\eta(w) \in E^+$, for $A \in \Sigma$. Applying the Hahn-Banach theorem to the set of averages

$$\left\{ \frac{1}{\eta(A)} \int_A F_w\, d\eta(w) : A \in \Sigma, \eta(A) > 0 \right\} \subseteq E^+,$$

with E^+ a closed, convex subset of E and F being E-valued, it follows from [6, Theorem 1.2.7] that

$$\text{(2)} \qquad F_w \in E^+, \quad \text{a.e. } w \in \Omega.$$

Step 2. Fix any $B \in \Sigma$ and set $B^c = \Omega \setminus B \in \Sigma$. Via the identification in Step 1 the bounded function $\chi_{B^c} \in E^*$ and so from (1), with $A := B$, we have

$$\big\langle \chi_{B^c}, \chi_B \big\rangle = \int_B \big\langle \chi_{B^c}, F_w \big\rangle\, d\eta(w).$$

Via the same identification $\big\langle \chi_{B^c}, \chi_B \big\rangle = \int_\Omega \chi_{B^c}(t)\chi_B(t)\, d\eta(t) = 0$, from which it follows that

$$\text{(3)} \qquad \int_B \big\langle \chi_{B^c}, F_w \big\rangle\, d\eta(w) = 0.$$

Regarding the function being integrated in (3), note that (2) implies that

$$w \mapsto \big\langle \chi_{B^c}, F_w \big\rangle = \int_{B^c} F_w(t)\, d\eta(t) \ge 0.$$

This, together with (3), implies that there exists $Z_B \in \Sigma$ with $Z_B \subseteq B$ and $\eta(Z_B) = 0$ such that

$$\text{(4)} \qquad \big\langle \chi_{B^c}, F_w \big\rangle = \int_{B^c} F_w(t)\, d\eta(t) = 0, \quad w \in B \setminus Z_B.$$

Again from (2) and (4) it follows that there exists $N_{B^c} \in \Sigma$ with $N_{B^c} \subseteq B^c$ and $\eta(N_{B^c}) = 0$ such that

$$\text{(5)} \qquad F_w(t) = 0, \quad t \in B^c \setminus N_{B^c}, \quad w \in B \setminus Z_B.$$

Step 3. Since (Ω, Σ, η) is non-atomic, we can define a double-indexed sequence of non-null sets

$$\Big\{ \Delta_n^k : n \in \mathbb{N}, 1 \le k \le 2^n \Big\} \subseteq \Sigma,$$

which is a dyadic decomposition of Ω. This is achieved as follows. We begin with $\Delta_0^1 := \Omega$ and continue inductively to define, for a fixed $n \in \mathbb{N}$ and $1 \le k \le 2^n$, the measurable sets

$$\Delta_{n+1}^{2k-1} \cup \Delta_{n+1}^{2k} = \Delta_n^k, \quad \Delta_{n+1}^{2k-1} \cap \Delta_{n+1}^{2k} = \emptyset, \quad \eta(\Delta_{n+1}^{2k-1}) = \eta(\Delta_{n+1}^{2k}).$$

Note, for each point $w \in \Omega$, that there is a unique sequence $\{\Delta_n^{k_n}\}$, with $1 \le k_n \le 2^n$, which satisfies

$$\Delta_{n+1}^{k_{n+1}} \subseteq \Delta_n^{k_n}, \quad w \in \Delta_n^{k_n}, n \in \mathbb{N}, \quad \bigcap_{n=1}^{\infty} \Delta_n^{k_n} = \{w\}.$$

Since η is non-atomic, the set $\bigcup_n (\Delta_n^{k_n})^c$ has full measure.

We apply (5) to each set Δ_n^k, for each $n \in \mathbb{N}$ and $1 \le k \le 2^n$, to deduce that

$$F_w(t) = 0, \quad t \in (\Delta_n^k)^c \setminus N_{n,k}, \quad w \in \Delta_n^k \setminus Z_{n,k}, \tag{6}$$

where the sets $N_{n,k}$ and $Z_{n,k}$ have measure zero. Define $Z = \bigcup_{n,k}(N_{n,k} \cup Z_{n,k})$, which has measure zero. Then from (6), for each $n \in \mathbb{N}$ and $1 \le k \le 2^n$, we have

$$F_w(t) = 0, \quad t \in (\Delta_n^k)^c \setminus Z, \quad w \in \Delta_n^k \setminus Z. \tag{7}$$

Fix $w_0 \in \Omega \setminus Z$ and let $\{\Delta_n^{k_n}\}$ be the corresponding unique sequence of sets as specified above with $\{w_0\} = \bigcap_{n=1}^{\infty} \Delta_n^{k_n}$. Then $w_0 \in \Delta_n^{k_n} \setminus Z$ for all $n \in \mathbb{N}$. By applying (7) to each set of this sequence we obtain

$$F_{w_0}(t) = 0, \quad t \in (\Delta_n^{k_n})^c \setminus Z, \quad n \in \mathbb{N}.$$

Since the set $\left(\bigcup_n (\Delta_n^{k_n})^c\right) \setminus Z$ has full measure, it follows that $F_{w_0}(t) = 0$ a.e. $t \in \Omega$, i.e., $F_{w_0} = 0 \in E$. But this holds for every $w_0 \in \Omega \setminus Z$, and hence $F_w = 0 \in E$ for a.e. $w \in \Omega$, which contradicts (1). This completes the proof if $\psi = \chi_\Omega$.

Step 4. For the general case, let $\psi \in E$ be the function given in the statement of the theorem and Δ be its support. There exists $\varepsilon > 0$ such that the set $\Delta' := \{w \in \Delta : |\psi(w)| \ge \varepsilon\}$ has non-zero measure. Then, for any $A \in \Sigma$ with $A \subseteq \Delta'$, we have $\chi_A \le \frac{1}{\varepsilon}|\psi|\chi_A$. Hence, for any partition (A_n) of Δ' it follows from (b) that

$$\sum_{n=1}^{\infty} \|\chi_{A_n}\|_E \le \frac{1}{\varepsilon} \sum_{n=1}^{\infty} \|\psi \chi_{A_n}\|_E \le \frac{M}{\varepsilon}.$$

Consider the continuous linear projection P acting in E by $f \mapsto P(f) := f\chi_{\Delta'} \in E$ and let $\tilde{E} := P(E) \subseteq E$ be its range. Then $\tilde{E}$ is a B.f.s. over the non-zero, finite, non-atomic measure space $(\Delta', \Sigma_{\Delta'}, \eta)$. Moreover, $\tilde{\psi} := \chi_{\Delta'}$ satisfies (b) in the statement of the theorem for the B.f.s. $\tilde{E}$. Applying Steps 1–3 to $\tilde{E}$ over $(\Delta', \Sigma_{\Delta'}, \eta)$ and for $\tilde{\psi}$, it follows that $\tilde{E}$ fails RNP. Since $\tilde{E}$ is a closed subspace of E, E also fails RNP. $\square$

We recall the optimal domain of a *kernel operator*. Given a measurable function $K\colon (x, y) \in [0, 1] \times [0, 1] \mapsto K(x, y) \in [0, \infty]$, its associated kernel operator T_K is defined by

$$T_K f(x) := \int_0^1 K(x, y) f(y)\, dy, \quad x \in [0, 1], \tag{8}$$

for any $f \in L^0$ for which it is meaningful to do so, i.e., such that $T_K f \in L^0$. Here L^0 is the space of all classes of a.e. $\mathbb{R}$-valued measurable functions defined on $[0, 1]$.

For X a B.f.s. space on $[0,1]$, the *optimal domain* of T_K (with values in X) is the linear space defined by

$$[T_K, X] := \Big\{ f \in L^0 : T_K(|f|) \in X \Big\},$$

which becomes a B.f.s. when endowed with the norm

$$\|f\|_{[T_K,X]} := \|T_K(|f|)\|_X, \quad f \in [T_K, X].$$

A B.f.s. X over (Ω, Σ, μ) satisfies the *Fatou property* if $\{f_n\} \subseteq X$ with $0 \le f_n \le f_{n+1} \uparrow f$ μ-a.e. and $\sup_n \|f_n\|_X < \infty$ imply that $f \in X$ and $\|f_n\|_X \to \|f\|_X$. The *associate space* X' of X consists of all measurable functions g satisfying $\int_\Omega |fg|\, d\mu < \infty$, for every $f \in X$. The space X' is a subspace of the Banach space dual X^* of X. The second associate space X'' of X is defined as $X'' = (X')'$. The B.f.s. X has the Fatou property if and only if $X'' = X$. In this case, X' is norming for X and $f \in X$ if and only if $\int_\Omega |fg|\, d\mu < \infty$, for every $g \in X'$. Recall that a *rearrangement invariant* (r.i.) space X on $[0,1]$ is a B.f.s. on $[0,1]$ such that if $g^* \le f^*$ and $f \in X$, then $g \in X$ and $\|g\|_X \le \|f\|_X$. Here f^* is the decreasing rearrangement of f, that is, the right continuous inverse of its distribution function: $\lambda_f(\tau) := \lambda(\{t \in [0,1] : |f(t)| > \tau\})$, where λ is Lebesgue measure. An r.i. space X always satisfies $L^\infty \subseteq X \subseteq L^1$. For further details concerning r.i. spaces we refer to [9].

Corollary 2. *Let $K\colon (x,y) \in [0,1] \times [0,1] \mapsto K(x,y) \in [0,\infty]$ be a measurable kernel and X an r.i. space on $[0,1]$ with the Fatou property such that each partial function $K_y \in X$, for $y \in [0,1]$, and the function $y \mapsto \|K_y(\cdot)\|_X$ is measurable on $[0,1]$, where $K_y\colon x \mapsto K(x,y)$. Let T_K be as in (8). Suppose there exist a non-null measurable set $\Delta \subseteq [0,1]$ and a measurable function ψ, supported in Δ, such that*

$$\int_\Delta |\psi(y)|\, \|K_y\|_X \, dy < \infty. \tag{9}$$

Then, for every r.i. space Y on $[0,1]$ with $X \subseteq Y$, the B.f.s. $[T_K, Y]$ does not have RNP.

Proof. Requirement (a) of Theorem 1, for $E = [T_K, X]$, is clear. Moreover, condition (9) implies that requirement (b) of Theorem 1 also holds. To see this let $A \subseteq \Delta$ be any measurable set. Via Fubini's theorem we have

$$\begin{aligned}
\sup_{g \in B^+_{X'}} \int_0^1 g(x) T_K(|\psi|\chi_A)(x)\, dx &= \sup_{g \in B^+_{X'}} \int_0^1 g(x) \int_A |\psi(y)| K(x,y)\, dy\, dx \\
&= \sup_{g \in B^+_{X'}} \int_A |\psi(y)| \int_0^1 g(x) K(x,y)\, dx\, dy \\
&\le \int_A |\psi(y)| \Big(\sup_{g \in B^+_{X'}} \int_0^1 g(x) K(x,y)\, dx \Big)\, dy \\
&= \int_A |\psi(y)|\, \|K_y\|_X \, dy < \infty.
\end{aligned}$$

Since X has the Fatou property, it follows that $T_K(|\psi|\chi_A) \in X$ and $\|T_K(|\psi|\chi_A)\|_X \le \int_A |\psi(y)|\, \|K_y\|_X \, dy$. Hence, $\psi\chi_A \in [T_K, X]$ with $\|\psi\chi_A\|_{[T_K,X]} \le \int_A |\psi(y)|\, \|K_y\|_X \, dy$.

Moreover, for any measurable partition $(A_n)_1^\infty$ of Δ, it follows that

$$\sum_{n=1}^\infty \|\psi\chi_{A_n}\|_{[T_K,X]} \leq \sum_{n=1}^\infty \int_{A_n} |\psi(y)|\,\|K_y\|_X\,dy = \int_\Delta |\psi(y)|\,\|K_y\|_X\,dy < \infty.$$

Suppose now that $X \subseteq Y$. Then $\|u\|_Y \leq M\,\|u\|_X$ for $u \in X$ and some $M > 0$. Hence, for each measurable set $A \subseteq \Delta$, we have

$$\|\psi\chi_A\|_{[T_K,Y]} = \|T_K(|\psi|\chi_A)\|_Y \leq M\|T_K(|\psi|\chi_A)\|_X = M\|\psi\chi_A\|_{[T_K,X]}.$$

Accordingly, requirement (b) of Theorem 1 also holds for $E = [T_K, Y]$. □

Concerning the measurability of $y \mapsto \|K_y\|_X$ in Corollary 2, if X has a.c. norm (a B.f.s. E has *absolutely continuous norm* if for every $f \in E$ we have $\lim_{\mu(A)\to 0} \|f\chi_A\|_E = 0$), then X is separable and $X^* = X'$, and so it suffices that $y \mapsto \int_0^1 K(x,y)g(x)\,dx$ is measurable for each $g \in X'$ [5, II Theorem 1.2]. Sometimes certain monotonicity properties of the kernel K ensure the measurability of $y \mapsto \|K_y\|_X$.

Corollary 2 applies to many different situations, for example, to the following well known kernels on $[0,1] \times [0,1]$:

(i) The Volterra kernel $K(x,y) := \chi_{[0,x]}(y)$.
(ii) The Poisson semigroup kernel $K(x,y) := \arctan(y/x)$ for $x \neq 0$ and $K(0,y) = \pi/2$.
(iii) The kernel associated with Sobolev's inequality in $\mathbb{R}^n$, namely $K(x,y) := y^{(1/n)-1}\chi_{[0,y]}(x)$ for $n \geq 2$.
(iv) The Riemann-Liouville fractional kernel $K(x,y) := |x-y|^{\alpha-1}$ for $0 < \alpha < 1$.
(v) The Cesàro kernel $K(x,y) := (1/x)\chi_{[0,x]}(y)$.
(vi) The generalized Cesàro kernel $K(x,y) := w(x)\chi_{[0,x]}(y)$.

We check condition (9) of Corollary 2 for each of these examples.

For case (i), given any r.i. space X on $[0,1]$ set $\Delta = [0,1]$ and $\psi = \chi_{[0,1]}$. Then $K_y \in L^\infty \subseteq X$ for $y \in [0,1]$. For the measurability condition observe that $y_1 \leq y_2$ implies that $K_{y_1} \geq K_{y_2} \geq 0$ pointwise for $x \in [0,1]$. Since $\|\cdot\|_X$ is a lattice norm, $\|K_{y_1}\|_X \geq \|K_{y_2}\|_X$. Thus, $y \mapsto \|K_y\|_X$ is monotone decreasing and so is measurable. This, together with

$$\int_0^1 \|K_y\|_X\,dy \leq \int_0^1 \|K_y\|_\infty\,dy < \infty,$$

shows that the corresponding optimal domain $[T_K, X]$ fails RNP.

Case (ii) follows in a similar way, with the only change being that the function $y \mapsto \|K_y\|_X$ is now monotone increasing.

For case (iii), set $\Delta = [0,1]$ and $\psi = \chi_{[0,1]}$. Note that $K_y = y^{(1/n)-1}\chi_{[0,y]} \in L^\infty$ and that $y \mapsto \|K_y\|_\infty = y^{(1/n)-1}$, for $0 < y \leq 1$, is a measurable function satisfying $\int_0^1 \|K_y\|_\infty\,dy < \infty$. Hence, for any r.i. space X on $[0,1]$, in which case $L^\infty \subseteq X$, the corresponding optimal domain $[T_K, X]$ fails RNP.

Concerning case (iv), let $p \geq 1$ satisfy $p(1-\alpha) < 1$. Then, for $\Delta = [0,1]$ and $\psi = \chi_{[0,1]}$, we have that $K_y \in L^p$ for $y \in [0,1]$ and

$$\int_0^1 \|K_y\|_p\,dy = \frac{1}{(1+p(\alpha-1))^{1/p}} \int_0^1 \left(y^{1+p(\alpha-1)} + (1-y)^{1+p(\alpha-1)}\right)^{1/p} dy < \infty.$$

So, if $L^p \subseteq X$ with X r.i., then the optimal domain $[T_K, X]$ fails RNP.

For case (v), consider any r.i. space X on $[0,1]$. Each function $K_y \in L^\infty \subseteq X$, for $y \in [0,1]$, and $y \mapsto \|K_y\|_X$ is monotone decreasing and so is measurable. Fix any $0 < a < 1$ and set $\Delta = [a,1]$ and $\psi = \chi_{[a,1]}$. Then

$$\begin{aligned} \int_a^1 \|K_y\|_X \, dy &= \int_a^1 \Big\| x \mapsto \frac{1}{x}\chi_{[y,1]}(x) \Big\|_X \, dy \\ &\le \int_a^1 \frac{1}{y} \big\|\chi_{[y,1]}\big\|_X \, dy \le \frac{1}{a}\big\|\chi_{[0,1-a]}\big\|_X . \end{aligned}$$

Hence, the optimal domain $[\mathcal{C}, X]$ fails RNP. Observe that $[\mathcal{C}, L^p] = Ces_p(I)$, $1 \le p < \infty$.

In case (vi), following [7, §3], we consider w to satisfy (I) $w > 0$ a.e. in $I = [0,l]$ for $l = 1$ or $l = \infty$; (II) $\int_a^l w(x)^p \, dx < \infty$ for all $a > 0$; (III) $\int_0^l w(x)^p \, dx = \infty$. Fix $0 < a < 1$ and set $\Delta = [a,1]$ and $\psi = \chi_{[a,1]}$. In view of condition (II) the function $K_y = w\chi_{[y,1]} \in L^p$ for $0 < y < 1$ and $y \mapsto \|K_y\|_p$ is measurable as it is monotone decreasing. Via conditions (I) and (II) we then have

$$\begin{aligned} \int_a^1 \|K_y\|_p \, dy &= \int_a^1 \big\| x \mapsto w(x)\chi_{[y,1]}(x) \big\|_p \, dy \\ &\le \int_a^1 \big\| x \mapsto w(x)\chi_{[a,1]}(x) \big\|_p \, dy \\ &= \int_a^1 \Big(\int_a^1 |w(x)|^p \, dx \Big)^{1/p} dy \\ &= (1-a)\Big(\int_a^1 |w(x)|^p \, dx \Big)^{1/p} < \infty. \end{aligned}$$

Hence, the optimal domain $Ces_{p,w}(I)$ fails to have RNP.

For further information concerning the B.f.s.'s $[T_K, X]$, with K as in (i)-(iv), see [11, §4.3] and the references therein. The B.f.s.'s $[T_K, X] = [\mathcal{C}, X]$ of (v) are thoroughly investigated in [4].

We recall briefly the construction of the space of real functions which are integrable with respect to a vector measure. Let (Ω, Σ) be a measurable space, X be a Banach space with dual space X^* and closed unit ball B_{X^*}, and $m\colon \Sigma \to X$ be a σ-additive vector measure. A measurable function $f\colon \Omega \to \mathbb{R}$ is called *scalarly m–integrable* if $f \in L^1(|x^*m|)$, for every $x^* \in X^*$. The function f is *m–integrable* if, in addition, for each $A \in \Sigma$ there exists a vector in X (denoted by $\int_A f \, dm$) such that $\langle x^*, \int_A f \, dm\rangle = \int_A f \, dx^*m$, for every $x^* \in X^*$. The m–integrable functions form a linear space in which

$$\|f\|_{L^1(m)} := \sup\Big\{ \int_\Omega |f| \, d|x^*m| : x^* \in B_{X^*} \Big\} \tag{10}$$

is a seminorm. Identifying functions which differ $\|m\|$-a.e., we obtain a Banach space (of classes) of m–integrable functions, denoted by $L^1(m)$. Here, $\|m\|$ is the *semivariation* of m defined by $\|m\|(A) := \sup\{|x^*m|(A) : x^* \in B_{X^*}\}$, for $A \in \Sigma$. The space $L^1(m)$ is a B.f.s. over (Ω, Σ, η) for η any Rybakov control measure for m. For further facts related to the spaces $L^1(m)$ see [11].

Returning briefly to kernel operators, if $L^\infty \subseteq [T_K, X]$ and $A \mapsto m_K(A) := T_K(\chi_A)$ is σ-additive in X, then the B.f.s. $L^1(m_K) \subseteq [T_K, X]$. This is the case for each kernel K listed in (i)–(v) above; see [4, Proposition 3.1], which also applies to the mentioned kernels. The containment $L^1(m_K) \subseteq [T_K, X]$ can be proper. For instance, this is so for the Cesàro kernel $K(x,y) := (1/x)\chi_{[0,x]}(y)$ in (v) above when X is the Marcinkiewicz space $L^{p,\infty}$, $1 < p < \infty$ [4, Proposition 3.2]. Choosing $X = L^\infty$ for this kernel it turns out that $L^\infty \subsetneq [T_K, L^\infty]$, but now m_K is only finitely additive and not σ-additive [4, §2]. Hence, the space "$L^1(m_K)$" is not available. The corresponding B.f.s. $[T_K, L^\infty] = Ces_\infty(I)$ is the well known Korenblyum-Kreĭn-Levin space. It can also happen that $L^\infty \nsubseteq [T_K, X]$, in which case the measure "m_K" is not available at all. Indeed, for the generalized Cesàro kernel $K(x,y) = w(x)\chi_{[0,x]}(y)$ in (vi) above, with $X = L^p$ for $1 \le p < \infty$, the weight $w(x) = 1/x^2$ satisfies conditions (I)–(III), but $\chi_{[0,1]} \notin [T_K, L^p]$.

Theorem 3. *Let (Ω, Σ) be a measurable space, X a Banach space and $m\colon \Sigma \to X$ a vector measure. Suppose that there exists a measurable set on which the variation measure of m is finite, non-zero and non-atomic. Then the B.f.s. $L^1(m)$ does not have RNP.*

Proof. The desired conclusion follows from Theorem 1 for $f = \chi_\Delta$, with Δ the set whose existence is asserted in the statement of the theorem, provided that we verify condition (b) in Theorem 1. This is achieved by showing that the $L^1(m)$-valued vector measure

$$A \in \Sigma \mapsto \tilde{m}(A) := \chi_A \in L^1(m)$$

has its variation measure $|\tilde{m}|$ equal to that of the X-valued vector measure m, and hence $|\tilde{m}|$ is finite.

Since $\|\tilde{m}(A)\|_{L^1(m)} = \|\chi_A\|_{L^1(m)} = \|m\|(A) \ge \|m(A)\|_X$, $A \in \Sigma$, it follows that $|\tilde{m}|(A) \ge |m|(A)$, $A \in \Sigma$. Concerning the reverse inequality, fix $A \in \Sigma$ and $\varepsilon > 0$. Then there is $x^* \in B_{X^*}$ such that $\|m\|(A) \le (1+\varepsilon)|x^*m|(A)$. The real measure x^*m has a Hahn decomposition, so that $|x^*m|(A) = (x^*m)(A^+) - (x^*m)(A^-)$, where the measurable sets A^+ and A^- are both contained in A, are disjoint and satisfy $A = A^+ \cup A^-$. So, $\|\tilde{m}(A)\|_{L^1(m)} = \|m\|(A) \le (1+\varepsilon)\big(\|m(A^+)\|_X + \|m(A^-)\|_X\big)$. Now, let $(A_n)_1^\infty$ be a measurable partition of $A \in \Sigma$. Then, also $(A_n^+)_1^\infty \cup (A_n^-)_1^\infty$ is a measurable partition of A and so

$$\sum_{n=1}^\infty \|\tilde{m}(A_n)\|_{L^1(m)} \le (1+\varepsilon) \sum_{n=1}^\infty \big(\|m(A_n^+)\|_X + \|m(A_n^-)\|_X\big) \le (1+\varepsilon)|m|(A).$$

Hence, $|\tilde{m}|(A) \le (1+\varepsilon)|m|(A)$, $A \in \Sigma$. This holds for every $\varepsilon > 0$, and hence $|\tilde{m}|(A) \le |m|(A)$. Thus, $|\tilde{m}|(A) = |m|(A)$, $A \in \Sigma$. □

The assumptions in Theorem 3 are best possible. Indeed, if E is any B.f.s. with a.c. norm and a weak unit (over a finite measure space), then the E-valued vector measure $m(A) := \chi_A$, for $A \in \Sigma$, has the property that $L^1(m) = E$ with equal norms (i.e., the identity map of $L^1(m)$ onto E is a surjective isometry); see the proof of Theorem 8 of [2]. For $L^1(m) = L^p([0,1])$, $1 < p < \infty$, with m as above, in which case $L^1(m)$ is reflexive, it follows that $L^1(m)$ has RNP. Of course, m is non-atomic but fails to have finite variation on every set of positive measure. Similarly, for each $1 < p < \infty$, it is known that $\ell^p = L^1(\nu)$ for a suitable ℓ^p-valued vector measure ν having finite variation [11, Example 3.69]. Again $L^1(\nu)$ has RNP. This time there is no set of positive measure on which ν is non-atomic.

As an application of Theorem 3, consider the vector measure m defined on $\mathcal{M}$, the Lebesgue measurable subsets of $[0,1]$, by

$$A \in \mathcal{M} \mapsto m(A) := \left(\int_{\frac{1}{n+1}}^{\frac{1}{n}} \chi_A(t) \frac{dt}{t} \right)_1^\infty \in c_0.$$

It is clearly well defined and finitely additive. For each $\xi = (\xi_n)_1^\infty \in \ell^1$ we have

$$A \mapsto \langle \xi, m(A) \rangle = \sum_{n=1}^\infty \xi_n \int_A \chi_{\left[\frac{1}{n+1}, \frac{1}{n}\right)}(t) \frac{dt}{t}, \quad A \in \mathcal{M}, \tag{11}$$

which is σ-additive. From the Orlicz-Pettis theorem it follows that m is also σ-additive. It has infinite variation since

$$\sum_{n=1}^\infty \left\| m\left(\left[\frac{1}{n+1}, \frac{1}{n}\right)\right) \right\|_\infty = \sum_{n=1}^\infty \int_{\frac{1}{n+1}}^{\frac{1}{n}} \frac{dt}{t} = \int_0^1 \frac{dt}{t} = \infty.$$

However, for $A \subset \left[\frac{1}{n+1}, \frac{1}{n}\right)$, we have $\|m(A)\|_\infty \le (n+1)\lambda(A)$. Hence, $|m|\left(\left[\frac{1}{n+1}, \frac{1}{n}\right)\right) < \infty$ and so m has σ-finite variation. It is routine to verify that $|m|(A) = \int_A \frac{dt}{t}$ for $A \in \mathcal{M}$. Since $|m|$ is non-atomic, so is m. It is immediate from (11), for a fixed $\xi \in \ell^1$, that $f \in L^0$ is $|\langle \xi, m \rangle|$-integrable if and only if

$$\int_0^1 |f| \, d|\langle \xi, m \rangle| = \sum_{n=1}^\infty |\xi_n| \int_{\frac{1}{n+1}}^{\frac{1}{n}} \frac{|f(t)|}{t} \, dt < \infty.$$

Hence, f is scalarly m-integrable precisely when $\left(\int_{\frac{1}{n+1}}^{\frac{1}{n}} \frac{|f(t)|}{t} \, dt \right)_1^\infty \in \ell^\infty$ and m-integrable if and only if $\left(\int_{\frac{1}{n+1}}^{\frac{1}{n}} \frac{|f(t)|}{t} \, dt \right)_1^\infty \in c_0$. Then Theorem 3 implies that the B.f.s.

$$L^1(m) = \left\{ f \in L^0 : \left(\int_{\frac{1}{n+1}}^{\frac{1}{n}} \frac{|f(t)|}{t} \, dt \right)_1^\infty \in c_0 \right\}$$

fails RNP where, from m being a positive vector measure, it follows via [11, Lemma 3.13] that the norm in $L^1(m)$ is given by

$$\|f\|_{L^1(m)} = \left\| \left(\int_{\frac{1}{n+1}}^{\frac{1}{n}} \frac{|f(t)|}{t} \, dt \right)_1^\infty \right\|_{c_0}, \quad f \in L^1(m). \tag{12}$$

It should be noted that $L^1(m)$ is not isomorphic to $L^1(\xi)$ for any positive measure ξ (from which it would follow directly that $L^1(m)$ fails RNP). This follows from the fact that, in such a case, the variation of m is necessarily finite [3, Proposition 2].

For a vector measure $m \colon \Sigma \to X$, let $L^1_w(m)$ denote the space of all (classes of) functions which are scalarly m-integrable. It is a B.f.s. for the same norm (10) as used in $L^1(m)$ and contains $L^1(m)$ as a closed subspace [11, Ch. 3, §1]. The containment $L^1(m) \subseteq L^1_w(m)$ can be proper [11, Example 3.34]. Whereas the B.f.s. $L^1(m)$ always has a.c. norm, the importance of $L^1_w(m)$ is that it is the *Fatou completion* of $L^1(m)$, that is, the smallest of all B.f.s.'s which contain $L^1(m)$ and have the Fatou property [13, §71, Theorem 2]. In other words, $L^1_w(m) = L^1(m)''$.

For $L^1([0,1])$, part (i) of the following result is classical.

Corollary 4. *Let (Ω, Σ) be a measurable space, X a Banach space and $m \colon \Sigma \to X$ a vector measure whose restriction to some set in Σ has non-zero, finite and non-atomic variation. Then the following assertions hold.*

(i) $L^1(m)$ *is not a dual Banach space.*
(ii) $L^1_w(m)$ *fails RNP.*

Proof. (i) Since the Banach space $L^1(m)$ is weakly compactly generated [2, Theorem 2] and fails RNP (by Theorem 3), it follows from a result of Kuo [8, Theorem 2.2] that $L^1(m)$ cannot be a dual Banach space.

(ii) Apply Theorem 3 and the fact that $L^1(m)$ is a closed subspace of $L^1_w(m)$. □

Returning to the vector measure $m\colon \mathcal{M} \to c_0$ given above, it was shown there that

$$L^1_w(m) = \left\{ f \in L^0 : \left(\int_{\frac{1}{n+1}}^{\frac{1}{n}} \frac{|f(t)|}{t}\, dt \right)_1^\infty \in \ell^\infty \right\}.$$

The norm in $L^1_w(m)$ is given by the right side of (12) with ℓ^∞ in place of c_0. Corollary 4(ii) shows that $L^1_w(m)$ fails RNP. A consideration of the function $f(t) = 1/t$ shows that the containment $L^1(m) \subseteq L^1_w(m)$ is proper.

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