PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 145, Number 4, April 2017, Pages 1717–1728
http://dx.doi.org/10.1090/proc/13367
Article electronically published on October 20, 2016

DAMPED STAR GRAPHS OF STIELTJES STRINGS

M. MÖLLER AND V. PIVOVARCHIK

(Communicated by Michael Hitrik)

Abstract. We consider a direct and an inverse problem arising in the theory of small transverse vibrations of a star graph of Stieltjes strings damped at the midpoint. The exterior vertices of the graph are supposed to be fixed. We give necessary and sufficient conditions on a sequence of complex numbers to be the spectrum of such a problem.

1. Introduction

It was shown in [7], [8] for a very wide class of inhomogeneous strings in the case where one of the ends of a vibrating string is damped (subject to viscous friction) one spectrum together with the length (if it is finite) of the string uniquely determine the density of the string, while in the case of an undamped string two spectra of boundary value problems are necessary to determine the density. An inverse problem for undamped Stieltjes strings (weightless threads bearing a finite number of point masses) was solved in [2]. For a Stieltjes string damped at one end the inverse problem was solved in [19], [20], [10] and in [13] for the inverse problem by parts of two spectra. An application was given in [9]. The same equation occurs if one considers longitudinal vibrations of point masses connected by springs and also occurs in the theory of synthesis of electrical circuits; see [12], [5, p. 129].

Boundary value problems on graphs consisting of Stieltjes strings are natural generalizations of boundary value problems on a single interval and are often used in the theory of vibrations of nets [3], [4], [12]. Direct and inverse problems on a star graph were considered in [1] for the case of simple eigenvalues, in [16] without the restriction to simple eigenvalues, and in [17] with prescribed numbers of masses on the edges. It was shown in [18] that restrictions on multiplicities of eigenvalues are independent on whether the strings of the star graph are smooth or not. An inverse problem on a tree of Stieltjes strings was solved in [15] for the case of simple eigenvalues.

In this paper we consider a direct and an inverse problem generated by the equations of small transverse vibrations of a plane graph of Stieltjes strings damped at the central vertex. We give necessary and sufficient conditions for a set of complex numbers to be the spectrum of such a problem. In Section 2 we give a physical motivation of such a problem and describe its spectrum, i.e., we solve the direct problem, and in Section 3 we solve the inverse problem.

Received by the editors December 19, 2015 and, in revised form, June 19, 2016.
2010 *Mathematics Subject Classification.* Primary 47B39; Secondary 34A55, 39A70.
Key words and phrases. Star graph, inverse problem, transverse vibrations, damped vibration, Dirichlet boundary condition, Neumann boundary condition, Hermite-Biehler polynomial, Nevanlinna function.

2. Formulation of the problem

Following [2], a thread, i.e., an elastic string of zero density, bearing a finite number of point masses is called a Stieltjes string. Suppose we have a plane star graph of $q > 2$ Stieltjes strings joined at the interior vertex. We assume all the other vertices to be fixed and a mass $M \geq 0$ to be placed at the interior vertex. In [16] the authors assumed such a plane net was stretched and considered two problems of small transverse vibrations of the graph: one with the mass M fixed at the central vertex, called the Dirichlet problem, and one with M free to move in the direction orthogonal to the equilibrium position of the strings, called the Neumann problem. In this paper it is assumed that the central vertex can move subject to damping with constant coefficient of damping $\alpha > 0$.

Let e_j, $j = 1, \dots, q$, be the j-th edge of the graph and let $n_j \geq 0$ be the number of point masses in the interior of e_j. The exterior and interior vertices of the Stieltjes string e_j will be denoted by $P_0^{(j)}$ and $P_{n_j+1}^{(j)}$, respectively. Observe that $P_{n_1+1}^{(1)}, \dots, P_{n_q+1}^{(q)}$ is the common interior vertex of the q Stieltjes strings. As we move from the exterior vertex to the interior vertex of e_j, the n_j point masses $m_k^{(j)} > 0$ in the interior of e_j will be encountered at the points $P_k^{(j)}$, $k = 1, \dots, n_j$. Denoting the distance between $P_k^{(j)}$ and $P_{k+1}^{(j)}$ by $l_k^{(j)}$, $k = 0, \dots, n_j$, it is clear that the length of the edge e_j is $l_j := \sum_{k=0}^{n_j} l_k^{(j)}$. For $j = 1, \dots, q$ and $k = 1, \dots, n_j$ denote by $v_k^{(j)}(t)$ the transverse displacements of the point mass $m_k^{(j)}$ at time t. We assume the threads to be stretched by forces each equal to 1.

Then small transverse vibrations of the edges e_j, $j = 1, \dots, q$, are governed by the equations

$$\frac{v_k^{(j)}(t) - v_{k+1}^{(j)}(t)}{l_k^{(j)}} + \frac{v_k^{(j)}(t) - v_{k-1}^{(j)}(t)}{l_{k-1}^{(j)}} + m_k^{(j)} v_k^{(j)\prime\prime}(t) = 0, \quad k = 1, \dots, n_j, \qquad t \in \mathbb{R}.$$

Continuity of the star graph at the interior vertex, where all edges join, yields

$$v_{n_1+1}^{(1)}(t) = \cdots = v_{n_q+1}^{(q)}(t), \qquad t \in \mathbb{R}.$$

The balance of forces at this interior vertex implies

$$\sum_{j=1}^{q} \frac{v_{n_j+1}^{(j)}(t) - v_{n_j}^{(j)}(t)}{l_{n_j}^{(j)}} = -M v_{n_1+1}^{(1)\prime\prime}(t) - \alpha v_{n_1+1}^{(1)\prime}(t), \quad t \in \mathbb{R}.$$

The exterior vertices are fixed, which means that we have Dirichlet boundary conditions

$$v_0^{(j)}(t) = 0, \quad j = 1, \dots, q, \quad t \in \mathbb{R}.$$

Substituting $v_k^{(j)}(t) = u_k^{(j)} e^{i\lambda t}$ we obtain the following recurrences for the amplitudes $u_k^{(j)}$:

$$\frac{u_k^{(j)} - u_{k+1}^{(j)}}{l_k^{(j)}} + \frac{u_k^{(j)} - u_{k-1}^{(j)}}{l_{k-1}^{(j)}} - m_k^{(j)} \lambda^2 u_k^{(j)} = 0, \quad j = 1, \dots, q, \ k = 1, \dots, n_j, \tag{2.1}$$

$$u_{n_1+1}^{(1)} = \cdots = u_{n_q+1}^{(q)}, \tag{2.2}$$

$$\sum_{j=1}^{q} \frac{u_{n_j+1}^{(j)} - u_{n_j}^{(j)}}{l_{n_j}^{(j)}} = (M\lambda^2 - i\alpha\lambda) u_{n_1+1}^{(1)}, \tag{2.3}$$

$$u_0^{(j)} = 0, \quad j = 1, \dots, q. \tag{2.4}$$

Problem (2.1)–(2.4) with $\alpha = 0$ is the Neumann problem (problem $\mathbf{N}$), and the problem with $\alpha > 0$ is called the *damped problem.*

If we clamp our strings at the interior vertex, then we obtain q problems $\mathbf{D}_j$, $j = 1, \dots, q$, on the edges with Dirichlet boundary conditions at both ends:

$$\frac{u_k^{(j)} - u_{k+1}^{(j)}}{l_k^{(j)}} + \frac{u_k^{(j)} - u_{k-1}^{(j)}}{l_{k-1}^{(j)}} - m_k^{(j)} \lambda^2 u_k^{(j)} = 0, \quad k = 1, \dots, n_j, \tag{2.5}$$

$$u_{n_j+1}^{(j)} = 0, \tag{2.6}$$

$$u_0^{(j)} = 0. \tag{2.7}$$

The union of problems (2.5)–(2.7) for $j = 1, \dots, q$ will be called the Dirichlet problem ($\mathbf{D}$). Observe that the Dirichlet problem is described by (2.1), (2.4), and (2.6) for $j = 1, \dots, q$.

According to [2, p. 188] we look for solutions of (2.1) in the form

$$u_k^{(j)} = R_{2k-2}^{(j)}(\lambda^2) u_1^{(j)}, \quad k = 1, \dots, n_j,$$

where $R_{2k-2}^{(j)}$ are polynomials of degree $k-1$. Defining

$$R_{2k-1}^{(j)}(\lambda^2) = \frac{R_{2k}^{(j)}(\lambda^2) - R_{2k-2}^{(j)}(\lambda^2)}{l_k^{(j)}}, \quad k = 1, \dots, n_j,$$

it follows from these definitions and from (2.1) and (2.4) that the polynomials $R_k^{(j)}$ have to satisfy the recurrence relations

$$R_{2k-1}^{(j)}(\lambda^2) = -\lambda^2 m_k^{(j)} R_{2k-2}^{(j)}(\lambda^2) + R_{2k-3}^{(j)}(\lambda^2), \quad k = 1, \dots, n_j, \tag{2.8}$$

$$R_{2k}^{(j)}(\lambda^2) = l_k^{(j)} R_{2k-1}^{(j)}(\lambda^2) + R_{2k-2}^{(j)}(\lambda^2), \quad k = 1, \dots, n_j, \tag{2.9}$$

$$R_{-1}^{(j)}(\lambda^2) = \frac{1}{l_0^{(j)}}, \quad R_0^{(j)}(\lambda^2) = 1. \tag{2.10}$$

Since (2.2) and (2.3) have to be satisfied simultaneously, they are equivalent to (2.2) and

$$\sum_{j=1}^{q} \frac{u_{n_j+1}^{(j)} - u_{n_j}^{(j)}}{l_{n_j}^{(j)}} = \sum_{j=1}^{q} \frac{1}{q} (M\lambda^2 - i\alpha\lambda) u_{n_j+1}^{(j)}.$$

Then (2.2), (2.3) and (2.6) for $j = 1, \dots, q$ can be written as

$$R_{2n_1}^{(1)}(\lambda^2)u_1^{(1)} = \cdots = R_{2n_q}^{(q)}(\lambda^2)u_1^{(q)}, \tag{2.11}$$

$$\sum_{j=1}^{q} R_{2n_j-1}^{(j)}(\lambda^2)u_1^{(j)} = \sum_{j=1}^{q} \frac{1}{q}(M\lambda^2 - i\alpha\lambda)R_{2n_j}^{(j)}(\lambda^2)u_1^{(j)}, \tag{2.12}$$

$$R_{2n_1}^{(j)}(\lambda^2)u_1^{(j)} = 0, \quad j = 1, \dots, q. \tag{2.13}$$

Recall that a complex number λ belongs to the spectrum of the Neumann problems (2.1)–(2.4) with $\alpha = 0$, of the Dirichlet problems (2.5)–(2.7), or of the damped problems (2.1)–(2.4) with $\alpha \neq 0$, respectively, if and only if this problem has a nontrivial solution $(u_k^{(j)})_{j=1,k=0}^{q,n_j+1} \neq 0$. Observing that $u_0^{(j)} = 0$ for $j = 1, \dots, q$ by (2.4) and (2.7), respectively, and that then $u_k^{(j)} = 0$ for $k = 2, \dots, n_j$ if $u_1^{(j)} = 0$ by (2.1) and (2.5), respectively, this means that λ belongs to the spectrum of the corresponding problem if and only if $u_1^{(j)} \neq 0$ for at least one $j \in \{1, \dots, q\}$.

We have seen that $(u_k^{(j)})_{j=1,k=0}^{q,n_j+1}$ is a solution of the Neumann problems (2.1)–(2.4) with $\alpha = 0$ if and only if (2.12) and (2.11) hold with $\alpha = 0$. This can be written as a system of linear equations

$$L_N(\lambda^2)(u_1^{(j)})_{j=1}^{q} = 0, \tag{2.14}$$

where the first row of the $q \times q$ matrix $L_N(\lambda^2)$ has the entries $R_{2n_j-1}^{(j)}(\lambda^2) - \frac{M}{q}\lambda^2 R_{2n_j}^{(j)}(\lambda^2)$, and for $j = 2, \dots, q$, the possibly nonzero entries of $L_N(\lambda^2)$ in the j-th row are $-R_{2n_{j-1}}^{(j-1)}(\lambda^2)$ in column $j-1$ and $R_{2n_j}^{(j)}(\lambda^2)$ in column j. Therefore, the spectrum of the Neumann problem is the set of those λ for which (2.14) has a nontrivial solution, i.e., for which $\phi_N(\lambda^2) := \det L_N(\lambda^2) = 0$. An expansion of this determinant with respect to its first row gives the explicit formula

$$\phi_N(\lambda^2) = \sum_{j=1}^{q} \left[R_{2n_j-1}^{(j)}(\lambda^2) - \frac{M}{q}\lambda^2 R_{2n_j}^{(j)}(\lambda^2) \right] \prod_{k=1,\, k\neq j}^{q} R_{2n_k}^{(k)}(\lambda^2). \tag{2.15}$$

The number λ belongs to the spectrum of the Dirichlet problems (2.5)–(2.7) if and only if (2.13) holds with at least one $u_1^{(j)} \neq 0$. Therefore the spectrum of the Dirichlet problem coincides with the set of zeros of

$$\phi_D(\lambda^2) = \prod_{j=1}^{q} R_{2n_j}^{(j)}(\lambda^2). \tag{2.16}$$

A reasoning as for the Neumann problem shows that the spectrum of the damped problem coincides with the set of zeros of

$$\phi(\lambda) = \sum_{j=1}^{q} \left[R_{2n_j-1}^{(j)}(\lambda^2) - \frac{1}{q}(M\lambda^2 - i\alpha\lambda)R_{2n_j}^{(j)}(\lambda^2) \right] \prod_{k=1,\, k\neq j}^{q} R_{2n_k}^{(k)}(\lambda^2),$$

which has the representation

$$\phi(\lambda) = \phi_N(\lambda^2) + i\alpha\lambda\phi_D(\lambda^2). \tag{2.17}$$

Remark 2.1. The degree of the polynomial ϕ_N is $1+\sum_{j=1}^{q} n_j$ if $M > 0$ and $\sum_{j=1}^{q} n_j$ if $M = 0$, the degree of ϕ_D is $\sum_{j=1}^{q} n_j$ in any case, while the degree of ϕ is $2 + 2\sum_{j=1}^{q} n_j$ if $M > 0$ and $1 + 2\sum_{j=1}^{q} n_j$ if $M = 0$.

Next we consider the damped problem in the case that none of the threads has an interior mass.

Example 2.2. Assume that $n_j = 0$ for all $j = 1, \dots, q$.

1. If $M = 0$, then the spectrum consists of one simple eigenvalue $\frac{i}{\alpha}\sum_{j=1}^{q}\frac{1}{l^{(j)}}$.
2. If $M > 0$, then the spectrum consists of two eigenvalues, counted with multiplicity, namely

$$\frac{i\alpha}{2M} \pm \left(\frac{1}{M}\sum_{j=1}^{q}\frac{1}{l^{(j)}} - \frac{\alpha^2}{4M^2}\right)^{\frac{1}{2}}.$$

Proof. For $n_j = 0$ we have $R_{2n_j}^{(j)} = R_0^{(j)} = 1$ and $R_{2n_j-1}^{(j)} = R_{-1} = \frac{1}{l^{(j)}}$. Therefore $\phi_D(\lambda^2) = 1$ and

$$\phi_N(\lambda^2) = \sum_{j=1}^{q}\left[\frac{1}{l^{(j)}} - \frac{M}{q}\lambda^2\right] = \sum_{j=1}^{q}\frac{1}{l^{(j)}} - M\lambda^2,$$

which gives

$$\phi(\lambda) = -M\lambda^2 + \sum_{j=1}^{q}\frac{1}{l^{(j)}} + i\alpha\lambda.$$

Solving $\phi(\lambda) = 0$ completes the proof. □

Definition 2.3. A complex-valued function ω defined on an open subset of $\mathbb{C}$ is said to be a Nevanlinna function if:

1) the domain of ω contains $\mathbb{C} \setminus \mathbb{R}$ and ω is analytic on $\mathbb{C} \setminus \mathbb{R}$;
2) $\omega(\overline{z}) = \overline{\omega(z)}$ for $z \in \mathbb{C} \setminus \mathbb{R}$;
3) $\operatorname{Im} z \operatorname{Im} \omega(z) \geq 0$ for $\operatorname{Im} z \neq 0$.

Definition 2.4. The Nevanlinna function ω is said to be an S-function if ω is defined and analytic on $\mathbb{C} \setminus [0, \infty)$ and if $\omega(z) > 0$ for $z \in (-\infty, 0)$.

Definition 2.5. A meromorphic S-function ω is said to be an S_0-function if 0 is not a pole of ω.

Theorem 2.6 ([16, Theorem 2.4])**.** *After cancellation of common factors (if any) in the numerator and in the denominator the function*

$$\frac{\phi_D}{\phi_N}$$

becomes an S_0-function.

Theorem 2.7 ([16, Theorem 2.5])**.** *Let $n = \sum_{j=1}^{q} n_j$ and assume that $n > 0$. Then the eigenvalues $(\nu_k)_{k=-n,k\neq 0}^{n}$ of problem* **D** *interlace with the eigenvalues $(\mu_k)_{k=-n,k\neq 0}^{n}$ or $(\mu_k)_{k=-n-1,k\neq 0}^{n+1}$ of problem* **N***:*

$$0 < \mu_1 < \nu_1 \leq \mu_2 \leq \cdots \leq \nu_n < \mu_{n+1} \quad \text{if } M > 0,$$
$$0 < \mu_1 < \nu_1 \leq \mu_2 \leq \cdots \leq \nu_n \quad \text{if } M = 0,$$

and satisfy $\nu_{-k} = -\nu_k$ and $\mu_{-k} = -\mu_k$ for all k.

Theorem 2.8 ([16, Theorem 2.5])**.** 1. *For $k \geq 2$ we have $\mu_k = \nu_k$ if and only if $\mu_k = \nu_{k-1}$.*

2. *The multiplicity of μ_k does not exceed $q - 1$.*

Definition 2.9 ([11], [14, Definition 5.1.4 and Remark 5.1.5])**.** A polynomial is said to be Hermite-Biehler if all its zeros lie in the open upper half-plane.

Definition 2.10 ([11], [14, Definition 5.1.8])**.** A polynomial is said to be generalized Hermite-Biehler if all its zeros lie in the closed upper half-plane.

Theorem 2.11. *The polynomial ϕ is a generalized Hermite-Biehler polynomial.*

Proof. Let ϕ_0 be the monic polynomial whose zeros (if any) are the common zeros of ϕ_N and ϕ_D, counted with multiplicity, and let

$$\tilde{\phi}_N = \frac{\phi_N}{\phi_0}, \quad \tilde{\phi}_D = \frac{\phi_D}{\phi_0}.$$

Then $\tilde{\phi}_N$ and $\tilde{\phi}_D$ have no common zeros, and

$$\frac{\tilde{\phi}_D}{\tilde{\phi}_N} = \frac{\phi_D}{\phi_N}$$

is an S_0-function by Theorem 2.6. In view of [14, Lemma 5.2.4,4],

$$\lambda \mapsto \alpha\lambda \frac{\tilde{\phi}_D(\lambda^2)}{\tilde{\phi}_N(\lambda^2)}$$

is a Nevanlinna function. Since the zeros of ϕ_N and ϕ_D are real, also the zeros of ϕ_0 are real, and hence $\phi_0(\lambda)$ is real for real λ. Then also $\alpha\lambda\tilde{\phi}_D(\lambda^2)$ and $\tilde{\phi}_N(\lambda^2)$ are real for real λ. Hence it follows from [14, Lemma 5.1.23] that

$$\tilde{\phi}(\lambda) := \tilde{\phi}_N(\lambda^2) + i\alpha\lambda\tilde{\phi}_D(\lambda^2)$$

describes a Hermite-Biehler function $\tilde{\phi}$. An application of [14, Theorem 5.1.9] proves that $\phi = \phi_0\tilde{\phi}$ is a generalized Hermite-Biehler function. □

Definition 2.12. A generalized Hermite-Biehler polynomial ω is said to be *symmetric* if $\omega(-\overline{z}) = \overline{\omega(z)}$ for all $z \in \mathbb{C}$.

It is clear that the generalized Hermite-Biehler polynomial ϕ is symmetric. In particular, if λ is a zero of ϕ, then also $-\overline{\lambda}$ is a zero of ϕ, and both have the same multiplicity. By Theorem 2.7, $\phi_N(0) \neq 0$, and therefore $\phi(0) \neq 0$. Hence the zeros of ϕ can be written as follows. The real zeros of ϕ are denoted by $(\beta_k)_{k=-p_1,k\neq 0}^{p_1}$, the pure imaginary zeros of ϕ are denoted by $(\gamma_k)_{k=1}^{p_2}$, and the remaining zeros of ϕ are denoted by $(\delta_k)_{k=-p_3,k\neq 0}^{p_3}$. Here we may assume that $\beta_{-k} = -\beta_k$ and that $\delta_{-k} = -\overline{\delta_k}$. The number p_2 of pure imaginary zeros is even (or zero) if $M > 0$ and odd if $M = 0$. Setting $m = 2p_1 + p_2 + 2p_3$ it is also clear that $m = 2\sum_{j=1}^{q} n_j + 2$ if $M > 0$ and $m = 2\sum_{j=1}^{q} n_j + 1$ if $M = 0$. With an obvious meaning of the union of sequences, the sequence of all zeros of ϕ will be denoted as $(\lambda_k)_{k=1}^{m} = (\beta_k)_{k=-p_1,k\neq 0}^{p_1} \cup (\gamma_k)_{k=1}^{p_2} \cup (\delta_k)_{k=-p_3,k\neq 0}^{p_3}$. Since $\phi(0) \neq 0$, it is clear that $\lambda_k \neq 0$ for all k.

In particular, for all $\lambda \in \mathbb{C}$,

$$\Phi(\lambda) = \prod_{k=-p_1,k\neq 0}^{p_1} \left(1 - \frac{\lambda}{\beta_k}\right) \prod_{k=-p_3,k\neq 0}^{p_3} \left(1 - \frac{\lambda}{\delta_k}\right) \prod_{k=1}^{p_2} \left(1 - \frac{\lambda}{\gamma_k}\right)$$

is well defined. From $\Phi(0) = 1$ it follows that $\phi = \phi(0)\Phi$, where $\phi(0) = \phi_N(0) \in \mathbb{R}$.

In the following, $m(\lambda_k)$, $m(\beta_k)$, $m(\gamma_k)$, $m(\delta_k)$ denote the multiplicities of λ_k, β_k, γ_k, δ_k, respectively, as zeros of ϕ, whereas $m(\mu_k)$ and $m(\nu_k)$ denote the multiplicities of μ_k and ν_k as zeros of ϕ_N and ϕ_D, respectively.

Theorem 2.13. 1. *If $M>0$, then $(\beta_k)_{k=-p_1,k\neq 0}^{p_1} = (\mu_k)_{k=-n-1,k\neq 0}^{n+1} \cap (\nu_k)_{k=-n,k\neq 0}^{n}$. If $M=0$, then $(\beta_k)_{k=-p_1,k\neq 0}^{p_1} = (\mu_k)_{k=-n,k\neq 0}^{n} \cap (\nu_k)_{k=-n,k\neq 0}^{n}$. Here $n=\sum_{j=1}^{q} n_j$.*

2. *If $\beta_k=\mu_p=\nu_p$, then $m(\mu_p)=m(\beta_k)$ and $m(\nu_p)=m(\beta_k)+1$. Therefore, $m(\beta_k)\le q-1$.*

3. *Let $\tilde{p}_1$ be the number of distinct positive real zeros of ϕ. Then $2\tilde{p}_1 \le p_2 + 2p_3 - 1$. The number p_2 is odd if and only if $M=0$.*

4. *$\operatorname{Im} \frac{d^s}{d\lambda^s}\Phi(\lambda)\big|_{\lambda=\beta_k} = 0$ for $s=0,\dots,m(\beta_k)$.*

Proof. 1. Since $\phi_N(\lambda^2)$ and $\phi_D(\lambda^2)$ are real for real λ, it follows for real λ that $\phi(\lambda)=0$ if and only if $\phi_N(\lambda^2)=0$ and $\phi_D(\lambda^2)=0$, which proves statement 1.

2. Without loss of generality let p be the smallest index such that $\beta_k=\nu_p=\mu_p$ and let r be the largest index such that $\beta_k=\nu_r=\mu_r$. In view of Theorems 2.7 and 2.8, part 1, we have $p\ge 2$ and

$$\mu_{p-1} < \nu_{p-1} = \mu_p = \nu_p = \cdots = \mu_r = \nu_r\,.$$

This proves $m(\nu_p)=m(\mu_p)+1$ if $r=n$. If $r<n$, then $\mu_r=\nu_r\le\mu_{r+1}\le\nu_{r+1}$ and $\nu_r<\nu_{r+1}$. But since $\nu_{r+1}=\mu_{r+1}$ if and only if $\nu_r=\mu_{r+1}$, we must have $\nu_{r+1}\neq\mu_{r+1}$ and $\nu_r\neq\mu_{r+1}$, so that also $m(\nu_p)=m(\mu_p)+1$ if $r<n$.

The Taylor series of $\lambda\mapsto\phi_N(\lambda^2)$ and $\lambda\mapsto\phi_D(\lambda^2)$ about $\beta_k=\mu_p=\nu_p$ show that

$$\phi(\lambda) = (\lambda-\mu_p)^{m(\mu_p)}\psi_1(\lambda) + i\alpha\lambda(\lambda-\nu_p)^{m(\nu_p)}\psi_2(\lambda),$$

where ψ_1 and ψ_2 are polynomials with $\psi_1(\beta_k)\neq 0$ and $\psi_2(\beta_k)\neq 0$. From $m(\nu_p)=m(\mu_p)+1$ we now conclude that $m(\beta_k)=m(\mu_p)=m(\nu_p)-1\le q-1$.

3. By part 2, every real zero of ϕ is the square root of a zero of multiplicity at least 2 of ϕ_D. In view of Remark 2.1 it follows that

$$2\tilde{p}_1 \le n < \frac{1}{2}(2p_1+p_2+2p_3).$$

Therefore

$$\tilde{p}_1 < p_1 - \tilde{p}_1 + \frac{1}{2}p_2 + p_3 \le \frac{1}{2}p_2 + p_3,$$

which implies $2\tilde{p}_1 \le p_2+2p_3-1$. Since the degree $2p_1+p_2+2p_3$ of ϕ is odd if and only if p_2 is odd, it follows from Remark 2.1 that p_2 is odd if and only if $M=0$.

4. By (2.17), $\operatorname{Im}\phi(\lambda)=\alpha\lambda\phi_D(\lambda^2)$ for real λ. By part 2, every β_k is a zero of multiplicity $m(\beta_k)+1$ of ϕ_D, and therefore $\operatorname{Im}\frac{d^s}{d\lambda^s}\phi(\lambda)\big|_{\lambda=\beta_k}=0$ for $s=0,\dots,m(\beta_k)$. Finally, we recall that Φ is a real multiple of ϕ. □

Remark 2.14. The function $\frac{\phi_N}{\phi_D}$ has the representation

$$\frac{\phi_N(z)}{\phi_D(z)} = \sum_{j=1}^{q} \frac{R_{2n_j-1}^{(j)}(z)}{R_{2n_j}^{(j)}(z)} - Mz \tag{2.18}$$

and according to [2, Supplement II, (18)] the fractions $\frac{R_{2n_j}^{(j)}(z)}{R_{2n_j-1}^{(j)}(z)}$ can be expanded into continued fractions

$$\frac{R_{2n_j}^{(j)}(z)}{R_{2n_j-1}^{(j)}(z)} = l_{n_j}^{(j)} + \cfrac{1}{-m_{n_j}^{(j)}z + \cfrac{1}{l_{n_j-1}^{(j)} + \cfrac{1}{-m_{n_j-1}^{(j)}z + \cdots + \cfrac{1}{l_1^{(j)} + \cfrac{1}{-m_1^{(j)}z + \cfrac{1}{l_0^{(j)}}}}}}}. \tag{2.19}$$

Using (2.18) and (2.19) we obtain

$$\frac{\phi_N(0)}{\phi_D(0)} = \sum_{j=1}^{q} l_j^{-1}. \tag{2.20}$$

3. Inverse problem

We will make use of

Theorem 3.1 ([16, Theorem 2.9]). *Let $q \in \mathbb{N}$, $q \geq 2$, $(l_j)_{j=1}^q \in (0,\infty)^q$, $n \in \mathbb{N}$, and suppose that $(\tau_k)_{k=-(n+1),\,k\neq 0}^{n+1} \in \mathbb{R}^{2(n+1)}$, $(\theta_k)_{k=-n,\,k\neq 0}^{n} \in \mathbb{R}^{2n}$ are such that*

0) $\tau_{-k} = -\tau_k, \quad \theta_{-k} = -\theta_k$,

1) $0 < \theta_1 < \tau_1 \leq \ldots \leq \theta_n \leq \tau_n < \theta_{n+1}$;

2) *for $k = 2, \ldots, n$, $\tau_{k-1} = \theta_k$ if and only if $\theta_k = \tau_k$;*

3) *the multiplicity of θ_k in the sequence $(\theta_k)_{k=-(n+1),k\neq 0}^{n+1}$ does not exceed $q-1$.*

Then there exists a star graph of q Stieltjes strings, i.e., sequences $(n_j)_{j=1}^q \in \mathbb{N}_0^q$ and $(m_k^{(j)})_{k=1}^{n_j} \in (0,\infty)^{n_j}$, $(l_k^{(j)})_{k=0}^{n_j} \in (0,\infty)^{n_j+1}$ for $j = 1, \ldots, q$ and and $M \in (0,\infty)$ with $n = \sum_{j=1}^q n_j$ and $\sum_{k=0}^{n_j} l_k^{(j)} = l_j$ such that the sequence of the zeros of $\lambda \mapsto \phi_N(\lambda^2)$ is $(\theta_k)_{k=-(n+1),\,k\neq 0}^{n+1}$ and the sequence of the zeros of $\lambda \mapsto \phi_D(\lambda^2)$ is $(\tau_k)_{k=-n,\,k\neq 0}^{n}$.

Now we come to the main theorem of this section.

Theorem 3.2. *Let $q \geq 2$ be an integer and let positive numbers l_j, $j = 1, \ldots, q$, be given. Let the sequences of numbers $(\beta_k)_{k=-p_1,k\neq 0}^{p_1}$, $(\gamma_k)_{k=1}^{p_2}$ and $(\delta_k)_{k=-p_3,k\neq 0}^{p_3}$ be given, where $p_1, p_2, p_3 \in \mathbb{N}_0$. Assume that the following conditions are satisfied:*

1) *For all indices k, $\operatorname{Im}\beta_k = 0$, $\beta_{-k} = -\beta_k$, $\beta_k \neq 0$, $2\tilde{p}_1 \leq p_2 + 2p_3 - 1$, where $\tilde{p}_1$ is the number of distinct elements of the sequence $(\beta_k)_{k=0}^{p_1}$.*

2) *The multiplicity of β_{-k} coincides with the multiplicity of β_k for each k and does not exceed $q-1$.*

3) *$\gamma_k = i|\gamma_k|$ and $\gamma_k \neq 0$ for all k.*

4) *$\operatorname{Im}\delta_k > 0$, $\operatorname{Re}\delta_k \neq 0$ and $\delta_{-k} = -\overline{\delta_k}$ for all k.*

5) *$\operatorname{Im}\frac{d^s}{d\lambda^s}\Phi(\lambda)\big|_{\lambda=\beta_k} = 0$ for all k and for $s = 0, \ldots, m(\beta_k)$, where $m(\beta_k)$ is the number of times β_k occurs in the sequence $(\beta_j)_{j=-p_1,k\neq 0}^{p_1}$ and where*

$$\Phi(\lambda) = \prod_{k=-p_1,k\neq 0}^{p_1}\left(1 - \frac{\lambda}{\beta_k}\right)\prod_{k=-p_3,k\neq 0}^{p_3}\left(1 - \frac{\lambda}{\delta_k}\right)\prod_{k=1}^{p_2}\left(1 - \frac{\lambda}{\gamma_k}\right). \tag{3.1}$$

Then there exists a collection of sequences $(m_k^{(j)})_{k=1}^{n_j}$ and $(l_k^{(j)})_{k=0}^{n_j}$ for $j = 1, \ldots, q$, numbers $M \geq 0$ and $\alpha > 0$ such that problems (2.1)–(2.4) with these data have spectrum $(\beta_k)_{k=-p_1,k\neq 0}^{p_1} \cup (\gamma_k)_{k=1}^{p_2} \cup (\delta_k)_{k=-p_3,k\neq 0}^{p_3}$. Here $M = 0$ if and only if p_2 is odd.

Proof. For $\lambda \in \mathbb{C}$ define

$$R(\lambda) = \prod_{k=1}^{p_1}\left(1 - \frac{\lambda}{\beta_k^2}\right), \quad \omega(\lambda) = \prod_{k=-p_3,k\neq 0}^{p_3}\left(1 - \frac{\lambda}{\delta_k}\right)\prod_{k=1}^{p_2}\left(1 - \frac{\lambda}{\gamma_k}\right).$$

By definition, $\Phi(\lambda) = R(\lambda^2)\omega(\lambda)$ for $\lambda \in \mathbb{C}$, and it is clear that ω is a Hermite-Biehler polynomial. Since $\omega(-\overline{\lambda}) = \overline{\omega(\lambda)}$, there are polynomials P and Q, which are real on the real axis, such that $Q(\lambda^2) = \frac{\omega(\lambda)-\omega(-\lambda)}{2i\lambda}$ and $P(\lambda^2) = \frac{\omega(\lambda)+\omega(-\lambda)}{2}$.

Then $\omega(\lambda) = P(\lambda^2) + i\lambda Q(\lambda^2)$ for $\lambda \in \mathbb{C}$. We observe that P and Q do not have common zeros. Indeed, if $P(\lambda^2) = Q(\lambda^2) = 0$ for some $\lambda \in \mathbb{C}$, then $\omega(-\lambda) = -\omega(\lambda)$ and $\omega(-\lambda) = \omega(\lambda)$, and therefore $\omega(-\lambda) = \omega(\lambda) = 0$. But this is impossible since all zeros of ω lie in the open upper half-plane. Furthermore, $P(0) = \omega(0) = 1$.

We can now conclude from [14, Lemma 5.1.23] that

$$\frac{\lambda Q(\lambda^2)}{P(\lambda^2)}$$

defines a Nevanlinna function. It follows from [14, Lemma 5.2.4,4] and $P(0) = 1$ that $\frac{Q}{P}$ is an S_0-function with $P(z) > 0$ and $Q(z) > 0$ for $z \in (-\infty, 0]$.

Case I (p_2 is even). The numbers $n = p_1 + \frac{1}{2}p_2 + p_3 - 1$ and $n - p_1$ are nonnegative integers because $\frac{1}{2}p_2 + p_3 > 0$ by condition 3).

First we consider the case that $n - p_1 = 0$. Then $p_2 + 2p_3 = 2(n - p_1) + 2 = 2$ and $2\tilde{p}_1 \le p_2 + 2p_3 - 1 = 1$ shows that $\tilde{p}_1 = 0$ and therefore $p_1 = 0$. Hence

$$\Phi(\lambda) = \left(1 - \frac{\lambda}{\lambda_1}\right)\left(1 - \frac{\lambda}{\lambda_2}\right) = 1 - \lambda\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right) + \frac{\lambda^2}{\lambda_1\lambda_2},$$

where λ_1 and λ_2 are nonzero complex numbers satisfying $\lambda_1 = i|\lambda_1|$ and $\lambda_2 = i|\lambda_2|$ or $\lambda_2 = -\overline{\lambda_1}$ and $\operatorname{Im}\lambda_1 > 0$. Then problem (2.1)–(2.4) with

$$M = -\frac{1}{\lambda_1\lambda_2}\sum_{j=1}^{q} l_j^{-1} > 0, \quad \alpha = i\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right)\sum_{j=1}^{q} l_j^{-1} > 0,$$

and $n_j = 0$ for $j = 0, \dots, q$ has the two eigenvalues (λ_1, λ_2) by Example 2.2.

Now let $n - p_1 > 0$. Since ω is a polynomial of even degree $p_2 + 2p_3$, the degree of Q is $\frac{1}{2}p_2 + p_3 - 1 = n - p_1$, and hence Q has $n - p_1$ zeros, which are all positive. The polynomial P has degree $\frac{1}{2}p_2 + p_3 = n + 1 - p_1$, and hence P has $n + 1 - p_1$ zeros, and P and Q do not have common zeros. Since zeros and poles of Nevanlinna functions are simple and interlace, there are positive numbers ν_k, $k = 1, \dots, n - p_1$, and μ_k, $k = 1, \dots, n + 1 - p_1$, such that $(\nu_k^2)_{k=1}^{n-p_1}$ is the sequence of the zeros of Q, such that $(\mu_k^2)_{k=1}^{n+1-p_1}$ is the sequence of the zeros of P and such that

$$0 < \mu_1^2 < \nu_1^2 < \mu_2^2 < \nu_2^2 < \cdots < \nu_{n-p_1}^2 < \mu_{n+1-p_1}^2.$$

We also put $\nu_{-k} = -\nu_k$ for $k = 1, \dots, n - p_1$ and $\mu_{-k} = -\mu_k$, $k = 1, \dots, n + 1 - p_1$.

Due to the condition 5) we see that each β_k is a zero of multiplicity at least $m(\beta_k) + 1$ of RQ, whereas β_k is a zero of R of multiplicity $m(\beta_k)$. Therefore $Q(\beta_k) = 0$, and each β_k coincides with some ν_r. For $\beta_k = \nu_r$ we set $\nu_r^{(j)} = \nu_r$, $j = 1, \dots, m(\beta_k)$, and $\mu_{r-1}^{(j)} = \nu_r$, $j = 1, \dots, m(\beta_k)$. Now we append all these $\nu_r^{(j)}$ to $(\nu_k)_{k=-n+p_1, k\neq 0}^{n-p_1}$ and $\mu_r^{(j)}$ to $(\mu_k)_{k=-n-1+p_1, k\neq 0}^{n+1-p_1}$ and obtain two sequences $(\tau_k)_{k=-n, k\neq 0}^{n}$ and $(\theta_k)_{k=-n-1, k\neq 0}^{n+1}$ which satisfy the assumptions of Theorem 3.1.

Thus, in view of Theorem 3.1, there exists a star graph of Stieljes strings with all quantities as indicated in the statement of this theorem, and in particular $M > 0$, except for α, such that $(\tau_k)_{k=-n, k\neq 0}^{n}$ and $(\theta_k)_{k=-n-1, k\neq 0}^{n+1}$ are the sequences of the eigenvalues of the Neuman and Dirichlet problems, respectively. Let ϕ_N and ϕ_D be the corresponding characteristic functions; see (2.15) and (2.16). The polynomials RP and ϕ_N are real on the real axis and have the same zeros. Hence there is a

nonzero real number α_N such that $RP = \alpha_N \phi_N$. In the same way we find a nonzero real number α_D such that $RQ = \alpha_D \phi_D$. Putting $\alpha = \alpha_D \alpha_N^{-1}$, it follows that

$$\Phi(\lambda) = R(\lambda^2)P(\lambda^2) + i\lambda R(\lambda^2)Q(\lambda^2) = \alpha_N[\phi_N(\lambda^2) + i\alpha\lambda\phi_D(\lambda^2)], \tag{3.2}$$

and hence the eigenvalues of the damped problem for this star graph and this α coincide with the zeros of Φ, that is, the given sequences. Finally, if α were negative, then replacing λ with $-\lambda$ and observing that $p_2 + p_3 > 0$ would give eigenvalues in the lower half-plane, which is impossible. Hence we have $\alpha > 0$.

Case II (p_2 is odd). The numbers $n = p_1 + \frac{1}{2}p_2 + p_3 - \frac{1}{2}$ and $n - p_1$ are nonnegative integers because $\frac{1}{2}p_2 + p_3 - \frac{1}{2} \geq 0$ by condition 3).

First we consider the case that $n - p_1 = 0$. Then $p_2 + 2p_3 = 2(n - p_1) + 1 = 1$ and $2\tilde{p}_1 \leq p_2 + 2p_3 - 1 = 0$ shows that $p_2 = 1$, $p_3 = 0$ and $\tilde{p}_1 = 0$, and therefore $p_1 = 0$. Hence

$$\Phi(\lambda) = 1 - \frac{\lambda}{\delta_1},$$

where $\delta_1 = i|\delta_1|$ and $\delta_1 \neq 0$. Then problem (2.1)–(2.4) with

$$M = 0, \quad \alpha = \frac{i}{\delta_1} \sum_{j=1}^{q} l_j^{-1} > 0,$$

and $n_j = 0$ for $j = 0, \ldots, q$ has the single eigenvalue δ_1 by Example 2.2.

Now let $n - p_1 > 0$. Since ω is a polynomial of odd degree $p_2 + 2p_3$, the degree of Q is $\frac{1}{2}p_2 + p_3 - \frac{1}{2} = n - p_1$, and hence Q has $n - p_1$ zeros, which are all positive. The polynomial P has degree $\frac{1}{2}p_2 + p_3 - \frac{1}{2} = n - p_1$, and hence P has $n - p_1$ zeros, and P and Q do not have common zeros. We know that $\lambda \mapsto \frac{\lambda Q(\lambda^2)}{P(\lambda^2)}$ is a Nevanlinna function, and hence the smallest positive zero of P interlaces with 0 and the smallest positive zero of Q. It follows that there are positive numbers ν_k, $k = 1, \ldots, n - p_1$ and μ_k, $k = 1, \ldots, n - p_1$ such that $(\nu_k^2)_{k=1}^{n-p_1}$ is the sequence of the zeros of Q, such that $(\mu_k^2)_{k=1}^{n+1-p_1}$ is the sequence of the zeros of P and such that

$$0 < \mu_1^2 < \nu_1^2 < \mu_2^2 < \nu_2^2 < \cdots < \nu_{n-p_1}^2.$$

Proceeding as in Case I we obtain a star graph of strings, but now with $M = 0$, and the constant $\alpha > 0$, such that the corresponding damped problem has the zeros of Φ, that is, the given sequences of numbers as eigenvalues. □

Remark 3.3. The star graphs and the damping coefficient in Theorem 3.2 can be constructed explicitly. A construction of star graphs is given in the proof of [16, Theorem 2.9] for $n - p_1 > 0$ and in the proof of Theorem 3.2 for $n - p_1 = 0$. To find α, we deduce from the proof of Theorem 3.2 that

$$\begin{aligned} \alpha = \alpha_D \alpha_n^{-1} &= \frac{R(0)Q(0)}{\phi_D(0)} \frac{\phi_N(0)}{R(0)P(0)} \\ &= -i \frac{\omega'(0)}{\phi_D(0)} \frac{\phi_N(0)}{\omega(0)}. \end{aligned}$$

Then we conclude from $\omega(0) = 1$,

$$\omega'(0) = -\sum_{k=-p_3, k\neq 0} \frac{1}{\delta_k} - \sum_{k=1}^{p_2} \frac{1}{\gamma_k}$$

and (2.20) that

$$\alpha = i\left(\sum_{k=-p_3,k\neq 0}\frac{1}{\delta_k}+\sum_{k=1}^{p_2}\frac{1}{\gamma_k}\right)\sum_{j=1}^{q} l_j^{-1}\,.$$

Remark 3.4. We have seen in Remark 2.1 that the number of eigenvalues uniquely determines the total number of masses in the interiors of the strings and whether or not there is a mass at the joint vertex. We have also seen in Remark 3.3 that the damping coefficient α is uniquely determined by the spectrum. Finally, it follows from (3.2) that

$$\frac{\phi_N(\lambda^2)}{\phi_D(\lambda^2)} = i\alpha\lambda\,\frac{\Phi(\lambda)-\Phi(-\lambda)}{\Phi(\lambda)+\Phi(-\lambda)}\,.$$

Hence we conclude from (2.18) and (2.19) that

$$\begin{aligned} M &= \lim_{\lambda\to\infty}\frac{\phi_N(\lambda^2)}{\lambda^2\phi_D(\lambda^2)} \\ &= i\alpha\lim_{\lambda\to\infty}\frac{\Phi(\lambda)-\Phi(-\lambda)}{\lambda(\Phi(\lambda)+\Phi(-\lambda))} \end{aligned}$$

which means that M is uniquely determined by the spectrum of the damped problem.

Acknowledgment

This work was partially supported by a grant of the NRF of South Africa, grant no. 80956.

References

[1] O. Boyko and V. Pivovarchik, *Inverse spectral problem for a star graph of Stieltjes strings*, Methods Funct. Anal. Topology **14** (2008), no. 2, 159–167. MR2432764

[2] F. P. Gantmacher and M. G. Krein, *Oscillation matrices and kernels and small vibrations of mechanical systems*, Revised edition, AMS Chelsea Publishing, Providence, RI, 2002. Translation based on the 1941 Russian original; Edited and with a preface by Alex Eremenko. MR1908601

[3] Joseph Genin and John S. Maybee, *Mechanical vibration trees*, J. Math. Anal. Appl. **45** (1974), 746–763. MR0339593

[4] G. M. L. Gladwell, *Inverse Problems in Vibration*, Kluwer Academic Publishers, Dordrecht, 2004

[5] Ernst A. Guillemin, *Synthesis of passive networks. Theory and methods appropriate to the realization and approximation problems*, John wiley and Sons, Inc., New York; Chapman and Hall, Ltd., London, 1958. MR0137480

[6] I. S. Kac and M. G. Krein, *On the spectral functions of the string*, Amer. Math. Transl. (2), 103 (1974), 19–102

[7] M. G. Krein and A. A. Nudelman, *Direct and inverse problems for the frequencies of boundary dissipation of a nonuniform string*, Dokl. Akad. Nauk SSSR, 247 (1979), n. 5, 1046–1049

[8] M. G. Kreĭn and A. A. Nudel′man, *Some spectral properties of a nonhomogeneous string with a dissipative boundary condition* (Russian), J. Operator Theory **22** (1989), no. 2, 369–395. MR1043733

[9] P. F. Kurchanov, A. D. Myshkis, and A. M. Filimonov, *Oscillations of a railroad train and Kronecker's theorem* (Russian), Prikl. Mat. Mekh. **55** (1991), no. 6, 989–995, DOI 10.1016/0021-8928(91)90140-P; English transl., J. Appl. Math. Mech. **55** (1991), no. 6, 870–876 (1992). MR1150440

[10] P. Lancaster and U. Prells, *Inverse problems for damped vibrating systems*, J. Sound Vibration **283** (2005), no. 3-5, 891–914, DOI 10.1016/j.jsv.2004.05.003. MR2136391

[11] B. Ja. Levin, *Distribution of zeros of entire functions*, Revised edition, Translations of Mathematical Monographs, vol. 5, American Mathematical Society, Providence, R.I., 1980. Translated from the Russian by R. P. Boas, J. M. Danskin, F. M. Goodspeed, J. Korevaar, A. L. Shields and H. P. Thielman. MR589888

[12] V. A. Marchenko, *Introduction to inverse problems of spectral analysis* (Russian), Akta, Kharkov, 2005

[13] Olga Martynuk, Vyacheslav Pivovarchik, and Christiane Tretter, *Inverse problem for a damped Stieltjes string from parts of spectra*, Appl. Anal. **94** (2015), no. 12, 2605–2619, DOI 10.1080/00036811.2014.996874. MR3401085

[14] Manfred Möller and Vyacheslav Pivovarchik, *Spectral theory of operator pencils, Hermite-Biehler functions, and their applications*, Operator Theory: Advances and Applications, vol. 246, Birkhäuser/Springer, Cham, 2015. MR3379142

[15] V. Pivovarchik, *Existence of a tree of Stieltjes strings corresponding to two given spectra*, J. Phys. A **42** (2009), no. 37, 375213, 16, DOI 10.1088/1751-8113/42/37/375213. MR2540422

[16] Vyacheslav Pivovarchik, Natalia Rozhenko, and Christiane Tretter, *Dirichlet-Neumann inverse spectral problem for a star graph of Stieltjes strings*, Linear Algebra Appl. **439** (2013), no. 8, 2263–2292, DOI 10.1016/j.laa.2013.07.003. MR3091304

[17] Vyacheslav Pivovarchik and Christiane Tretter, *Location and multiplicities of eigenvalues for a star graph of Stieltjes strings*, J. Difference Equ. Appl. **21** (2015), no. 5, 383–402, DOI 10.1080/10236198.2014.992425. MR3334518

[18] Vyacheslav Pivovarchik and Harald Woracek, *Sums of Nevanlinna functions and differential equations on star-shaped graphs*, Oper. Matrices **3** (2009), no. 4, 451–501, DOI 10.7153/oam-03-26. MR2597674

[19] Krešimir Veselić, *On linear vibrational systems with one-dimensional damping*, Appl. Anal. **29** (1988), no. 1-2, 1–18, DOI 10.1080/00036818808839770. MR960575

[20] K. Veselić, *On linear vibrational systems with one-dimensional damping. II*, Integral Equations Operator Theory **13** (1990), no. 6, 883–897, DOI 10.1007/BF01198923. MR1073858

The John Knopfmacher Centre for Applicable Analysis and Number Theory, School of Mathematics, University of the Witwatersrand, WITS 2050, Johannesburg, South Africa

E-mail address: manfred.moller@wits.ac.za

Department of Higher Mathematics and Statistics, South Ukrainian National Pedagogical University, 26, Staroportofrankovskaya str., Odessa, 65020, Ukraine

E-mail address: vpivovarchik@gmail.com