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# FURSTENBERG ENTROPY VALUES FOR NONSINGULAR ACTIONS OF GROUPS WITHOUT PROPERTY (T)

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ABSTRACT. Let $G$ be a discrete countable infinite group that does not have Kazhdan's property (T) and let $\kappa$ be a generating probability measure on $G$. Then for each $t > 0$, there is a type $III_1$ ergodic free nonsingular $G$-action whose $\kappa$-entropy (or the Furstenberg entropy) is $t$.

## 0. INTRODUCTION

Let $G$ be a discrete countable infinite group. A probability measure $\kappa$ on $G$ is called *generating* if the support of $\kappa$ generates $G$ as a semigroup. Let $T = (T_g)_{g\in G}$ be a nonsingular action of $G$ on a standard probability space $(X, \mathfrak{B}, \mu)$. The *Furstenberg entropy* (or *$\kappa$-entropy*) of $T$ is defined by

$$h_\kappa(T,\mu) := -\sum_{g\in G}\kappa(g)\int_X \log\frac{d\mu\circ T_g}{d\mu}(x)d\mu(x)$$

(see [9]). Jensen's inequality implies that $h_\kappa(T,\mu) \geq 0$ and that (for generating measures) equality holds if and only if $\mu$ is invariant under $T$. Of course, the $\kappa$-entropy is invariant under conjugacy. If $\sum_{g\in G}\kappa(g)\frac{d\mu\circ T_g}{d\mu}(x) = 1$ for a.e. $x \in X$, then $T$ is called *$\kappa$-stationary*. The *Furstenberg entropy realization problem* is to describe all values that $\kappa$-entropy takes on the set of $\kappa$-stationary actions. The problem appears quite difficult. Some progress was achieved in recent papers [15], [14], [3], [12]. To state one of the results on the entropy realization problem we first recall that $G$ has Kazhdan's *property (T)* if every unitary representation of $G$ which has almost invariant vectors admits a nonzero invariant vector (see [2]). It was shown in [14] that if $G$ has property (T), then for every generating measure $\kappa$, the pair $(G, \kappa)$ has an *entropy gap*; i.e. there exists some constant $\epsilon = \epsilon(G, \kappa) > 0$ such that the $h_\kappa(T) > \epsilon$ for each purely infinite ergodic stationary $G$-action $T$. We recall that an ergodic action is called *purely infinite* if it does not admit an equivalent invariant probability measure. In [4] the converse statement was proved: if $T$ does not have property (T), then for each generating measure $\kappa$,

$$\inf\{h_\kappa(T,\mu) \mid T \text{ is purely infinite, ergodic, } \mu\text{-nonsingular action of } G\} = 0. \tag{0-1}$$

We note that the authors of [4] consider $\kappa$-entropy values on arbitrary (not only stationary, as in the other aforementioned papers) purely infinite nonsingular actions. They also show that the entropy gap for $(G, \kappa)$ established in [14] for the stationary actions holds also for all (purely infinite) ergodic nonsingular actions.

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In this connection we note that if a purely infinite ergodic $G$-action is stationary, then the space of the action is nonatomic. However, in the general (nonstationary) case considered in [4], there exist purely infinite transitive $G$-actions on purely atomic measure spaces. In particular, the action of $G$ on itself via rotations is free, nonconservative, ergodic and purely infinite. We consider such actions as *pathological*. Unfortunately, the proof the main result from [4] does not exclude appearance of pathological actions in (0-1).

Our purpose in the present paper is to refine the main result from [4] in two aspects: to examine all possible values for the $\kappa$-entropy and "get rid" of possible pathological actions on which such values are attained. In fact, we show more.

**Main Theorem.** *Let $G$ not have property (T). Let $\kappa$ be a generating measure on $G$. Then the following are satisfied.*

(1) *For each real $t \in (0, +\infty)$, there is a type $III_1$ ergodic free nonsingular action $T = (T_g)_{g\in G}$ on a standard probability space $(X, \mu)$ such that $h_\kappa(T, \mu) = t$.*
(2) *For each real $t \in (0, +\infty)$, there is $\lambda \in (0, 1)$ and a type $III_\lambda$ ergodic free nonsingular action $T = (T_g)_{g\in G}$ on a standard probability space $(X, \mu)$ such that $h_\kappa(T, \mu) = t$.*

The construction is based on the idea of Bowen, Hartman and Tamuz [4] to consider skew products of nonstrongly ergodic $G$-actions with action of $\bigoplus_{n\in\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$. The proof utilizes the measurable orbit theory (see [8], [16] and a survey [7]) and cohomology properties of nonstrongly ergodic actions [17], [11].

## 1. Some background on orbit theory

Let $T$ be an ergodic free nonsingular action of $G$ on a standard nonatomic probability space $(X, \mathfrak{B}, \mu)$. Denote by $\mathcal{R}$ the $T$-orbit equivalence relation on $X$. We recall that the *full group* $[\mathcal{R}]$ of $\mathcal{R}$ consists of all one-to-one nonsingular transformations $r$ of $(X, \mu)$ such that the graph of $r$ is a subset of $\mathcal{R}$. Given a locally compact second countable group $H$, denote by $\lambda_H$ a left Haar measure on $H$. A Borel map $\alpha : \mathcal{R} \to H$ is called a *cocycle* of $\mathcal{R}$ if $\alpha(x, y) = \alpha(x, z)\alpha(z, y)$ for all points $x, y, z$ from a $\mu$-conull subset of $X$ such that $x \sim_\mathcal{R} y \sim_\mathcal{R} z$. By $T(\alpha) = (T(\alpha)_g)_{g\in G}$ we denote the *$\alpha$-skew product extension of $T$*, i.e. a $G$-action on the product space $(X \times H, \mu \times \lambda_H)$:

$$T(\alpha)_g(x, h) = (T_g x, \alpha(T_g x, x)h).$$

It is obvious that $T(\alpha)$ is $(\mu \times \lambda_H)$-nonsingular. We say that $\alpha$ is *ergodic* if $T(\alpha)$ is ergodic. Consider the $H$-action on $(X \times H, \mu \times \lambda_H)$ by rotations (from the right) along the second coordinate. It commutes with $T(\alpha)$. The restriction of this action to the sub-$\sigma$-algebra of $T(\alpha)$-invariant Borel subsets is called *the action of $H$ associated with $\alpha$*. It is ergodic. It is trivial if and only if $\alpha$ is ergodic. If $H = \mathbb{R}^*_+$ and $\alpha(T_g x, x) := \log \frac{d\mu\circ T_g}{d\mu}(x)$ at a.e. $x$ for each $g \in G$, then $\alpha$ is called *the Radon-Nikodym cocycle* of $\mathcal{R}$. It does not depend on the choice of nonsingular group action generating $\mathcal{R}$. The corresponding associated action of $\mathbb{R}^*_+$ is called *the associated flow* of $T$. If two group actions are orbit equivalent, then their associated flows are isomorphic. The associated flow is transitive and free if and only if $T$ admits an $\sigma$-finite invariant $\mu$-equivalent measure. In this case $T$ is said to be *of type $II$*. If the invariant measure is finite, $T$ is said to be of type $II_1$. If the invariant measure is infinite, $T$ is said to be of type $II_\infty$. If $T$ does not admit an invariant equivalent measure, then $T$ is said to be *of type $III$*. Type $III$ admits further classification into subtypes $III_\lambda$, $0 \le \lambda \le 1$. If the associated flow of $T$ is periodic with period $-\log\lambda$ for some $\lambda \in (0,1)$, then $T$ is said to be *of type $III_\lambda$*. If the associated flow is trivial (on a singleton), then $T$ is said to be *of type $III_1$*. Equivalently, $T$ is of type $III_1$ if and only if the Radon-Nikodym cocycle of $T$ is ergodic. If $T$ is of type $III$ but not of type $III_\lambda$ for any $\lambda \in (0,1]$, then $T$ is said to be *of type $III_0$*.

**Lemma 1.1.** *Let $\mu$ be invariant under $T$. Let $H$ be discrete and countable. Let $\alpha : \mathcal{R} \to H$ be an ergodic cocycle. Then the following hold.*

(i) *The subrelation $\mathcal{R}_0 := \{(x,y) \in \mathcal{R} \mid \alpha(x,y) = 1\}$ of $\mathcal{R}$ is ergodic.*
(ii) *For each $h \in H$, there is an element $r_h \in [\mathcal{R}]$ such that $\alpha(r_h x, x) = h$ for $\mu$-a.e. $x \in X$.*

*Idea of the proof.* Pass to the ergodic skew product extension $T(\alpha)$ and use the following Hopf lemma: If $D = (D_h)_{h\in H}$ is an ergodic $H$-action of type $II$ and $\lambda$ is a $D$-invariant equivalent measure, then for all subsets $A$ and $B$ with $\lambda(A) = \lambda(B)$, there is a Borel bijection $\tau : A \to B$ such that the graph of $\tau$ is a subset of the $D$-orbit equivalence relation. $\square$

Let $S$ be a nonsingular $H$-action on a standard probability space $(Y, \mathfrak{Y}, \nu)$. Given a cocycle $\alpha : \mathcal{R} \to H$, we can form a *skew product action* $T(\alpha, S) = (T(\alpha,S)_g)_{g\in G}$ of $G$ on the product space $(X \times Y, \mu \times \nu)$ by setting

$$T(\alpha,S)_g(x,y) := (T_g x, S_{\alpha(T_g x, x)} y).$$

Then $T(\alpha, S)$ is $(\mu\times\nu)$-nonsingular.

**Lemma 1.2.** *Let $T, \mu, H$ be as in Lemma* 1.1. *If $\alpha$ is ergodic and $S$ is ergodic, then $T(\alpha,S)$ is also ergodic. The associated flow of $T(\alpha,S)$ is isomorphic to the associated flow of $S$. In particular, the type of $T(\alpha,S)$ equals the type of $S$.*

*Proof.* Let $F : X \times Y \to \mathbb{R}$ be a Borel function. If $F$ is $T(\alpha,S)$-invariant, then $F(x,y) = F(x',y)$ if $(x',x) \in \mathcal{R}_0$ for a.e. $y$. By Lemma 1.1(i), $F(x,y) = f(y)$ for some Borel function $f : Y \to \mathbb{R}$. Lemma 1.1(ii) now yields that $f$ is invariant under $S$. Since $S$ is ergodic, $f$ is constant mod $\nu$. Thus $F$ is constant mod $\mu\times\nu$. Hence $T(\alpha,S)$ is ergodic.

The second claim of the lemma follows from Lemma 1.1, the fact that $\mathcal{R}$ is generated by $\mathcal{R}_0$ and the family of transformations $(r_h)_{h\in H}$ and that $r_h[\mathcal{R}_0]r_h^{-1} = [\mathcal{R}_0]$ for each $h \in H$. $\square$

We now recall the definition of strongly ergodic actions (see [5], [11] and references therein). Let $T$ be an ergodic nonsingular $G$-action on nonatomic probability space $(X, \mathfrak{B}, \mu)$. A sequence $(B_n)_{n\in\mathbb{N}}$ in $\mathfrak{B}$ is called *asymptotically invariant* if $\lim_{n\to\infty} \mu(B_n \triangle T_g B_n) = 0$ for every $g \in G$. If every asymptotically invariant sequence $(B_n)_{n\in\mathbb{N}}$ is trivial, i.e. $\lim_{n\to\infty} \mu(B_n)(1-\mu(B_n)) = 0$, then $T$ is called *strongly ergodic*. We note that the strong ergodicity is invariant under orbit equivalence. We will need the following lemma.

**Lemma 1.3.**

(i) *If $G$ does not have property (T), then there is an ergodic probability preserving free action $T$ of $G$ which is not strongly ergodic* [5].

(ii) *If $T$ is an ergodic nonsingular free action of $G$ which is not strongly ergodic, then for each countable discrete Abelian group $A$, there is an ergodic cocycle of the $T$-orbit equivalence relation with values in $A$ (see* [16, Corollary 1.5] *and Theorem* A2 *below).*[1]

## 2. Proof of the main result

The following lemma is almost a literal repetition of [4, Lemma 4.1], where it was proved under an additional assumption that $T$ is measure preserving.

**Lemma 2.1** (Entropy addition formula)**.** *Let $\kappa$ be a probability measure on $G$ and let $T$ be a nonsingular action of $G$ on a standard probability space $(X, \mathfrak{B}, \mu)$. Given a discrete countable group $H$ and a nonsingular action $S = (S_h)_{h\in H}$ of $H$ on a standard probability space $(Y, \mathfrak{F}, \nu)$, let $\kappa_x$ denote the pushforward of $\kappa$ under the map $G \ni g \mapsto \alpha(T_g x, x) \in H$ for each $x \in X$. Then*

$$h_\kappa(T(\alpha), \mu \times \nu) = h_\kappa(T, \mu) + \int_X h_{\kappa_x}(S, \nu) d\mu(x).$$

*Proof.*

$$\begin{aligned} h_\kappa(T(\alpha, S), \mu \times \nu) &= -\sum_{g\in G} \kappa(g) \int_{X\times Y} \log\left(\frac{d(\mu\times\nu)\circ T_g(\alpha)}{d(\mu\times\nu)}(x,y)\right) d\mu(x)d\nu(y) \\ &= h_\kappa(T,\mu) - \int_X \sum_{g\in G}\kappa(g)\int_Y \log\left(\frac{d\nu\circ S_{\alpha(T_g x,x)}}{d\nu}(y)\right) d\nu(y)d\mu(x) \\ &= h_\kappa(T,\mu) - \int_X \sum_{h\in H}\kappa_x(h)\int_Y \log\left(\frac{d\nu\circ S_h}{d\nu}(y)\right) d\nu(y)d\mu(x) \\ &= h_\kappa(T,\mu) + \int_X h_{\kappa_x}(S,\nu)d\mu(x). \end{aligned}$$

□

*Proof of Main Theorem.* We will proceed in two steps. In the first step, for each $\epsilon > 0$, we construct an ergodic nonsingular $G$-action of type $III_1$ (or of type $III_\lambda$ for some $\lambda \in (0, 1)$) whose $\kappa$-entropy is less then $\epsilon$. In the second step we show how to change the quasiinvariant measure for the action constructed in the first step with appropriate equivalent measures to force the $\kappa$-entropy to attain all the values from the interval $(\epsilon, +\infty)$.

*Step* 1. Fix $\epsilon > 0$ and $\lambda \in (0, 1)$. Fix an enumeration $G = \{g_n \mid n \in \mathbb{N}\}$ and a sequence of integers $1 = l_1 \le l_2 \le \cdots$ such that $l_{n+1} - l_n \le 1$ for all $n \in \mathbb{N}$, $l_n \to \infty$ and

$$\sum_{n=1}^{\infty} \kappa(g_n)(l_n + 1) < 2. \tag{2-1}$$

By Lemma 1.3(i), there is a measure preserving free action $T$ of $G$ on a standard probability space $(X, \mathfrak{B}, \mu)$ which is not strongly ergodic. Denote by $\mathcal{R}$ the $T$-orbit equivalence relation. Let $F := \bigoplus_{n\in\mathbb{N}} \mathbb{Z}/2\mathbb{Z}$. We consider the elements of $F$ as $\mathbb{Z}/2\mathbb{Z}$-valued functions on $\mathbb{N}$ with finite support. Given $f \in F$, we let

$$\|f\| := \max\{j \in \mathbb{N} \mid f(j) \neq 0\}.$$

[1] Since the proof of [16, Corollary 1.5] was not completed there, we provide a complete proof of it in Appendix A.

By Lemma 1.3(ii), there exists an ergodic cocycle $\alpha : \mathcal{R} \to F$. For each $n \in \mathbb{N}$, we can choose $M_n \in \mathbb{N}$ such that

$$\mu\left(\left\{x \in X \;\middle|\; \max_{1\le i\le n} \|\alpha(T_{g_i}x, x)\| < M_n\right\}\right) > 1 - \frac{1}{n2^n}. \tag{2-2}$$

Without loss of generality we may assume that $M_n = M_{n+1}$ if and only if $l_n = l_{n+1}$ for each $n \in \mathbb{N}$. Let $N := \{f \in F \mid f(M_n) = 0 \text{ for each } n \in \mathbb{N}\}$. Then $N$ is a subgroup of $F$. The quotient group $F/N$ is identified naturally with the "complimentary to $F$" subgroup $\{f \in F \mid f(n) = 0 \text{ for each } n \neq M_1, M_2, \ldots\}$, which is, in turn, isomorphic to $F$ in a natural way. Hence passing from $\alpha$ to the quotient cocycle

$$\alpha + N : \mathcal{R} \ni (x, y) \mapsto \alpha(x, y) + N \in F/N$$

means that we may assume without loss of generality that $M_n = l_n$ for each $n \in \mathbb{N}$ in (2-2). (We use here a simple fact that $\alpha + N$ is ergodic whenever $\alpha$ is.) Therefore applying (2-2) we obtain that

$$\begin{aligned}
\int_X \|\alpha(T_{g_n}x, x)\|\, d\mu(x) &= \sum_{s=1}^{\infty} s\mu(\{x \in X \mid \|\alpha(T_{g_n}x, x)\| = s\}) \\
&\le l_n + \sum_{s>l_n} s\mu(\{x \in X \mid \|\alpha(T_{g_n}x, x)\| = s\}) \\
&\le l_n + \sum_{s>l_n} \frac{1}{2^s} \\
&\le l_n + 1. \qquad (2\text{-}3)
\end{aligned}$$

The second inequality here follows from the fact that for each $s > l_n$, we have $s = l_m$ for some $m \ge n$ and hence

$$\mu(\{x \in X \mid \|\alpha(T_{g_n}x, x)\| = s\}) \le \frac{1}{m2^m} \le \frac{1}{s2^s}$$

because $m \ge l_m = s$.

Now we consider $F$ as a (dense) subgroup of the compact Abelian group $K := (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ of all $\mathbb{Z}/2\mathbb{Z}$-valued functions on $\mathbb{N}$. Denote by $S$ the action of $F$ on $K$ by translations. Let $\nu_n$ denote the distribution on $Z/2\mathbb{Z}$ such that

$$\nu_n(0) = \frac{1}{1 + e^{\epsilon_n}}, \quad \nu_n(1) = \frac{e^{\epsilon_n}}{1 + e^{\epsilon_n}}$$

for some sequence $(\epsilon_n)_{n\in\mathbb{N}}$ of reals such that $\lim_{n\to\infty} \epsilon_n = 0$, $\sum_{n\in\mathbb{N}} \epsilon_n^2 = \infty$ and $\max\{|\epsilon_n| \mid n \in \mathbb{N}\} < \epsilon$. Let $\nu = \bigotimes_{n\in\mathbb{N}} \nu_n$. Then $S$ is $\nu$-nonsingular, ergodic and of type $III_1$ [1]. Therefore by Lemma 1.2, the skew product $G$-action $T(\alpha, S)$ on $(X \times K, \mu \times \nu)$ is ergodic and of type $III_1$. Hence $h_\kappa(T(\alpha, S)) > 0$. To estimate $h_\kappa(T(\alpha, S))$ from above, we first let $N_f := \{n \in \mathbb{N} \mid f(n) \neq 0\}$ for $f \in F$. It is obvious that $\#N_f \le \|f\|$. Since

$$\begin{aligned}
-\int_Y \log\left(\frac{d\nu \circ S_f}{d\nu}\right)(y) d\nu(y) &= -\sum_{n\in N_f} \int_{\mathbb{Z}/2\mathbb{Z}} \log\left(\frac{\nu_n(y_n + 1)}{\nu_n(y_n)}\right) d\nu_n(y_n) \\
&= \sum_{n\in N_f} (\nu_n(1) - \nu_n(0)) \log \frac{\nu_n(1)}{\nu_n(0)},
\end{aligned}$$

we obtain that for each probability $\xi$ on $F$,

$$\begin{aligned} h_\xi(S,\nu) &= \sum_{f\in F}\xi(f)\sum_{n\in N_f}(\nu_n(1)-\nu_n(0))\log\frac{\nu_n(1)}{\nu_n(0)} \\ &\le \sum_{f\in F}\xi(f)\sum_{n\in N_f}|\epsilon_n| \\ &\le \epsilon\|f\|\sum_{f\in F}\xi(f). \end{aligned} \tag{2-4}$$

Since $T$ preserves $\kappa$, it follows that $h_\kappa(T,\mu)=0$. Then Lemma 2.1, (2-4) and (2-3) yield that

$$\begin{aligned} h_\kappa(T(\alpha,S),\mu\times\nu) &\le \epsilon\int_X\sum_{f\in F}\kappa_x(f)\|f\|\,d\mu(x) \\ &= \epsilon\sum_{g\in G}\kappa(g)\int_X\|\alpha(T_gx,x)\|\,d\mu(x) \\ &\le \epsilon\sum_{n=1}^{\infty}\kappa(g_n)(l_n+1). \end{aligned}$$

It now follows from (2-1) that $h_\kappa(T(\alpha,S),\mu\times\nu)\le 2\epsilon$. Hence

$$\inf\{h_\kappa(A)\mid A \text{ is a type } III_1 \text{ ergodic free action of } G\}=0.$$

In a similar way we may show that

$$\inf\{h_\kappa(A)\mid A \text{ is a type } III_\lambda \text{ ergodic free action of } G,\ \lambda\in(0,1)\}=0. \tag{2-5}$$

For that we argue as above but with a different measure $\nu$. Indeed, let $\nu_n$ denote the distribution on $\mathbb{Z}/2\mathbb{Z}$ such that

$$\nu_n(0)=\frac{1}{1+e^\epsilon},\quad \nu_n(1)=\frac{e^\epsilon}{1+e^\epsilon}.$$

Let $\nu=\bigotimes_{n\in\mathbb{N}}\nu_n$. Then $S$ is $\nu$-nonsingular, ergodic and of type $III_{e^{-\epsilon}}$ [1]. Therefore by Lemma 1.2, the skew product $G$-action $T(\alpha,S)$ on $(X\times K,\mu\times\nu)$ is ergodic and of type $III_{e^{-\epsilon}}$. Hence $h_\kappa(T(\alpha,S))>0$. As in the $III_1$-case considered above, we obtain that $h_\kappa(T(\alpha,S),\mu\times\nu)<2\epsilon$ and hence (2-5) follows.

*Step* 2. Given $\epsilon>0$, let $\nu$ be a measure on $K$ such that

$$h_\kappa(T(\alpha,S),\mu\times\nu)<\epsilon. \tag{2-6}$$

We choose $n_0>0$ such that

$$\int_X\kappa_x(\{f\in F\mid f(n_0)\ne 0\})\,d\mu(x)>0. \tag{2-7}$$

It exists because otherwise we would have that $\kappa_x$ is supported at 0 for a.e. $x\in X$. The latter yields that $\alpha(T_gx,x)=0$ at a.e. $x$ for all $g$ from the support of $\kappa$. Since $\kappa$ is generating, it follows that $\alpha$ is trivial, a contradiction.

Let $\omega$ be a probability on $\mathbb{Z}/2\mathbb{Z}$ supported at 1.[2] For each $\theta\in(0,1]$, we let

$$\nu_n^\theta=\begin{cases}\nu_n, & \text{if } n\ne n_0,\\ \theta\nu_n+(1-\theta)\omega, & \text{if } n=n_0,\end{cases}$$

[2]We consider the group $\mathbb{Z}/2\mathbb{Z}$ as $\{0,1\}$ with addition mod 2.

and $\nu^\theta := \bigotimes_{n\in\mathbb{N}} \nu_n^\theta$. Then $\nu^\theta$ is equivalent to $\nu$, and hence $\mu \times \nu^\theta$ is equivalent to $\mu \times \nu$. Therefore the dynamical systems $(T(\alpha, S), \mu \times \nu^\theta)$ and $(T(\alpha, S), \mu \times \nu)$ are of the same Krieger's type. It follows from the equality in (2-4) that

$$h_{\kappa_x}(S, \nu) - h_{\kappa_x}(S, \nu^\theta) = \kappa_x(\{f \in F \mid f(n_0) \neq 0\})(\Phi(\nu_{n_0}(0)) - \Phi(\nu_{n_0}^\theta(0))), \tag{2-8}$$

where $\Phi(t) := (1 - 2t) \log \frac{1-t}{t}$, if $t \in (0, 1)$. Therefore the map

$$(0, 1] \ni \theta \mapsto h_\kappa(T(\alpha, S), \mu \times \nu^\theta) = \int_X h_{\kappa_x}(S, \nu^\theta)\, d\mu(x) \in \mathbb{R}$$

is continuous. In view of (2-7) and (2-8), this map goes to infinity as $\theta \to 0$. Since $\nu^1 = \nu$ and (2-6) holds, it follows that

$$\{h_\kappa(T(\alpha, S), \mu \times \nu^\theta) \mid \theta \in (0, 1]\} \supset (\epsilon, \infty),$$

as desired. □

## Appendix A

Let $\mathcal{R}$ be an ergodic measure preserving countable equivalence relation on a nonatomic standard probability space $(X, \mathfrak{B}, \mu)$ and let $G$ be a locally compact second countable group. A cocycle $\rho : \mathcal{R} \to G$ is called *regular* if the action of $G$ associated with $\rho$ is transitive. For instance, an ergodic cocycle is regular. A coboundary is also regular.

**Proposition A1.** *Let $A$ be an amenable discrete countable group and let $H$ be a locally compact second countable amenable group. Let $\alpha : \mathcal{R} \to A$ be a cocycle. If $\alpha$ is not regular, then there is an ergodic cocycle of $\mathcal{R}$ with values in $H$.*

*Proof.* Let $\mathcal{A}$ stand for the "transitive" equivalence relation on $A$, i.e. $\mathcal{A} = A \times A$. Let $\lambda$ be a probability measure on $A$ which is equivalent to Haar measure. Then the equivalence relation $\mathcal{R} \times \mathcal{A}$ is an ergodic equivalence relation on the probability space $(X \times A, \mu \times \lambda)$. Let $V = (V_a)_{a\in A}$ denote the nonsingular action of $A$ on $(X \times A, \mu \times \lambda)$ by right rotations along the second coordinate. Then $\mathcal{R} \times \mathcal{A}$ is generated by a subrelation $\mathcal{R}(\alpha)$ and $V$; i.e. two points $z_1, z_2 \in X \times A$ are $(\mathcal{R} \times \mathcal{A})$-equivalent if and only if the points $V_{a_1} z_1$ and $V_{a_2} z_2$ are $\mathcal{R}(\alpha)$-equivalent for some $a_1, a_2 \in A$. Let $W = (W_a)_{a\in A}$ stand for the action of $A$ associated with $\alpha$. Denote by $(\Omega, \nu)$ the space of this action. Then we can assume that there is a Borel map $\pi : X \times A \to \Omega$ such that $\nu = (\mu \times \lambda) \circ \pi^{-1}$ and $\pi \circ V_a = W_a \circ \pi$ for each $a \in A$. We observe that $\pi$ is the $\mathcal{R}(\alpha)$-ergodic decomposition of $X \times A$. Since $A$ is amenable, the $W$-orbit equivalence relation $\mathcal{I}$ on $(\Omega, \nu)$ is hyperfinite [6]. It is ergodic. Since $\alpha$ is not regular, $\mathcal{I}$ is nontransitive. Hence there is an ergodic cocycle $\beta : \mathcal{I} \to H$ (see [13] and [10]). We now define a cocycle $\beta^* : \mathcal{R} \times \mathcal{A} \to H$ by setting

$$\beta^*(z_1, z_2) := \beta(\pi(z_1), \pi(z_2)).$$

Then $\beta^*$ is well defined. Since $\pi$ is the $\mathcal{R}(\alpha)$-ergodic decomposition, it follows that $\beta^*$ is ergodic. Restricting $\mathcal{R} \times \mathcal{A}$ and $\beta^*$ to the subset $X \times \{1_A\}$ we obtain $\mathcal{R}$ and a cocycle of $\mathcal{R}$ with values in $H$ respectively. Of course, this cocycle is also ergodic. □

Let $A$ be an Abelian locally compact noncompact second countable group. For a cocycle $\alpha : \mathcal{R} \to A$, we denote by $E(\alpha) \subset A \sqcup \{\infty\}$ the essential range of $\alpha$ (see [16, Definition 3.1]). The following theorem provides a complete proof of

[17, Corollary 1.5] (it was assumed additionally in [17] that $A$ in the statement of the theorem below is Abelian).

**Theorem A2.** *Let $T = (T_g)_{g\in G}$ be an ergodic measure preserving action of $G$ on a standard nonatomic probability space $(X, \mathfrak{B}, \mu)$. Let $A$ be a countable amenable group. If $T$ is not strongly ergodic, then there is an ergodic cocycle of the $T$-orbit equivalence relation $\mathcal{R}$ with values in $A$.*

*Proof.* Let $\mathcal{R}$ denote the $T$-orbit equivalence relation. Let $K$, $F$, $S$, $N_f$ and $\|.\|$ denote the same objects as in the proof of the Main Theorem. Let $\lambda_K$ stand for the Haar measure on $K$. For each $n \in \mathbb{N}$, denote by $f_n$ the element of $F$ such that $N_{f_n} = \{n\}$. By [11, Lemma 2.4], there are a Borel map $\pi : X \to K$ and a sequence $(V_n)_{n\in\mathbb{N}}$ of transformations in $[\mathcal{R}]$ such that $\mu \circ \pi^{-1} = \lambda_K$,

$$\{\pi(T_g x) \mid g \in G\} = \{S_f \pi(x) \mid f \in F\}$$

and $\pi(V_n x) = S_{f_n}\pi(x)$ for a.a. $x \in X$. Denote by $\mathcal{S}$ the $S$-orbit equivalence relation on $K$. We define a cocycle $\beta : \mathcal{S} \to F$ by setting $\beta(S_f y, y) := f$, $y \in Y$, $f \in F$. By [16, Proposition 3.15], $E(\beta) = \{0, +\infty\}$. We now define a cocycle $\beta^* : \mathcal{R} \to F$ by setting

$$\beta^*(T_g x, x) := \beta(\pi(T_g x), \pi(x)),$$

$x \in X$, $g \in G$. Since $E(\beta^*) \subset E(\beta)$, we obtain that either $E(\beta^*) = \{0, \infty\}$ or $E(\beta^*) = \{0\}$. In the latter case, $\beta^*$ is a coboundary [16]. Hence there is a Borel map $\xi : X \to K$ such that $\beta^*(V_n x, x) = \xi(V_n x) - \xi(x)$ for each $n$ at a.e. $x \in X$. There is a constant $C > 0$ and a subset $X_C \subset X$ such that $\mu(X_C) > 3/4$ and $\|\xi(x)\| < C$ for all $x \in X_C$. It follows that for each $n$,

$$\sup_{x\in X_C\cap V_n^{-1}X_C} \|\beta^*(V_n x, x)\| < C.$$

This contradicts the fact that $\|\beta^*(V_n x, x)\| = n$ for a.a. $x \in X$ and $n \in \mathbb{N}$. Hence $E(\beta^*) = \{0, \infty\}$. This yields that $\beta^*$ is not regular. It remains to apply Proposition A1. $\square$

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