

THE EGH CONJECTURE AND THE SPERNER PROPERTY OF COMPLETE INTERSECTIONS

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ABSTRACT. Let A be a graded complete intersection over a field and B the monomial complete intersection with the generators of the same degrees as A . The EGH conjecture says that if I is a graded ideal in A , then there should be an ideal J in B such that B/J and A/I have the same Hilbert function. We show that if the EGH conjecture is true, then it can be used to prove that every graded complete intersection over any field has the Sperner property.

1. INTRODUCTION

The Sperner theory for finite posets was started by the paper of Sperner [16] and was well established by the 1960s. The theory served as a model for the theory of the weak and strong Lefschetz properties of Artinian rings, which has been intensively studied in recent years by many authors ([6], [7], [10], [12], [13], [14], [15]). For the detail of the Sperner theory of finite posets we refer the reader to Anderson [2], Aigner [3], Bollobás [4], Greene–Kleitman [9], for example.

A graded Artinian algebra A is said to have the Sperner property if $\max\{\mu(\mathfrak{a})\}$, where \mathfrak{a} runs over all ideals in A , is equal to the dimension of a homogeneous component A_i of A of the maximal size. It is known that the weak Lefschetz property immediately implies the Sperner property of A ([18] Corollary, [17] Prop 3.2.(3)).

In this paper we introduce the notion of the matching property of Gorenstein algebras (Definition 3.2). This is a ring theoretic version of the matching property of posets as defined in Aigner [3] VIII §3.

Then we will prove that Gorenstein algebras with the matching property enjoy the Sperner property (Proposition 4.1), as naturally expected from the definition. A reason we are interested in the matching property of Gorenstein algebras is that it has a relation to the EGH (Eisenbud–Green–Harris) conjecture ([8]) on complete intersections. In Theorem 5.2 of the present paper we prove that if the EGH conjecture is true it implies that all graded complete intersections with a standard grading have the matching property. As a consequence it implies that all complete intersections for which the EGH conjecture have been proved true have the Sperner property. Theorem 5.3 is such a case.

It would be a good theorem if one could prove that all complete intersections have the Sperner property. Being weaker than the WLP, the Sperner property has one advantage over the WLP; we can expect it does not depend on the characteristic of the ground field, while the WLP and SLP inescapably depend on the characteristic.

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Throughout this paper a field means any field (including finite fields) unless otherwise specified.

2. THE DILWORTH NUMBER OF ARTINIAN GRADED RINGS

Notation 2.1. Let $V = \bigoplus_{i=0}^{\infty} V_i$ be a graded vector space, where $\dim_K V_i < \infty$ for all i . The map $i \mapsto H(V, i) := \dim_K V_i$ is called the Hilbert function of V .

Definition 2.2. Let $A = \bigoplus_{i=0}^c A_i$ be an Artinian graded algebra over a field $A_0 = K$, and $\mathfrak{m} = \bigoplus_{i=1}^c A_i$ the maximal ideal of A . The **Dilworth number**, denoted $d(A)$, is defined to be

$$d(A) := \max\{\mu(\mathfrak{a}) \mid \text{ideals } \mathfrak{a} \subset A\}.$$

We say that A has the **Sperner property** if

$$d(A) = \max\{\dim_K A_i \mid i = 0, 1, 2, \dots\}.$$

We denote by $\mu(\mathfrak{a})$ and $\tau(\mathfrak{a})$ respectively the minimal number of generators and the type of \mathfrak{a} . Namely $\mu(\mathfrak{a}) = \dim_K \mathfrak{a}/\mathfrak{m}\mathfrak{a}$ and $\tau(\mathfrak{a}) = \dim_K(\mathfrak{a} : \mathfrak{m})/\mathfrak{a}$. (These may apply to any local ring (A, \mathfrak{m}) not necessarily Artinian.)

For an Artinian local ring (A, \mathfrak{m}) we introduce two families of ideals:

$$\mathcal{F}(A) := \{\mathfrak{a} \mid \mu(\mathfrak{a}) = d(A)\}, \quad \mathcal{G}(A) := \{\mathfrak{a} \mid \tau(\mathfrak{a}) = d(A)\}.$$

If A is graded, we assume that \mathfrak{a} runs over all graded ideals of A (although it does not make much difference). We need the following result.

Proposition 2.3. *Assume that (A, \mathfrak{m}) is an Artinian local ring. Then $\mathcal{F}(A)$ and $\mathcal{G}(A)$ are posets with the inclusion as an order, and moreover these are lattices with respect to $+$ and \cap as join and meet, and they are isomorphic as lattices via the correspondence:*

$$\begin{aligned} \mathcal{F}(A) &\leftrightarrow \mathcal{G}(A) \\ \mathcal{F}(A) \ni \mathfrak{a} &\mapsto \mathfrak{m}\mathfrak{a} \in \mathcal{G}(A), \end{aligned}$$

and

$$\mathcal{G}(A) \ni \mathfrak{a} \mapsto \mathfrak{a} : \mathfrak{m} \in \mathcal{F}(A).$$

Assume furthermore that (A, \mathfrak{m}) is Gorenstein. Then the correspondence $\mathfrak{a} \mapsto 0 : \mathfrak{a}$ gives an order reversing isomorphism of lattices in both directions:

$$\mathcal{F}(A) \leftrightarrow \mathcal{G}(A).$$

Proof. Except for the last assertion this is proved in Ikeda-Watanabe [11]. Assume that A is Gorenstein. Then we have $0 : (0 : \mathfrak{a}) = \mathfrak{a}$ for any ideal $\mathfrak{a} \subset A$. Furthermore we have $0 : (a_1, a_2, \dots, a_m) = \bigcap_{i=1}^m (0 : a_i)$ and the generators (a_1, \dots, a_m) are a minimal set of generators if and only if the intersection $\bigcap (0 : a_i)$ is an irredundant intersection of irreducible ideals. This proves the last assertion. \square

3. MATCHING PROPERTY OF MONOMIAL COMPLETE INTERSECTIONS

Notation 3.1. Let $A = \bigoplus_{i=0}^{\infty} A_i$ be a graded algebra over a field $K = A_0$. For a subset $V \subset A_k$, we denote by $A_1 \cdot V$ the vector subspace in A_{k+1} spanned by the set

$$\{xv \mid x \in A_1, v \in V\}.$$

Definition 3.2. Let $A = \bigoplus_{i=0}^c A_i$ be a graded Artinian Gorenstein algebra over a field $K = A_0$. We say that A has the **matching property** if we have

$$\dim_K V \leq \dim_K(A_1 \cdot V)$$

for any vector space $V \subset A_j$ with j such that $\dim_K A_j \leq \dim_K A_{j+1}$.

Over a field of characteristic zero all monomial complete intersections have the WLP, which implies trivially the matching property of monomial complete intersections. In Proposition 3.4 we show that the matching property holds for all monomial complete intersections over a field of any characteristic.

Let N be a positive integer, and let $P(N)$ be the set of all positive divisors of N . Recall that a divisor lattice is the set $P(N)$ for some N endowed with a structure of a poset where the partial order is the divisibility. It is a poset with rank function, hence we have the rank decomposition $P = \bigsqcup_{i=0}^c P_i$. For details see [2] or [3].

Proposition 3.3. *Let $P = \bigsqcup_{i=0}^c P_i$ be a divisor lattice. For a subset $S \subset P_j$, define the neighbor $N(S)$ of S to be*

$$N(S) = \{y \in P_{j+1} \mid \exists x \in S \text{ such that } x < y\}.$$

Then if $|P_j| \leq |P_{j+1}|$, then for any $S \subset P_j$, we have

$$|S| \leq |N(S)|.$$

Proof. This follows from the theorem which says P has a symmetric chain decomposition. (See deBruijn et al. [5].) This also follows from the fact that the monomial complete intersections over a field of characteristic zero have the WLP. (See [10].) □

Proposition 3.4. *Any monomial complete intersection over a field has the matching property.*

Proof. Let $R = K[x_1, x_2, \dots, x_n]$ be the polynomial ring over a field K and write $A = R/\mathfrak{a} = \bigoplus_{i=0}^c A_i$, $\mathfrak{a} = (x_1^{a_1}, x_2^{a_2}, \dots, x_n^{a_n})$. Let $V \subset A_j$ be a vector space. We have to show that $\dim_K V \leq \dim_K(A_1 \cdot V)$ if $\dim_K A_j \leq \dim_K A_{j+1}$. Let I be the ideal of A which is generated by V . Then we have

$$\dim_K V = H(I, j), \quad \dim_K(A_1 \cdot V) = H(I, j + 1).$$

Note that there exists a monomial ideal J in A such that A/I and A/J have the same Hilbert function. To see this, let $\phi : R \rightarrow A$ be the natural surjection. Then $\mathfrak{b} := \phi^{-1}(I)$ contains \mathfrak{a} . Let $\text{In}(\mathfrak{b})$ be the ideal generated by the set of monomials that occur as the head term of an element in \mathfrak{b} w.r.t. some monomial order, and let J be the image of $\text{In}(\mathfrak{b})$ under ϕ . It is well known that R/\mathfrak{b} and $R/\text{In}(\mathfrak{b})$ have the same Hilbert function. Furthermore we have that $\text{In}(\mathfrak{b}) \supset \mathfrak{a}$, since monomials are head terms of themselves. Since $R/\mathfrak{b} = A/I$ and $R/\text{In}(\mathfrak{b}) = A/J$, it suffices to prove $H(J, j) \leq H(J, j + 1)$. Note that A has the monomial basis $P = \bigsqcup_{i=0}^c P_i$, which has a structure of the divisor lattice. Assume that $\dim_K A_j \leq \dim_K A_{j+1}$, or equivalently, $|P_j| \leq |P_{j+1}|$. Put $S = J \cap P_j$ and let J' be the ideal generated by S . Then by Proposition 3.3, we have $\mu(J') = |S| \leq |N(S)| = \mu(\mathfrak{m}J')$, hence $H(J, j) \leq H(J, j + 1)$. Thus we have proved that $H(I, j) \leq H(I, j + 1)$. □

4. THE SPERNER PROPERTY OF ARTINIAN GORENSTEIN RINGS

Proposition 4.1. *Let $A = \bigoplus_{i=0}^c A_i$ be a graded Artinian Gorenstein algebra over a field $A_0 = K$ with a unimodal Hilbert function. Assume that A has the matching property. Then A has the Sperner property.*

Proof. Let $I \subset A$ be an ideal with $\mu(I) = d(A)$, the Dilworth number of A . So $I \in \mathcal{F}(A)$. Let $j_0 = \min\{j \mid \dim_K A_j > \dim_K A_{j+1}\}$. We are going to show that $\mu(I) = \dim_K A_{j_0}$. By Watanabe [17] Lemma 2.4 we may assume that I is graded.

Let α be the initial degree of I . We treat two cases; $\alpha \geq j_0$ and $\alpha < j_0$.

Case 1. Assume that $\alpha \geq j_0$. Then $\mathfrak{m}^{j_0} \supset I$ and $\mathfrak{m}^{j_0+1} \supset \mathfrak{m}I$. Put $J = (0 : \mathfrak{m}I)$. Then, bearing in mind that A is Gorenstein, $J \supset 0 : \mathfrak{m}^{j_0+1} = \mathfrak{m}^{c-j_0}$, and thanks to Proposition 2.3, we have $J \in \mathcal{F}(A)$. Since the Hilbert function of A is unimodal, we have $c - j_0 \leq j_0$. This means that $J \supset \mathfrak{m}^{j_0}$ and the initial degree of J is smaller than or equal to j_0 . If the initial degree of J equals j_0 , then $J = \mathfrak{m}^{j_0}$. Otherwise we can use Sublemma 4.2 repeatedly to obtain an ideal $J' \in \mathcal{F}(A)$ whose initial degree is j_0 which contains \mathfrak{m}^{j_0} . This means that $J' = \mathfrak{m}^{j_0}$ and $\mu(J') = \dim_K A_{j_0}$.

Case 2. Assume that $\alpha < j_0$. Then we use again Sublemma 4.2 repeatedly and we obtain an ideal $I' \in \mathcal{F}(A)$ such that the initial degree of I' is at least j_0 . So this case reduces to Case 1. \square

Sublemma 4.2. *With the same notation and assumption as Proposition 4.1 suppose that $I = \bigoplus_{j \geq \alpha} I_j$ is a graded ideal of A and α is the initial degree. Put $I' = \bigoplus_{j \geq \alpha+1} I_j$. Let $j_0 = \min\{j \mid \dim_K A_j > \dim_K A_{j+1}\}$ and assume that $\alpha < j_0$. Then we have $\mu(I) \leq \mu(I')$. Hence if $I \in \mathcal{F}(A)$, then $I' \in \mathcal{F}(A)$.*

Proof. Note that

$$\mu(I) = \dim_K I_\alpha + \dim_K I_{\alpha+1}/(A_1 \cdot I_\alpha) + \cdots + \dim_K I_c/(A_1 \cdot I_{c-1}),$$

and

$$\mu(I') = \dim_K I_{\alpha+1} + \dim_K I_{\alpha+2}/(A_1 \cdot I_{\alpha+1}) + \cdots + \dim_K I_c/(A_1 \cdot I_{c-1}).$$

Since A has the matching property, we have

$$\begin{aligned} \mu(I') - \mu(I) &= \dim_K I_{\alpha+1} - \{\dim_K I_\alpha + \dim_K I_{\alpha+1}/(A_1 \cdot I_\alpha)\} \\ &= \{\dim_K I_{\alpha+1} - \dim_K I_{\alpha+1}/(A_1 \cdot I_\alpha)\} - \dim_K I_\alpha \\ &= \dim_K A_1 \cdot I_\alpha - \dim_K I_\alpha \geq 0. \end{aligned}$$

\square

5. MAIN RESULT

In this section we show that the EGH conjecture implies the Sperner property of complete intersections. Recall that the EGH conjecture is as follows:

Conjecture 5.1 (EGH conjecture [8]). *Let $R = K[x_1, x_2, \dots, x_n]$ be the polynomial ring over a field K . If I is a graded ideal in R containing a regular sequence f_1, f_2, \dots, f_n of degrees a_1, a_2, \dots, a_n respectively, then I has the same Hilbert function as an ideal containing $(x_1^{a_1}, x_2^{a_2}, \dots, x_n^{a_n})$.*

Theorem 5.2. *Suppose that the Eisenbud-Green-Harris conjecture is true for a graded complete intersection A over a field K . Then A has the matching property. Consequently A has the Sperner property.*

Proof. Let $A = \bigoplus_{i=0}^c A_i$ be a graded complete intersection defined by the ideal generated by homogeneous elements

$$f_1, f_2, \dots, f_n \in R = K[x_1, x_2, \dots, x_n]$$

of degrees a_1, a_2, \dots, a_n respectively. Let j be an integer such that $\dim_K A_j \leq \dim_K A_{j+1}$ and let $V \subset A_j$ be any vector subspace. By Proposition 4.1 it is enough to show that

$$\dim_K V \leq \dim_K (A_1 \cdot V).$$

Let $I = VA$, namely, I is the ideal in A generated by V . Then

$$\dim_K V = \mu(I), \quad \dim_K (A_1 \cdot V) = \mu(\mathbf{m}I).$$

Since $\mu(\mathbf{m}^i I) = \dim_K I_{i+j}$, we have

$$(1) \quad \mu(I) = \dim_K V = H(A, j) - H(A/I, j),$$

and

$$(2) \quad \mu(\mathbf{m}I) = \dim_K (A_1 \cdot V) = H(A, j + 1) - H(A/I, j + 1).$$

Let $\mathfrak{a} \subset R$ be a graded ideal such that $A/I = R/\mathfrak{a}$, where $\mathfrak{a} \supset (f_1, f_2, \dots, f_n)$. Now we invoke the EGH conjecture for A , which is to assume the following: There is an ideal \mathfrak{b} containing $(x_1^{a_1}, \dots, x_n^{a_n})$ such that R/\mathfrak{b} has the same Hilbert function as that of R/\mathfrak{a} . Put $B = R/(x_1^{a_1}, \dots, x_n^{a_n})$ and $J = \mathfrak{b}/(x_1^{a_1}, \dots, x_n^{a_n})$. We regard J as an ideal in B . By definition I has initial degree j , so j is the initial degree of J . Let J' be the ideal of B generated by the elements of J_j .

We want to show that $\mu(\mathbf{m}I) - \mu(I) \geq 0$. By (1) and (2)

$$\mu(\mathbf{m}I) - \mu(I) = \{H(A, j + 1) - H(A/I, j + 1)\} - \{H(A, j) - H(A/I, j)\}.$$

Since A and B have the same Hilbert function and so do A/I and B/J , this is equal to

$$(3) \quad \mu(\mathbf{m}I) - \mu(I) = \{H(B, j + 1) - H(B/J, j + 1)\} - \{H(B, j) - H(B/J, j)\}.$$

As well as for I , we have

$$(4) \quad \mu(\mathbf{m}J') - \mu(J') = \{H(B, j + 1) - H(B/J', j + 1)\} - \{H(B, j) - H(B/J', j)\},$$

where we used the same \mathbf{m} for the maximal ideal of B . Note that $\mu(J') = \dim_K J'_j$ and $\mu(\mathbf{m}J') = \dim_K (A_1 \cdot J')$. Since B has the matching property (Proposition 3.4), we have $\mu(\mathbf{m}J') - \mu(J') \geq 0$, which implies that the RHS of (4) is ≥ 0 . Since $J' \subset J$, we have

$$H(B/J', j + 1) - H(B/J, j + 1) \geq 0.$$

Add this number to the RHS of (4). Then we obtain

$$\{H(B, j + 1) - H(B/J, j + 1)\} - \{H(B, j) - H(B/J', j)\} \geq 0.$$

Since $J'_j = J_j$, we have

$$\{H(B, j + 1) - H(B/J, j + 1)\} - \{H(B, j) - H(B/J, j)\} \geq 0,$$

which implies that the LHS of (3) is ≥ 0 . Thus we have proved that

$$\dim_K V \leq \dim_K (A_1 \cdot V),$$

as desired. □

Theorem 5.3. *Let $R = K[x_1, x_2, \dots, x_n]$ be the polynomial ring. Suppose that I is a complete intersection ideal $I = (L_1, L_2, \dots, L_n)$, where each L_j is a product of linear forms. Then $A := R/I$ has the Sperner property.*

Proof. By [1] Corollary 3.3, the EGH conjecture is true for A . So Theorem 5.2 applies. \square

Remark 5.4. In the above theorem we may require only that $(n-2)$ of the generators are products of linear forms and the other two are arbitrary homogeneous elements. Indeed, to prove the theorem we may assume that K is algebraically closed. The proof is carried out by induction on n . If $n = 2$, then any homogeneous form factors into a product of linear forms over K .

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