INEQUALITIES FOR THE RATIO OF COMPLETE ELLIPTIC INTEGRALS

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Abstract. We present various inequalities for the complete elliptic integral of the first kind,

$$K(r) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - r^2 \sin^2(t)}} \, dt \quad (0 < r < 1).$$

Among others, we prove that the inequalities

$$\frac{1}{1 + \frac{1}{4} r} < \frac{K(r)}{K(\sqrt{r})} \quad \text{and} \quad \frac{K(\sqrt{1 - r^2})}{K(\sqrt{1 - r})} < \frac{2}{1 + \sqrt{r}}$$

are valid for all $r \in (0, 1)$. These estimates refine results published by Anderson, Vamanamurthy, and Vuorinen in 1990.

1. Introduction

The complete elliptic integrals of the first and second kind are defined for $r \in (0, 1)$ by

$$K(r) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - r^2 \sin^2(t)}} \, dt = \pi \sum_{n=0}^{\infty} \frac{a_n r^{2n}}{n!}$$

and

$$E(r) = \int_0^{\pi/2} \sqrt{1 - r^2 \sin^2(t)} \, dt = \pi \sum_{n=0}^{\infty} \frac{a_n 1 - 2n r^{2n}}{1 - 2n}$$

with

$$a_n = \left( \frac{(1/2)_n}{n!} \right)^2 = \frac{1}{\pi} \left( \frac{\Gamma(n + 1/2)}{\Gamma(n + 1)} \right)^2.$$ 

Both functions can be expressed in terms of the Gaussian hypergeometric function, namely

$$K(r) = \frac{\pi}{2} F(1/2, 1/2; 1; r^2) \quad \text{and} \quad E(r) = \frac{\pi}{2} F(-1/2, 1/2; 1; r^2).$$

The associate complete elliptic integrals are given by

$$K'(r) = K(r') \quad \text{and} \quad E'(r) = E(r'), \quad \text{where} \quad r' = \sqrt{1 - r^2}.$$
These functions have remarkable applications in the theory of quasiconformal mappings as well as in geometry, mechanics, geodesy and other fields. The main properties of \( K \) and \( E \) are collected, for instance, in the books [6] and [8]. We also refer to the research articles [2], [9], [10], [11], [14], [15], [16] and the references cited therein.

In the next sections we make use of the formulas

\begin{equation}
K\left(\frac{1-\sqrt{r}}{1+\sqrt{r}}\right) = \frac{1+r}{2} \frac{K'(r)}{K}\tag{1.3}
\end{equation}

and

\begin{equation}
K\left(\frac{2\sqrt{r}}{1+r}\right) = (1+r)K(r)\tag{1.4}
\end{equation}

which are known as Landen identities and we apply the limit relations

\begin{equation}
\lim_{r \to 1} \left(K(r) + \frac{1}{2} \log(1 - r^2)\right) = \log(4)\tag{1.5}
\end{equation}

and

\begin{equation}
\lim_{r \to 0} \frac{E(r) - (1 - r^2)K(r)}{r^2} = \frac{\pi}{4};\tag{1.6}
\end{equation}

see [8] (164.02), (112.01)] [11 (17.3.27)].

The work on this paper has been inspired by the following elegant inequalities which were published by Anderson et al. [4] in 1990:

\begin{equation}
\frac{1}{1+r} < \frac{K(r)}{K(\sqrt{r})}\tag{1.7}
\end{equation}

and

\begin{equation}
\frac{K'(r)}{K'(\sqrt{r})} < \frac{2}{1+r}.\tag{1.8}
\end{equation}

Both inequalities are valid for all \( r \in (0, 1) \).

Here we present various new inequalities for the ratio of complete elliptic integrals. Among others, we prove that the following refinements of (1.7) and (1.8) hold for all \( r \in (0, 1) \):

\begin{equation}
\frac{1}{1+\frac{1}{4}r} < \frac{K(r)}{K(\sqrt{r})}\quad \text{and} \quad \frac{K'(r)}{K'(\sqrt{r})} < \frac{2}{1+\sqrt{r}}.
\end{equation}

An application of the second inequality leads to an improvement of the well-known arithmetic mean - geometric mean inequality

\begin{equation}
0 < \frac{1+r}{2} - \sqrt{r} \quad (0 < r < 1).
\end{equation}

We show that the lower bound 0 can be replaced by a positive expression written in terms of \( K \).

In the next section we provide three lemmas. They play a role in the proofs of our main results which are given in Section [9].
2. Lemmas

The first lemma provides properties of the logarithmic derivative of the gamma function; see [1, (6.3.8), (6.4.10)].

**Lemma 2.1.** The function \( \psi = \Gamma'/\Gamma \) is strictly concave on \((0, \infty)\) and satisfies for \( x > 0 \) the duplication formula
\[
\psi(2x) = \frac{1}{2} \psi(x) + \frac{1}{2} \psi\left(x + \frac{1}{2}\right) + \log(2).
\]

Moreover, we need several limit relations. They are collected in the following lemmas.

**Lemma 2.2.** We have
\[
\lim_{r \to 1} \frac{F(a, b; a + b; r^2)}{F(a, b; a + b; r)} = 1 \quad \text{and} \quad \lim_{r \to 0} \frac{F(a, b; a + b; 1 - r^2)}{F(a, b; a + b; 1 - r)} = 2.
\]

**Proof.** Using [12, eq. (2), p. 74] gives the asymptotic relation
\[
F(a, b; a + b; x) \sim -\frac{1}{B(a, b)} \log(1 - x) \quad (x \to 1),
\]
where \( B(a, b) \) denotes Euler’s beta function. This leads to (2.1). \( \square \)

**Remark 2.3.** Setting \( a = b = 1/2 \) in (2.1) yields
\[
(2.2) \quad \lim_{r \to 1} \frac{K(r)}{K(\sqrt{r})} = 1
\]
and
\[
(2.3) \quad \lim_{r \to 0} \frac{K'(r)}{K'(\sqrt{r})} = 2.
\]

**Lemma 2.4.** Let
\[
u(r) = K'(\sqrt{r}), \quad v(r) = r \frac{d}{dr} u(r), \quad w(r) = u(r) + \frac{1}{2} \log(r)
\]
and
\[
A(r) = v(r)w(r^2) - 2v(r^2)w(r), \quad B(r) = v(r^2) - v(r),
\]
\[
C(r) = \left(\frac{w(r^2)}{\log(r)} - 1\right) (2w(r) - w(r^2)).
\]

Then,
\[
\lim_{r \to 0} A(r) = \log(2), \quad \lim_{r \to 0} B(r) = 0, \quad \lim_{r \to 0} C(r) = -\log(4), \quad \lim_{r \to 1} v(r) = -\frac{\pi}{8}.
\]

**Proof.** From (1.5) we obtain
\[
(2.5) \quad \lim_{r \to 0} w(r) = \log(4).
\]

Thus,
\[
(2.6) \quad \lim_{r \to 0} v(r) = \lim_{r \to 0} \frac{\mathcal{E}(\sqrt{1 - r}) - rw(r) + \frac{1}{2} r \log(r)}{2(r - 1)} = \frac{1 - 0 + 0}{2 \cdot (0 - 1)} = -\frac{1}{2}.
\]
Using (2.5) and (2.6) leads to

\[
\lim_{r \to 0} A(r) = -\frac{1}{2} \cdot \log(4) + 2 \cdot \frac{1}{2} \cdot \log(4) = \log(2),
\]

\[
\lim_{r \to 0} B(r) = -\frac{1}{2} + \frac{1}{2} = 0, \quad \lim_{r \to 0} C(r) = (0 - 1) \cdot (2 \cdot \log(4) - \log(4)) = -\log(4).
\]

Applying (1.6) gives

\[
\lim_{r \to 1} v(r) = \lim_{r \to 1} \left( -\frac{1}{2} \cdot \frac{\mathcal{E}(\sqrt{1 - r}) - r \mathcal{K}(\sqrt{1 - r})}{1 - r} \right) = -\frac{1}{2} \cdot \frac{\pi}{4} = -\frac{\pi}{8}.
\]

\[\square\]

3. Inequalities

Our first theorem offers an improvement of inequality (1.7).

**Theorem 3.1.** For all \( r \in (0, 1) \) we have

\[
\frac{1}{1 + \lambda r} < \frac{\mathcal{K}(r)}{\mathcal{K}(\sqrt{r})} < \frac{1}{1 + \mu r}
\]

with the best possible constant factors \( \lambda = 1/4 \) and \( \mu = 0 \).

**Proof.** Let \( r \in (0, 1) \). We define

\[
f(r) = \frac{2}{\pi} \left( 1 + \frac{1}{4} r \right) \mathcal{K}(r) \quad \text{and} \quad g(r) = \frac{2}{\pi} \mathcal{K}(\sqrt{r}).
\]

Applying (1.1) and (1.2) yields the series representations

\[
f(r) = \sum_{n=0}^{\infty} \left( a_n + \frac{1}{4} a_n r \right) r^{2n}
\]

and

\[
g(r) = \sum_{n=0}^{\infty} \left( a_{2n} + a_{2n+1} r \right) r^{2n}.
\]

Let

\[
P_n(r) = \frac{a_n + \frac{1}{4} a_n r}{a_{2n} + a_{2n+1} r}.
\]

Then, \( P_0(r) = 1 \). We obtain

\[
P_n(r) = \frac{1 + \frac{1}{4} r}{1 + \left( \frac{4n+1}{4n+2} \right)^2 r} \frac{a_n}{a_{2n}} > \frac{1 + \frac{1}{4}}{1 + \left( \frac{4n+1}{4n+2} \right)^2} \frac{a_n}{a_{2n}}.
\]

Let \( x > 0 \) and

\[
Q(x) = \frac{\Gamma(x + 1/2) \Gamma(2x + 1)}{\Gamma(x + 1) \Gamma(2x + 1/2)}.
\]

Using Lemma 2.7 and Jensen’s inequality gives

\[
\frac{Q'(x)}{Q(x)} = \psi(x + 1/2) + 2\psi(2x + 1) - \psi(x + 1) - 2\psi(2x + 1/2)
\]

\[= 2\psi(x + 1/2) - \left[ \psi(x + 1/4) + \psi(x + 3/4) \right] \geq 2\psi(x + 1/2) - 2\psi(x + 1/2) = 0.
\]

It follows that \( Q \) is increasing on \((0, \infty)\). Thus, for \( n \geq 1 \),

\[
\frac{a_n}{a_{2n}} = Q^2(n) \geq Q^2(1) = \frac{16}{9}.
\]
This yields

\[ P_n(r) > \frac{16}{9} \left( 1 + \frac{1}{1 + \left( \frac{2n+1}{2n+2} \right)^2} \right) = 1 + \frac{32n^2 + 104n + 35}{9(32n^2 + 24n + 5)} > 1. \]

Hence, we have \( f(r) > g(r) \) which is equivalent to the left-hand side of (3.1) with \( \lambda = 1/4 \).

Since \( \mathcal{K} \) is strictly increasing on \((0,1)\), we obtain \( \mathcal{K}(r)/\mathcal{K}(\sqrt{r}) < 1 \). This leads to the right-hand side of (3.1) with \( \mu = 0 \).

It remains to show that the given constants are best possible. Double-inequality (3.1) can be written as

(3.2) \[ \mu < S(r) = \left( \frac{\mathcal{K}(\sqrt{r})}{\mathcal{K}(r)} - 1 \right) \frac{1}{r} < \lambda. \]

Since

\[ S(r) = \frac{a_1 + (a_2 - a_1)r + \ldots}{a_0 + a_1r^2 + \ldots}, \]

we obtain

(3.3) \[ \lim_{r \to 0} S(r) = \frac{a_1}{a_0} = \frac{1}{4}. \]

Using (2.2) gives

(3.4) \[ \lim_{r \to 1} S(r) = 0. \]

From (3.2), (3.3) and (3.4) we conclude that the constants \( \lambda = 1/4 \) and \( \mu = 0 \) are sharp.

**Note added in proof.** After the paper was accepted for publication we discovered that G.D. Anderson, M.K. Vamanamurthy and M. Vuorinen [2] proved the inequalities

(3.5) \[ \frac{1}{\sqrt{1 + r}} < \frac{\mathcal{K}(r)}{\mathcal{K}(\sqrt{r})} < \min\left\{ \frac{\sqrt{2}, 1/\sqrt{r'}}{\sqrt{1 + r}} \right\}, \quad (0 < r < 1). \]

The lower bound in (3.5) improves the left-hand side of (3.1) with \( \lambda = 1/4 \), whereas the upper bounds given in (3.5) and (3.1) cannot be compared in general. We also refer to S.-L. Qiu and M.K. Vamanamurthy [13] who showed that the function \( r \mapsto \sqrt{1 + r} \mathcal{K}(r)/\mathcal{K}(\sqrt{r}) \) is strictly increasing from \([0,1)\) onto \([1, \sqrt{2})\).

An application of (3.1) leads to the following result.

**Corollary 3.2.** For all \( r \in (0,1) \) we have

(3.6) \[ 0 < \frac{\mathcal{K}(\sqrt{r}) - \mathcal{K}(r)}{\mathcal{K}(\frac{2\sqrt{r}}{1 + r}) - \mathcal{K}(\sqrt{r})} < \frac{1}{3}. \]

Both bounds are sharp.

**Proof.** Let \( r \in (0,1) \). We denote the ratio in (3.6) by \( T(r) \). Since \( \mathcal{K} \) is strictly increasing on \((0,1)\), we conclude from \( 0 < r < \sqrt{r} < 2\sqrt{r}/(1 + r) < 1 \) that

\[ 0 < \mathcal{K}(\sqrt{r}) - \mathcal{K}(r) \quad \text{and} \quad 0 < \mathcal{K}(\frac{2\sqrt{r}}{1 + r}) - \mathcal{K}(\sqrt{r}). \]
Thus, \( T(r) > 0 \). Using (1.4) and the left-hand side of (3.1) with \( \lambda = 1/4 \) gives
\[
\left( \frac{K(2\sqrt{r})}{1 + r} - K(\sqrt{r}) \right)(1 - 3T(r)) = \frac{(4 + r)K(r) - 4K(\sqrt{r})}{K(\sqrt{r})} > \frac{(4 + r)K(r) - 4 \left(1 + \frac{1}{4}r\right)K(r) = 0}.
\]
This leads to \( T(r) < 1/3 \).

Applying (2.2) gives
\[
(3.7) \lim_{r \to 1} T(r) = \lim_{r \to 1} \frac{1 - \frac{K(r)}{K(\sqrt{r})}}{(1 + r)\frac{K(r)}{K(\sqrt{r})}} - 1 = \frac{1}{2} - 1 = 0.
\]
We have
\[
T(r) = \frac{S(r)}{1 - S(r)},
\]
where \( S \) is defined in (5.2). Using (3.3) leads to
\[
(3.8) \lim_{r \to 0} T(r) = \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{1}{3}.
\]
From (3.7) and (3.8) we conclude that the constant bounds given in (3.6) are best possible.

Now, we present a refinement of (1.8).

**Theorem 3.3.** Let \( \alpha, \beta \in \mathbb{R} \). The inequalities
\[
\frac{2}{1 + r^{\alpha}} < \frac{K'(r)}{K'(\sqrt{r})} < \frac{2}{1 + r^{\beta}}
\]
are valid for all \( r \in (0, 1) \) if and only if \( \alpha \leq 0 \) and \( \beta \geq 1/2 \).

**Proof.** First, we prove that if \( \alpha = 0 \) and \( \beta = 1/2 \), then (3.9) holds for all \( r \in (0, 1) \). Since \( K' \) is strictly decreasing on \((0, 1)\), we obtain
\[
(3.10) \quad \frac{2}{1 + r^0} = 1 < \frac{K'(r)}{K'(\sqrt{r})}.
\]
Let
\[
\delta(r) = \frac{1 - \sqrt{r}}{1 + \sqrt{r}}.
\]
We replace in (1.4) \( r \) by \( \delta^2(r) \) and obtain
\[
(3.11) \quad K(\frac{1 - \sqrt{r}}{1 + r}) = (1 + \delta^2(r))K(\delta^2(r)).
\]
Using (1.3) and (3.11) gives
\[
(3.12) \quad \frac{K'(r)}{K'(\sqrt{r})} = \frac{1 + \sqrt{r}}{1 + r} \frac{K(\frac{1 - \sqrt{r}}{1 + r})}{K(\delta^2(r))} = \frac{1 + \sqrt{r}}{1 + r} \left(1 + \delta^2(r)\right) \frac{K(\delta^2(r))}{K(\delta^2(r))}.
\]
Since \( K \) is strictly increasing on \((0, 1)\), we get
\[
(3.13) \quad \frac{K'(r)}{K'(\sqrt{r})} < \frac{1 + \sqrt{r}}{1 + r} \left(1 + \delta^2(r)\right) = \frac{2}{1 + \sqrt{r}}.
\]
If \( r \in (0, 1) \), then the function \( t \mapsto 2/(1 + r^t) \) is increasing on \( \mathbb{R} \). It follows from (3.10) and (3.13) that if \( \alpha \leq 0 \) and \( \beta \geq 1/2 \), then (3.9) is valid for all \( r \in (0, 1) \).
Next, we assume that (3.9) holds for all \( r \in (0, 1) \). Then, we obtain
\[
(3.14) \quad \alpha < F(r) < \beta
\]
with
\[
F(r) = \frac{\log(G(r))}{\log(r)} \quad \text{and} \quad G(r) = \frac{\mathcal{K}'(\sqrt{r})}{\mathcal{K}'(r)} - 1.
\]
Using (2.3) leads to
\[
\lim_{r \to 0} G(r) = 0.
\]
Thus,
\[
(3.15) \quad \lim_{r \to 0} F(r) = \lim_{r \to 0} \frac{rG_1(r)}{G(r)}
\]
with \( G_1(r) = (d/dr)G(r) \). We have
\[
\lim_{r \to 0} \frac{rG_1(r)}{G(r)} = 2 \cdot \frac{A(r) + B(r)}{C(r)}
\]
where \( A, B \) and \( C \) are defined in (2.4). Applying Lemma 2.4 gives
\[
(3.16) \quad \lim_{r \to 0} \frac{rG_1(r)}{G(r)} = 2 \cdot \frac{0 + 0}{-\log(4)} = 0.
\]
From (3.15) and (3.16) we conclude that
\[
(3.17) \quad \lim_{r \to 0} F(r) = 0.
\]
Since \( G(1) = 1 \), we obtain
\[
(3.18) \quad \lim_{r \to 1} F(r) = \lim_{r \to 1} \frac{rG_1(r)}{G(r)} = \lim_{r \to 1} G_1(r).
\]
We have
\[
G_1(r) = \frac{2u(r^2)v(r) - 4u(r)v(r^2)}{ru^2(r^2)},
\]
with \( u(r) = \mathcal{K}'(\sqrt{r}) \). Using \( u(1) = \pi/2 \) and Lemma 2.4 yields
\[
(3.19) \quad \lim_{r \to 1} G_1(r) = \frac{1}{2}.
\]
From (3.18) and (3.19) we get
\[
(3.20) \quad \lim_{r \to 1} F(r) = \frac{1}{2}.
\]
Applying (3.14) and the limit relations (3.17) and (3.20) leads to \( \alpha \leq 0 \) and \( \beta \geq 1/2 \). This completes the proof of Theorem 3.3. \( \square \)

**Remark 3.4.** From (3.12) and the left-hand side of (3.1) with \( \lambda = 1/4 \) we obtain for \( r \in (0, 1) \):
\[
\frac{\mathcal{K}'(r)}{\mathcal{K}'(\sqrt{r})} > \frac{1 + \sqrt{r}}{1 + \frac{\delta^2(r)}{1 + \frac{1}{4}\delta^2(r)}} = \frac{2(1 + \sqrt{r})}{(1 + \sqrt{r})^2 + \frac{1}{4}(1 - \sqrt{r})^2} > 1.
\]
This is a refinement of the first inequality in (3.9) with \( \alpha = 0 \).
Remark 3.5. The following companion of (3.9) holds for \( r \in (0, 1) \):

\[
0 < \frac{2}{1 + \sqrt{r}} - \frac{K'(r)}{K'(\sqrt{r})} \leq 0.2254 \ldots .
\]

The constant bounds are sharp. Does there exist a closed expression for the given numerical upper bound?

In the literature we can find many refinements of the classical arithmetic mean - geometric mean inequality; see, for example, [7] II.3. The second inequality in (3.9) plays a key role in the proof of the following new refinement.

**Corollary 3.6.** For all \( r \in (0, 1) \) we have

\[
0 < \frac{K\left(\frac{1-r}{1+r}\right) - K\left(\frac{1-\sqrt{r}}{1+\sqrt{r}}\right)}{K(\sqrt{1-r})} < \frac{1 + r}{2} - \sqrt{r}.
\]

**Proof.** Let \( r \in (0, 1) \). Since

\[
0 < \frac{1 - \sqrt{r}}{1 + \sqrt{r}} < \frac{1 - r}{1 + r} < 1,
\]

we obtain

\[
K'\left(\frac{1 - \sqrt{r}}{1 + \sqrt{r}}\right) < K'\left(\frac{1 - r}{1 + r}\right).
\]

This leads to the left-hand side of (3.21). Applying the second inequality in (3.9) with \( \beta = 1/2 \) gives

\[
\frac{K'(r)}{K'(\sqrt{r})} < \frac{2}{1 + \sqrt{r}} < \frac{2 + r - \sqrt{r}}{1 + r}.
\]

It follows that

\[
(1 + r)K'(r) < (2 + r - \sqrt{r})K'(\sqrt{r}) = (1 + \sqrt{r})K'(\sqrt{r}) + (1 + r - 2\sqrt{r})K'(\sqrt{r}).
\]

Using (1.3) we find

\[
2K\left(\frac{1 - r}{1 + r}\right) < 2K\left(\frac{1 - \sqrt{r}}{1 + \sqrt{r}}\right) + (1 + r - 2\sqrt{r})K(\sqrt{1-r}).
\]

This is equivalent to the right-hand side of (3.21). \( \square \)

4. **Concluding remarks**

The generalized elliptic integrals are defined for \( a \in (0, 1/2] \) and \( r \in (0, 1) \) by

\[
K_a(r) = \frac{\pi}{2} F\left(a, 1-a; 1; r^2\right) \quad \text{and} \quad E_a(r) = \frac{\pi}{2} F\left(a-1, 1-a; 1; r^2\right).
\]

The special case \( a = 1/2 \) gives

\[
K_{1/2} = K \quad \text{and} \quad E_{1/2} = E.
\]

For more information on these functions we refer the reader to [3].

The referee pointed out that Theorems 3.1 and 3.3 seem to hold not only for the complete elliptic integral \( K \) but also for \( K_a \). Indeed, using similar methods as in the proof of Theorem 3.1 we obtain the following result.
Theorem 4.1. Let $a \in (0, 1/2]$. For all $r \in (0, 1)$ we have
\[ \frac{1}{1 + \lambda_a r} < \frac{\mathcal{K}_a(r)}{\mathcal{K}_a(\sqrt{r})} < \frac{1}{1 + \mu_a r} \]
with the best possible factors $\lambda_a = a(1 - a)$ and $\mu_a = 0$.

It remains an open problem to find a corresponding generalization of Theorem 3.3. M.E.H. Ismail even asked whether our inequalities can be extended to the zero-balanced hypergeometric function $F(a, b; a + b; r^2)$.

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References


