DAMPED STAR GRAPHS OF STIELTJES STRINGS

M. MÖLLER AND V. PIVOVARCHIK

(Communicated by Michael Hitrik)

Abstract. We consider a direct and an inverse problem arising in the theory of small transverse vibrations of a star graph of Stieltjes strings damped at the midpoint. The exterior vertices of the graph are supposed to be fixed. We give necessary and sufficient conditions on a sequence of complex numbers to be the spectrum of such a problem.

1. Introduction

It was shown in [7], [8] for a very wide class of inhomogeneous strings in the case where one of the ends of a vibrating string is damped (subject to viscous friction) one spectrum together with the length (if it is finite) of the string uniquely determine the density of the string, while in the case of an undamped string two spectra of boundary value problems are necessary to determine the density. An inverse problem for undamped Stieltjes strings (weightless threads bearing a finite number of point masses) was solved in [2]. For a Stieltjes string damped at one end the inverse problem was solved in [19], [20], [10] and in [13] for the inverse problem by parts of two spectra. An application was given in [9]. The same equation occurs if one considers longitudinal vibrations of point masses connected by springs and also occurs in the theory of synthesis of electrical circuits; see [12], [5, p. 129].

Boundary value problems on graphs consisting of Stieltjes strings are natural generalizations of boundary value problems on a single interval and are often used in the theory of vibrations of nets [5], [4], [12]. Direct and inverse problems on a star graph were considered in [1] for the case of simple eigenvalues, in [16] without the restriction to simple eigenvalues, and in [17] with prescribed numbers of masses on the edges. It was shown in [18] that restrictions on multiplicities of eigenvalues are independent on whether the strings of the star graph are smooth or not. An inverse problem on a tree of Stieltjes strings was solved in [15] for the case of simple eigenvalues.

In this paper we consider a direct and an inverse problem generated by the equations of small transverse vibrations of a plane graph of Stieltjes strings damped at the central vertex. We give necessary and sufficient conditions for a set of complex numbers to be the spectrum of such a problem. In Section 2 we give a physical motivation of such a problem and describe its spectrum, i.e., we solve the direct problem, and in Section 3 we solve the inverse problem.

Received by the editors December 19, 2015 and, in revised form, June 19, 2016.

2010 Mathematics Subject Classification. Primary 47B39; Secondary 34A55, 39A70.

Key words and phrases. Star graph, inverse problem, transverse vibrations, damped vibration, Dirichlet boundary condition, Neumann boundary condition, Hermite-Biehler polynomial, Nevanlinna function.

©2016 American Mathematical Society
2. Formulation of the problem

Following [2], a thread, i.e., an elastic string of zero density, bearing a finite number of point masses is called a Stieltjes string. Suppose we have a plane star graph of \( q > 2 \) Stieltjes strings joined at the interior vertex. We assume all the other vertices to be fixed and a mass \( M \geq 0 \) to be placed at the interior vertex. In [16] the authors assumed such a plane net was stretched and considered two problems of small transverse vibrations of the graph: one with the mass \( M \) fixed at the central vertex, called the Dirichlet problem, and one with \( M \) free to move in the direction orthogonal to the equilibrium position of the strings, called the Neumann problem. In this paper it is assumed that the central vertex can move subject to damping with constant coefficient of damping \( \alpha > 0 \).

Let \( e_j, j = 1, \ldots, q, \) be the \( j \)-th edge of the graph and let \( n_j \geq 0 \) be the number of point masses in the interior of \( e_j \). The exterior and interior vertices of the Stieltjes string \( e_j \) will be denoted by \( P^{(j)}_{n_j+1} \) and \( P^{(j)}_{n_j} \), respectively. Observe that \( P^{(1)}_{n_1+1}, \ldots, P^{(q)}_{n_q+1} \) is the common interior vertex of the \( q \) Stieltjes strings. As we move from the exterior vertex to the interior vertex of \( e_j \), the \( n_j \) point masses \( m_k^{(j)} > 0 \) in the interior of \( e_j \) will be encountered at the points \( P^{(j)}_{k} \), \( k = 1, \ldots, n_j \). Denoting the distance between \( P^{(j)}_{k} \) and \( P^{(j)}_{k+1} \) by \( l^{(j)}_{k} \), it is clear that the length of the edge \( e_j \) is \( l_j := \sum_{k=0}^{n_j} l^{(j)}_{k} \). For \( j = 1, \ldots, q \) and \( k = 1, \ldots, n_j \) denote by \( v_k^{(j)}(t) \) the transverse displacements of the point mass \( m_k^{(j)} \) at time \( t \). We assume the threads to be stretched by forces each equal to 1.

Then small transverse vibrations of the edges \( e_j, j = 1, \ldots, q, \) are governed by the equations

\[
\frac{v_k^{(j)}(t) - v_{k+1}^{(j)}(t)}{l^{(j)}_{k}} + \frac{v_k^{(j)}(t) - v_{k-1}^{(j)}(t)}{l^{(j)}_{k-1}} + m_k^{(j)} v_k^{(j)''}(t) = 0, \quad k = 1, \ldots, n_j, \quad t \in \mathbb{R}.
\]

Continuity of the star graph at the interior vertex, where all edges join, yields

\[
v^{(1)}_{n_1+1}(t) = \cdots = v^{(q)}_{n_q+1}(t), \quad t \in \mathbb{R}.
\]

The balance of forces at this interior vertex implies

\[
\sum_{j=1}^{q} \frac{v_{n_j+1}^{(j)}(t) - v_{n_j}^{(j)}(t)}{l^{(j)}_{n_j}} = -M v^{(1)''}_{n_1+1}(t) - \alpha v^{(1)''}_{n_1+1}(t), \quad t \in \mathbb{R}.
\]

The exterior vertices are fixed, which means that we have Dirichlet boundary conditions

\[
v_0^{(j)}(t) = 0, \quad j = 1, \ldots, q, \quad t \in \mathbb{R}.
\]
Substituting \( v^{(j)}_k(t) = u^{(j)}_k e^{i\lambda t} \) we obtain the following recurrences for the amplitudes \( u^{(j)}_k \):

\[
\begin{align*}
(2.1) & \quad \frac{u^{(j)}_k - u^{(j)}_{k+1}}{l^{(j)}_k} + \frac{u^{(j)}_k - u^{(j)}_{k-1}}{l^{(j)}_{k-1}} - m^{(j)}_k \lambda^2 u^{(j)}_k = 0, \quad j = 1, \ldots, q, \quad k = 1, \ldots, n_j, \\
(2.2) & \quad u^{(1)}_{n_1+1} = \cdots = u^{(q)}_{n_q+1}, \\
(2.3) & \quad \sum_{j=1}^q \frac{u^{(j)}_{n_j+1} - u^{(j)}_{n_j}}{l^{(j)}_{n_j}} = (M\lambda^2 - i\alpha\lambda)u^{(j)}_{n_1+1}, \\
(2.4) & \quad u^{(j)}_{0} = 0, \quad j = 1, \ldots, q.
\end{align*}
\]

Problem \((2.1) \sim (2.4)\) with \( \alpha = 0 \) is the Neumann problem (problem N), and the problem with \( \alpha > 0 \) is called the damped problem.

If we clamp our strings at the interior vertex, then we obtain \( q \) problems \( D_j, \ j = 1, \ldots, q \), on the edges with Dirichlet boundary conditions at both ends:

\[
\begin{align*}
(2.5) & \quad \frac{u^{(j)}_k - u^{(j)}_{k+1}}{l^{(j)}_k} + \frac{u^{(j)}_k - u^{(j)}_{k-1}}{l^{(j)}_{k-1}} - m^{(j)}_k \lambda^2 u^{(j)}_k = 0, \quad k = 1, \ldots, n_j, \\
(2.6) & \quad u^{(j)}_{n_j+1} = 0, \\
(2.7) & \quad u^{(j)}_{0} = 0.
\end{align*}
\]

The union of problems \((2.5) \sim (2.7)\) for \( j = 1, \ldots, q \) will be called the Dirichlet problem (D). Observe that the Dirichlet problem is described by \((2.1), (2.4)\), and \((2.6)\) for \( j = 1, \ldots, q \).

According to [2, p. 188] we look for solutions of \((2.1)\) in the form

\[
\begin{align*}
u^{(j)}_k = R^{(j)}_{2k-2}(\lambda^2)u^{(j)}_1, \quad k = 1, \ldots, n_j,
\end{align*}
\]

where \( R^{(j)}_{2k-2} \) are polynomials of degree \( k - 1 \). Defining

\[
R^{(j)}_{2k-1}(\lambda^2) = \frac{R^{(j)}_{2k}(\lambda^2) - R^{(j)}_{2k-2}(\lambda^2)}{l^{(j)}_k}, \quad k = 1, \ldots, n_j,
\]

it follows from these definitions and from \((2.1)\) and \((2.4)\) that the polynomials \( R^{(j)}_k \) have to satisfy the recurrence relations

\[
\begin{align*}
(2.8) & \quad R^{(j)}_{2k-1}(\lambda^2) = -\lambda^2 m^{(j)}_k R^{(j)}_{2k-2}(\lambda^2) + R^{(j)}_{2k-3}(\lambda^2), \quad k = 1, \ldots, n_j, \\
(2.9) & \quad R^{(j)}_{2k}(\lambda^2) = i^{(j)}_k R^{(j)}_{2k-1}(\lambda^2) + R^{(j)}_{2k-2}(\lambda^2), \quad k = 1, \ldots, n_j, \\
(2.10) & \quad R^{(j)}_{-1}(\lambda^2) = \frac{1}{l^{(j)}_0}, \quad R^{(j)}_0(\lambda^2) = 1.
\end{align*}
\]

Since \((2.2)\) and \((2.3)\) have to be satisfied simultaneously, they are equivalent to \((2.2)\) and

\[
\begin{align*}
\sum_{j=1}^q \frac{u^{(j)}_{n_j+1} - u^{(j)}_{n_j}}{l^{(j)}_{n_j}} = \sum_{j=1}^q \frac{1}{q} (M\lambda^2 - i\alpha\lambda)u^{(j)}_{n_j+1}.
\end{align*}
\]
Then \((2.2), (2.3)\) and \((2.6)\) for \(j = 1, \ldots, q\) can be written as
\begin{align}
(2.11) \quad & R_{2n_1}^{(1)}(\lambda^2)u_1^{(1)} = \cdots = R_{2n_q}^{(q)}(\lambda^2)u_1^{(q)}, \\
(2.12) \quad & \sum_{j=1}^{q} R_{2n_j-1}^{(j)}(\lambda^2)u_1^{(j)} = \sum_{j=1}^{q} \frac{1}{q} (M\lambda^2 - i\alpha\lambda) R_{2n_j}^{(j)}(\lambda^2)u_1^{(j)}, \\
(2.13) \quad & R_{2n_1}^{(j)}(\lambda^2)u_1^{(j)} = 0, \quad j = 1, \ldots, q.
\end{align}

Recall that a complex number \(\lambda\) belongs to the spectrum of the Neumann problems \((2.1)-(2.4)\) with \(\alpha = 0\), of the Dirichlet problems \((2.5)-(2.7)\), or of the damped problems \((2.1)-(2.4)\) with \(\alpha \neq 0\), respectively, if and only if this problem has a nontrivial solution \((u_k^{(j)})_{j=1,k=0}^{q,n_j+1} \neq 0\). Observing that \(u_0^{(j)} = 0\) for \(j = 1, \ldots, q\) by \((2.4)\) and \((2.7)\), respectively, and that then \(u_k^{(j)} = 0\) for \(k = 2, \ldots, n_j\) if \(u_1^{(j)} = 0\) by \((2.1)\) and \((2.5)\), respectively, this means that \(\lambda\) belongs to the spectrum of the corresponding problem if and only if \(u_1^{(j)} \neq 0\) for at least one \(j \in \{1, \ldots, q\}\).

We have seen that \((u_k^{(j)})_{j=1,k=0}^{q,n_j+1} = 0\) is a solution of the Neumann problems \((2.1)-(2.4)\) with \(\alpha = 0\) if and only if \((2.12)\) and \((2.11)\) hold with \(\alpha = 0\). This can be written as a system of linear equations
\begin{equation}
(2.14) \quad L_N(\lambda^2)(u_1^{(j)})_{j=1}^{q} = 0,
\end{equation}
where the first row of the \(q \times q\) matrix \(L_N(\lambda^2)\) has the entries \(R_{2n_j-1}^{(j)}(\lambda^2) - \frac{M}{q} \lambda^2 R_{2n_j}^{(j)}(\lambda^2)\), and for \(j = 2, \ldots, q\), the possibly nonzero entries of \(L_N(\lambda^2)\) in the \(j\)-th row are \(-R_{2n_j-1}^{(j-1)}(\lambda^2)\) in column \(j-1\) and \(R_{2n_j}^{(j)}(\lambda^2)\) in column \(j\). Therefore, the spectrum of the Neumann problem is the set of those \(\lambda\) for which \((2.14)\) has a nontrivial solution, i.e., for which \(\phi_N(\lambda^2) := \det L_N(\lambda^2) = 0\). An expansion of this determinant with respect to its first row gives the explicit formula
\begin{equation}
(2.15) \quad \phi_N(\lambda^2) = \sum_{j=1}^{q} \left[ R_{2n_j-1}^{(j)}(\lambda^2) - \frac{M}{q} \lambda^2 R_{2n_j}^{(j)}(\lambda^2) \right] \prod_{k=1, k \neq j}^{q} R_{2n_k}^{(k)}(\lambda^2).
\end{equation}

The number \(\lambda\) belongs to the spectrum of the Dirichlet problems \((2.5)-(2.7)\) if and only if \((2.13)\) holds with at least one \(u_1^{(j)} \neq 0\). Therefore the spectrum of the Dirichlet problem coincides with the set of zeros of
\begin{equation}
(2.16) \quad \phi_D(\lambda^2) = \prod_{j=1}^{q} R_{2n_j}^{(j)}(\lambda^2).
\end{equation}

A reasoning as for the Neumann problem shows that the spectrum of the damped problem coincides with the set of zeros of
\begin{equation}
\phi(\lambda) = \sum_{j=1}^{q} \left[ R_{2n_j-1}^{(j)}(\lambda^2) - \frac{1}{q} (M\lambda^2 - i\alpha\lambda) R_{2n_j}^{(j)}(\lambda^2) \right] \prod_{k=1, k \neq j}^{q} R_{2n_k}^{(k)}(\lambda^2),
\end{equation}
which has the representation
\begin{equation}
(2.17) \quad \phi(\lambda) = \phi_N(\lambda^2) + i\alpha\lambda\phi_D(\lambda^2).
\end{equation}

**Remark 2.1.** The degree of the polynomial \(\phi_N\) is \(1 + \sum_{j=1}^{q} n_j\) if \(M > 0\) and \(\sum_{j=1}^{q} n_j\) if \(M = 0\), the degree of \(\phi_D\) is \(\sum_{j=1}^{q} n_j\) in any case, while the degree of \(\phi\) is \(2 + 2\sum_{j=1}^{q} n_j\) if \(M > 0\) and \(1 + 2\sum_{j=1}^{q} n_j\) if \(M = 0\).
Next we consider the damped problem in the case that none of the threads has an interior mass.

**Example 2.2.** Assume that \( n_j = 0 \) for all \( j = 1, \ldots, q \).
1. If \( M = 0 \), then the spectrum consists of one simple eigenvalue \( \frac{i\alpha}{2M} \sum_{j=1}^{q} \frac{1}{l(j)} \).
2. If \( M > 0 \), then the spectrum consists of two eigenvalues, counted with multiplicity, namely

\[
\frac{i\alpha}{2M} \pm \left( \frac{1}{M} \sum_{j=1}^{q} \frac{1}{l(j)} - \frac{\alpha^2}{4M^2} \right)^{1/2}
\]

**Proof.** For \( n_j = 0 \) we have \( R_{2n,j}^{(j)} = R_0^{(j)} = 1 \) and \( R_{2n,j-1}^{(j)} = R_{-1} = \frac{1}{l(j)} \). Therefore \( \phi_D(\lambda^2) = 1 \) and

\[
\phi_N(\lambda^2) = \sum_{j=1}^{q} \left[ \frac{1}{l(j)} - \frac{M}{q} \lambda^2 \right] = \sum_{j=1}^{q} \frac{1}{l(j)} - M\lambda^2,
\]

which gives

\[
\phi(\lambda) = -M\lambda^2 + \sum_{j=1}^{q} \frac{1}{l(j)} + i\alpha\lambda.
\]

Solving \( \phi(\lambda) = 0 \) completes the proof.

**Definition 2.3.** A complex-valued function \( \omega \) defined on an open subset of \( \mathbb{C} \) is said to be a Nevanlinna function if:
1) the domain of \( \omega \) contains \( \mathbb{C} \setminus \mathbb{R} \) and \( \omega \) is analytic on \( \mathbb{C} \setminus \mathbb{R} \);
2) \( \omega(\overline{z}) = \overline{\omega(z)} \) for \( z \in \mathbb{C} \setminus \mathbb{R} \);
3) \( \text{Im } z \text{Im } \omega(z) \geq 0 \text{ for } \text{Im } z \neq 0 \).

**Definition 2.4.** The Nevanlinna function \( \omega \) is said to be an \( S \)-function if \( \omega \) is defined and analytic on \( \mathbb{C} \setminus [0, \infty) \) and if \( \omega(z) > 0 \) for \( z \in (-\infty, 0) \).

**Definition 2.5.** A meromorphic \( S \)-function \( \omega \) is said to be an \( S_0 \)-function if \( 0 \) is not a pole of \( \omega \).

**Theorem 2.6** ([16, Theorem 2.4]). After cancellation of common factors (if any) in the numerator and in the denominator the function

\[
\frac{\phi_D}{\phi_N}
\]

becomes an \( S_0 \)-function.

**Theorem 2.7** ([16, Theorem 2.5]). Let \( n = \sum_{j=1}^{q} n_j \) and assume that \( n > 0 \). Then the eigenvalues \((\nu_k)_{k=-n,k\neq0}^{n}\) of problem \( D \) interlace with the eigenvalues \((\mu_k)_{k=-n,k\neq0}^{n+1}\) of problem \( N \):

\[
0 < \mu_1 < \nu_1 \leq \mu_2 \leq \cdots \leq \nu_n < \mu_{n+1} \quad \text{if } M > 0,
\]

\[
0 < \mu_1 < \nu_1 \leq \mu_2 \leq \cdots \leq \nu_n \quad \text{if } M = 0,
\]

and satisfy \( \nu_k = -\nu_k \) and \( \mu_k = -\mu_k \) for all \( k \).

**Theorem 2.8** ([16, Theorem 2.5]). 1. For \( k \geq 2 \) we have \( \mu_k = \nu_k \) if and only if \( \mu_k = \nu_{k-1} \).
2. The multiplicity of \( \mu_k \) does not exceed \( q - 1 \).
Definition 2.9 ([11], [14] Definition 5.1.4 and Remark 5.1.5). A polynomial is said to be Hermite-Biehler if all its zeros lie in the open upper half-plane.

Definition 2.10 ([11], [14] Definition 5.1.8). A polynomial is said to be generalized Hermite-Biehler if all its zeros lie in the closed upper half-plane.

Theorem 2.11. The polynomial \( \phi \) is a generalized Hermite-Biehler polynomial.

Proof. Let \( \phi_0 \) be the monic polynomial whose zeros (if any) are the common zeros of \( \phi_N \) and \( \phi_D \), counted with multiplicity, and let

\[
\tilde{\phi}_N = \frac{\phi_N}{\phi_0}, \quad \tilde{\phi}_D = \frac{\phi_D}{\phi_0}.
\]

Then \( \tilde{\phi}_N \) and \( \tilde{\phi}_D \) have no common zeros, and

\[
\frac{\tilde{\phi}_D}{\tilde{\phi}_N} = \frac{\phi_D}{\phi_N}
\]

is an \( S_0 \)-function by Theorem 2.7. In view of [14] Lemma 5.2.4,4,

\[
\lambda \mapsto \alpha \frac{\phi_D(\lambda^2)}{\phi_N(\lambda^2)}
\]

is a Nevanlinna function. Since the zeros of \( \phi_N \) and \( \phi_D \) are real, also the zeros of \( \phi_0 \) are real, and hence \( \phi_0(\lambda) \) is real for real \( \lambda \). Then also \( \alpha \lambda \phi_D(\lambda^2) \) and \( \tilde{\phi}_N(\lambda^2) \) are real for real \( \lambda \). Hence it follows from [14] Lemma 5.1.23 that

\[
\tilde{\phi}(\lambda) := \tilde{\phi}_N(\lambda^2) + i\alpha \lambda \phi_D(\lambda^2)
\]
describes a Hermite-Biehler function \( \tilde{\phi} \). An application of [14] Theorem 5.1.9 proves that \( \phi = \phi_0 \tilde{\phi} \) is a generalized Hermite-Biehler function.

Definition 2.12. A generalized Hermite-Biehler polynomial \( \omega \) is said to be symmetric if \( \omega(-z) = \omega(z) \) for all \( z \in \mathbb{C} \).

It is clear that the generalized Hermite-Biehler polynomial \( \phi \) is symmetric. In particular, if \( \lambda \) is a zero of \( \phi \), then also \( -\lambda \) is a zero of \( \phi \), and both have the same multiplicity. By Theorem 2.7, \( \phi_N(0) \neq 0 \), and therefore \( \phi(0) \neq 0 \). Hence the zeros of \( \phi \) can be written as follows. The real zeros of \( \phi \) are denoted by \( (\beta_k)_{k=-p_1, k \neq 0}^{p_1} \), the pure imaginary zeros of \( \phi \) are denoted by \( (\gamma_k)_{k=1}^{p_2} \), and the remaining zeros of \( \phi \) are denoted by \( (\delta_k)_{k=-p_3, k \neq 0}^{p_3} \). Here we may assume that \( \beta_{-k} = -\beta_k \) and that \( \delta_{-k} = -\delta_k \). The number \( p_2 \) of pure imaginary zeros is even (or zero) if \( M > 0 \) and odd if \( M = 0 \). Setting \( m = 2p_1 + p_2 + 2p_3 \) it is also clear that \( m = 2 \sum_{j=1}^q n_j + 2 \) if \( M > 0 \) and \( m = 2 \sum_{j=1}^q n_j + 1 \) if \( M = 0 \). With an obvious meaning of the union of sequences, the sequence of all zeros of \( \phi \) will be denoted as \( (\lambda_k)_{k=-1}^{m} = (\beta_k)_{k=-p_1, k \neq 0}^{p_1} \cup (\gamma_k)_{k=1}^{p_2} \cup (\delta_k)_{k=-p_3, k \neq 0}^{p_3} \). Since \( \phi(0) \neq 0 \), it is clear that \( \lambda_k \neq 0 \) for all \( k \).

In particular, for all \( \lambda \in \mathbb{C} \),

\[
\Phi(\lambda) = \prod_{k=-p_1, k \neq 0}^{p_1} \left(1 - \frac{\lambda}{\beta_k}\right) \prod_{k=-p_3, k \neq 0}^{p_3} \left(1 - \frac{\lambda}{\delta_k}\right) \prod_{k=1}^{p_2} \left(1 - \frac{\lambda}{\gamma_k}\right)
\]
is well defined. From \( \Phi(0) = 1 \) it follows that \( \phi = \phi(0) \Phi \), where \( \phi(0) = \phi_N(0) \in \mathbb{R} \).

In the following, \( m(\lambda_k), m(\beta_k), m(\gamma_k), m(\delta_k) \) denote the multiplicities of \( \lambda_k, \beta_k, \gamma_k, \delta_k \), respectively, as zeros of \( \phi \), whereas \( m(\mu_k) \) and \( m(\nu_k) \) denote the multiplicities of \( \mu_k \) and \( \nu_k \) as zeros of \( \phi_N \) and \( \phi_D \), respectively.
Theorem 2.13. 1. If $M > 0$, then $(\beta_k)_{k=p_1, k \neq 0} = (\mu_k)_{k=p_1, k \neq 0} = (\nu_k)_{k=p_1, k \neq 0}$. If $M = 0$, then $(\beta_k)_{k=p_1, k \neq 0} = (\mu_k)_{k=p_1, k \neq 0} \cap (\nu_k)_{k=p_1, k \neq 0}$. Here $n = \sum_{j=1}^{q} n_j$.

2. If $\beta_k = \mu_p = \nu_p$, then $m(\mu_p) = m(\beta_k)$ and $m(\nu_p) = m(\beta_k) + 1$. Therefore, $m(\beta_k) \leq q - 1$.

3. Let $p_1$ be the number of distinct positive real zeros of $\phi$. Then $2\tilde{p}_1 \leq p_2 + 2p_3 - 1$. The number $p_2$ is odd if and only if $M = 0$.

4. $\text{Im} \frac{d^r}{dx^r} \Phi(\lambda) \big|_{\lambda=\beta_k} = 0$ for $s = 0, \ldots, m(\beta_k)$.

Proof. 1. Since $\phi_N(\lambda^2)$ and $\phi_D(\lambda^2)$ are real for real $\lambda$, it follows for real $\lambda$ that $\phi(\lambda) = 0$ if and only if $\phi_N(\lambda^2) = 0$ and $\phi_D(\lambda^2) = 0$, which proves statement 1.

2. Without loss of generality let $p$ be the smallest index such that $\beta_k = \nu_p = \mu_p$ and let $r$ be the largest index such that $\beta_k = \nu_r = \mu_r$. In view of Theorems 2.7 and 2.8 part 1, we have $p \geq 2$ and

$$m_p - 1 < \nu_p - 1 = \mu_p = \nu_p = \cdots = \mu_r = \nu_r.$$ 

This proves $m(\nu_p) = m(\mu_p) + 1$ if $r = n$. If $r < n$, then $\mu_r = \nu_r \leq \mu_{r+1} \leq \nu_{r+1}$ and $\nu_r < \nu_{r+1}$. But since $\nu_{r+1} = \mu_{r+1}$ if and only if $\nu_r = \mu_{r+1}$, we must have $\nu_{r+1} \neq \mu_{r+1}$ and $\nu_r \neq \mu_{r+1}$, so that also $m(\nu_p) = m(\mu_p) + 1$ if $r < n$.

The Taylor series of $\lambda \mapsto \phi_N(\lambda^2)$ and $\lambda \mapsto \phi_D(\lambda^2)$ about $\beta_k = \mu_p = \nu_p$ show that

$$\phi(\lambda) = (\lambda - \mu_p)^m(\mu_p) \psi_1(\lambda) + i\alpha(\lambda - \nu_p)^m(\nu_p) \psi_2(\lambda),$$

where $\psi_1$ and $\psi_2$ are polynomials with $\psi_1(\beta_k) \neq 0$ and $\psi_2(\beta_k) \neq 0$. From $m(\nu_p) = m(\mu_p) + 1$ we now conclude that $m(\beta_k) = m(\mu_p) = m(\nu_p) - 1 \leq q - 1$.

3. By part 2, every real zero of $\phi$ is the square root of a zero of multiplicity at least 2 of $\phi_D$. In view of Remark 2.1 it follows that

$$2\tilde{p}_1 \leq n < \frac{1}{2}(2p_1 + p_2 + 2p_3).$$

Therefore

$$\tilde{p}_1 < p_1 - \tilde{p}_1 + \frac{1}{2}p_2 + p_3 \leq \frac{1}{2}p_2 + p_3,$$

which implies $2\tilde{p}_1 \leq p_2 + 2p_3 - 1$. Since the degree $2p_1 + p_2 + 2p_3$ of $\phi$ is odd if and only if $p_2$ is odd, it follows from Remark 2.1 that $p_2$ is odd if and only if $M = 0$.

4. By (2.17), $\text{Im} \phi(\lambda) = \alpha \lambda \phi_D(\lambda^2)$ for real $\lambda$. By part 2, every $\beta_k$ is a zero of multiplicity $m(\beta_k) + 1$ of $\phi_D$, and therefore $\text{Im} \frac{d^r}{dx^r} \phi(\lambda) \big|_{\lambda=\beta_k} = 0$ for $s = 0, \ldots, m(\beta_k)$. Finally, we recall that $\Phi$ is a real multiple of $\phi$.

Remark 2.14. The function $\frac{\phi_N}{\phi_D}$ has the representation

$$\frac{\phi_N(z)}{\phi_D(z)} = \sum_{j=1}^{q} R_{2n_j-1}^{(j)}(z) - Mz$$

and according to Supplement II, (18) the fractions $\frac{R_{2n_j}^{(j)}(z)}{R_{2n_j-1}^{(j)}(z)}$ can be expanded into continued fractions

$$\frac{R_{2n_j}^{(j)}(z)}{R_{2n_j-1}^{(j)}(z)} = \frac{1}{n_j} + \frac{1}{-m_{n_j}^{(j)}z + \frac{1}{l_1^{(j)} + \frac{1}{l_2^{(j)} + \cdots}}}. $$

and into continued fractions
Using (2.18) and (2.19) we obtain
\[
\frac{\phi_N(0)}{\phi_D(0)} = \sum_{j=1}^{q} l_j^{-1}.
\]

3. INVERSE PROBLEM

We will make use of

**Theorem 3.1.** (Theorem 2.9). Let \(q \in \mathbb{N}, q \geq 2, (l_j)^q_{j=1} \in (0, \infty)^q, n \in \mathbb{N}, n \geq 2, \) \((\tau_k)^n_{k=-(n+1),k \neq 0} \in \mathbb{R}^{2(n+1)}, (\theta_k)^n_{k=-n,k \neq 0} \in \mathbb{R}^{2n}\) are such that
\begin{enumerate}
\item \(\tau_k = -\tau_k, \quad \theta_k = -\theta_k,\)
\item \(0 < \theta_1 < \tau_1 \leq \ldots \leq \theta_n \leq \tau_n < \theta_{n+1};\)
\item for \(k = 2, \ldots, n, \tau_{k-1} = \theta_k\) if and only if \(\theta_k = \tau_k;\)
\item the multiplicity of \(\theta_k\) in the sequence \((\theta_k)^n_{k=-(n+1),k \neq 0}\) does not exceed \(q - 1\).
\end{enumerate}

Then there exists a star graph of \(q\) Stieltjes strings, i.e., sequences \((n_j)^q_{j=1} \in \mathbb{N}_0^q\) and \((m_k)^{n_j}_{j=1} \in (0, \infty)^{n_j}, (l_k)^{n_j}_{j=0} \in (0, \infty)^{n_j+1}\) for \(j = 1, \ldots, q\) and \(M \in (0, \infty)\) with \(n = \sum_{j=1}^{q} n_j\) and \(\sum_{k=0}^{n_j} (l_k)^j = l_j\) such that the sequence of the zeros of \(\lambda \mapsto \phi_N(\lambda^2)\) is \((\theta_k)^n_{k=-(n+1),k \neq 0}\) and the sequence of the zeros of \(\lambda \mapsto \phi_D(\lambda^2)\) is \((\tau_k)^n_{k=-n,k \neq 0}\).

Now we come to the main theorem of this section.

**Theorem 3.2.** Let \(q \geq 2\) be an integer and let positive numbers \(l_j, j = 1, \ldots, q,\) be given. Let the sequences of numbers \((\beta_k)^{p_1}_{k=-p_1,k \neq 0}, (\gamma_k)^{p_2}_{k=1}\) and \((\delta_k)^{p_3}_{k=-p_3,k \neq 0}\) be given, where \(p_1, p_2, p_3 \in \mathbb{N}_0.\) Assume that the following conditions are satisfied:
\begin{enumerate}
\item For all indices \(k, \Im \beta_k = 0, \beta_{-k} = -\beta_k, \beta_k \neq 0, 2\tilde{p}_1 \leq p_2 + 2p_3 - 1,\) where \(\tilde{p}_1\) is the number of distinct elements of the sequence \((\beta_k)^{p_1}_{k=-p_1,k \neq 0}\) and \(\beta_{-k}\) is the multiplicity of \(\beta_k\) for each \(k\) and does not exceed \(q - 1.\)
\item \(\gamma_k = |\gamma_k|\) and \(\gamma_k \neq 0\) for all \(k.\)
\item \(\Im \delta_k > 0, \Re \delta_k \neq 0\) and \(\delta_{-k} = -\delta_k\) for all \(k.\)
\item \(\Im \frac{d^s}{d\lambda^s} \Phi(\lambda)|_{\lambda = \beta_k} = 0\) for all \(k\) and for \(s = 0, \ldots, m(\beta_k),\) where \(m(\beta_k)\) is the number of times \(\beta_k\) occurs in the sequence \((\beta_k)^{p_1}_{k=-p_1,k \neq 0}\) and where
\[
(3.1) \quad \Phi(\lambda) = \prod_{k=-p_1,k \neq 0}^{p_1} \left(1 - \frac{\lambda}{\beta_k}\right) \prod_{k=-p_3,k \neq 0}^{p_3} \left(1 - \frac{\lambda}{\delta_k}\right) \prod_{k=1}^{p_2} \left(1 - \frac{\lambda}{\gamma_k}\right).
\]
\end{enumerate}

Then there exists a collection of sequences \((m_k)^{n_j}_{k=1}\) and \((l_k)^{n_j}_{k=0}\) for \(j = 1, \ldots, q,\) numbers \(M \geq 0\) and \(\alpha > 0\) such that problems (2.1)–(2.4) with these data have spectrum \((\beta_k)^{p_1}_{k=-p_1,k \neq 0} \cup (\gamma_k)^{p_2}_{k=1} \cup (\delta_k)^{p_3}_{k=-p_3,k \neq 0}.\) Here \(M = 0\) if and only if \(p_2\) is odd.

**Proof.** For \(\lambda \in \mathbb{C}\) define
\[
R(\lambda) = \prod_{k=1}^{p_1} \left(1 - \frac{\lambda}{\beta_k^2}\right), \quad \omega(\lambda) = \prod_{k=-p_3,k \neq 0}^{p_3} \left(1 - \frac{\lambda}{\delta_k}\right) \prod_{k=1}^{p_2} \left(1 - \frac{\lambda}{\gamma_k}\right).
\]

By definition, \(\Phi(\lambda) = R(\lambda^2)\omega(\lambda)\) for \(\lambda \in \mathbb{C},\) and it is clear that \(\omega\) is a Hermite-Biehler polynomial. Since \(\omega(-\lambda) = \overline{\omega(\lambda)}\), there are polynomials \(P\) and \(Q,\) which are real on the real axis, such that \(Q(\lambda^2) = \frac{\omega(\lambda) - \omega(-\lambda)}{2\lambda}\) and \(P(\lambda^2) = \frac{\omega(\lambda) + \omega(-\lambda)}{2}.\)
Then $\omega(\lambda) = P(\lambda^2) + i\lambda Q(\lambda^2)$ for $\lambda \in \mathbb{C}$. We observe that $P$ and $Q$ do not have common zeros. Indeed, if $P(\lambda^2) = Q(\lambda^2) = 0$ for some $\lambda \in \mathbb{C}$, then $\omega(-\lambda) = -\omega(\lambda)$ and $\omega(-\lambda) = \omega(\lambda)$, and therefore $\omega(-\lambda) = \omega(\lambda) = 0$. But this is impossible since all zeros of $\omega$ lie in the open upper half-plane. Furthermore, $P(0) = \omega(0) = 1$.

We can now conclude from [14, Lemma 5.1.23] that

$$\frac{\lambda Q(\lambda^2)}{P(\lambda^2)}$$

defines a Nevanlinna function. It follows from [14, Lemma 5.2.4.4] and $P(0) = 1$ that $\frac{Q}{P}$ is an $S_0$-function with $P(z) > 0$ and $Q(z) > 0$ for $z \in (-\infty, 0]$.

Case I ($p_2$ is even). The numbers $n = p_1 + \frac{1}{2}p_2 + p_3 - 1$ and $n - p_1$ are nonnegative integers because $\frac{1}{2}p_2 + p_3 > 0$ by condition 3).

First we consider the case that $n - p_1 = 0$. Then $p_2 + 2p_3 = 2(n - p_1) + 2 = 2$ and $2p_1 \leq p_2 + 2p_3 - 1 = 1$ shows that $p_1 = 0$ and therefore $p_1 = 0$. Hence

$$\Phi(\lambda) = \left(1 - \frac{\lambda}{\lambda_1}\right)\left(1 - \frac{\lambda}{\lambda_2}\right) = 1 - \lambda \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right) + \frac{\lambda^2}{\lambda_1\lambda_2},$$

where $\lambda_1$ and $\lambda_2$ are nonzero complex numbers satisfying $\lambda_1 = i|\lambda_1|$ and $\lambda_2 = i|\lambda_2|$ or $\lambda_2 = -\lambda_1$ and $\text{Im} \lambda_1 > 0$. Then problem (2.1)-(2.2) with

$$M = -\frac{1}{\lambda_1\lambda_2} \sum_{j=1}^{q} l_j^{-1} > 0, \quad \alpha = i \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right) \sum_{j=1}^{q} l_j^{-1} > 0,$$

and $n_j = 0$ for $j = 0, \ldots, q$ has the two eigenvalues $(\lambda_1, \lambda_2)$ by Example 2.2.

Now let $n - p_1 > 0$. Since $\omega$ is a polynomial of even degree $p_2 + 2p_3$, the degree of $Q$ is $\frac{1}{2}p_2 + p_3 - 1 = n - p_1$, and hence $Q$ has $n - p_1$ zeros, which are all positive. The polynomial $P$ has degree $\frac{1}{2}p_2 + p_3 = n + 1 - p_1$, and hence $P$ has $n + 1 - p_1$ zeros, and $P$ and $Q$ do not have common zeros. Since zeros and poles of Nevanlinna functions are simple and interlace, there are positive numbers $\nu_k$, $k = 1, \ldots, n - p_1$, and $\mu_k$, $k = 1, \ldots, n + 1 - p_1$, such that $(\nu_k^{n-p_1})_{k=1}^{n-p_1}$ is the sequence of the zeros of $Q$, such that $(\mu_k^{n+1-p_1})_{k=1}^{n+1-p_1}$ is the sequence of the zeros of $P$ and such that

$$0 < \mu_1^2 < \nu_1^2 < \mu_2^2 < \nu_2^2 < \cdots < \nu_{n-p_1}^2 < \mu_{n+1-p_1}^2.$$

We also put $\nu_{-k} = -\nu_k$ for $k = 1, \ldots, n - p_1$ and $\mu_{-k} = -\mu_k$, $k = 1, \ldots, n + 1 - p_1$.

Due to the condition 5) we see that each $\beta_k$ is a zero of multiplicity at least $m(\beta_k) + 1$ of $RQ$, whereas $\beta_k$ is a zero of $R$ of multiplicity $m(\beta_k)$. Therefore $Q(\beta_k) = 0$, and each $\beta_k$ coincides with some $\nu_r$. For $\beta_k = \nu_r$ we set $\nu_r^{(j)} = \nu_r$, $j = 1, \ldots, m(\beta_k)$, and $\mu_r^{(j)} = \nu_r$, $j = 1, \ldots, m(\beta_k)$. Now we append all these $\nu_r^{(j)}$ to $(\nu_k^{n-p_1})_{k=-n+p_1,k\neq0}^{n-p_1}$ and $\mu_r^{(j)}$ to $(\mu_k^{n+1-p_1})_{k=-n-1+p_1,k\neq0}^{n+1-p_1}$ and obtain two sequences $(\tau_k)_{k=-n,k\neq0}^{n}$ and $(\theta_k)_{k=-n-1,k\neq0}^{n+1}$ which satisfy the assumptions of Theorem 3.1.

Thus, in view of Theorem 3.1 there exists a star graph of Stieljes strings with all quantities as indicated in the statement of this theorem, and in particular $M > 0$, except for $\alpha$, such that $(\tau_k)_{k=-n,k\neq0}^{n}$ and $(\theta_k)_{k=-n-1,k\neq0}^{n+1}$ are the sequences of the eigenvalues of the Neuman and Dirichlet problems, respectively. Let $\phi_N$ and $\phi_D$ be the corresponding characteristic functions; see (2.15) and (2.16). The polynomials $RP$ and $\phi_N$ are real on the real axis and have the same zeros. Hence there is a
nonzero real number \( \alpha_N \) such that \( RP = \alpha_N \phi_N \). In the same way we find a nonzero real number \( \alpha_D \) such that \( RQ = \alpha_D \phi_D \). Putting \( \alpha = \alpha_D \alpha_N^{-1} \), it follows that

\[
\Phi(\lambda) = R(\lambda^2)P(\lambda^2) + i\lambda R(\lambda^2)Q(\lambda^2) = \alpha_N[\phi_N(\lambda^2) + i\alpha \phi_D(\lambda^2)],
\]

and hence the eigenvalues of the damped problem for this star graph and this \( \alpha \) coincide with the zeros of \( \Phi \), that is, the given sequences. Finally, if \( \alpha \) were negative, then replacing \( \lambda \) with \(-\lambda\) and observing that \( p_2 + p_3 > 0 \) would give eigenvalues in the lower half-plane, which is impossible. Hence we have \( \alpha > 0 \).

Case II (\( p_2 \) is odd). The numbers \( n = p_1 + \frac{1}{2}p_2 + p_3 - \frac{1}{2} \) and \( n - p_1 \) are nonnegative integers because \( \frac{1}{2}p_2 + p_3 - \frac{1}{2} \geq 0 \) by condition 3).

First we consider the case that \( n - p_1 = 0 \). Then \( p_2 + 2p_3 = 2(n - p_1) + 1 = 1 \) and \( 2\tilde{p}_1 \leq p_2 + 2p_3 - 1 = 0 \) shows that \( p_2 = 1 \), \( p_3 = 0 \) and \( \tilde{p}_1 = 0 \), and therefore \( p_1 = 0 \). Hence

\[
\Phi(\lambda) = 1 - \frac{\lambda}{\delta_1},
\]

where \( \delta_1 = i|\delta_1| \) and \( \delta_1 \neq 0 \). Then problem (2.1)–(2.4) with

\[
M = 0, \quad \alpha = \frac{i}{\delta_1} \sum_{j=1}^{q} t_j^{-1} > 0,
\]

and \( n_j = 0 \) for \( j = 0, \ldots, q \) has the single eigenvalue \( \delta_1 \) by Example 2.2.

Now let \( n - p_1 > 0 \). Since \( \omega \) is a polynomial of odd degree \( p_2 + 2p_3 \), the degree of \( Q \) is \( \frac{1}{2}p_2 + p_3 - \frac{1}{2} = n - p_1 \), and hence \( Q \) has \( n - p_1 \) zeros, which are all positive. The polynomial \( P \) has degree \( \frac{1}{2}p_2 + p_3 - \frac{1}{2} = n - p_1 \), and hence \( P \) has \( n - p_1 \) zeros, and \( P \) and \( Q \) do not have common zeros. We know that \( \lambda \mapsto \frac{\lambda Q(\lambda^2)}{P(\lambda^2)} \) is a Nevanlinna function, and hence the smallest positive zero of \( P \) interlaces with 0 and the smallest positive zero of \( Q \). It follows that there are positive numbers \( \nu_k \), \( k = 1, \ldots, n - p_1 \) and \( \mu_k \), \( k = 1, \ldots, n - p_1 \) such that \( (\nu_k^2)_{k=1}^{n-p_1} \) is the sequence of the zeros of \( Q \), such that \( (\mu_k^2)_{k=1}^{n-p_1} \) is the sequence of the zeros of \( P \) and such that

\[
0 < \mu_1^2 < \nu_1^2 < \mu_2^2 < \nu_2^2 < \cdots < \nu_{n-p_1}^2.
\]

Proceeding as in Case I we obtain a star graph of strings, but now with \( M = 0 \), and the constant \( \alpha > 0 \), such that the corresponding damped problem has the zeros of \( \Phi \), that is, the given sequences of numbers as eigenvalues.

\[
\text{Remark 3.3.} \quad \text{The star graphs and the damping coefficient in Theorem 3.2 can be constructed explicitly. A construction of star graphs is given in the proof of [16, Theorem 2.9] for } n - p_1 > 0 \text{ and in the proof of Theorem 3.2 for } n - p_1 = 0. \]

To find \( \alpha \), we deduce from the proof of Theorem 3.2 that

\[
\alpha = \alpha_D \alpha_N^{-1} = \frac{R(0)Q(0)}{\phi_D(0)} \frac{\phi_N(0)}{R(0)P(0)} = \frac{-1}{\omega'(0)} \frac{\phi_N(0)}{\phi_D(0)} \omega(0).
\]

Then we conclude from \( \omega(0) = 1 \),

\[
\omega'(0) = -\sum_{k=-p_3, k \neq 0}^{p_2} \frac{1}{\delta_k} - \sum_{k=1}^{p_2} \frac{1}{\gamma_k}
\]
and (2.20) that

$$\alpha = i \left( \sum_{k=-p_3, k \neq 0}^{1} \frac{1}{\delta k} + \sum_{k=1}^{p_2} \frac{1}{\gamma k} \right) \sum_{j=1}^{q} l_j^{-1}. $$

**Remark 3.4.** We have seen in Remark 2.1 that the number of eigenvalues uniquely determines the total number of masses in the interiors of the strings and whether or not there is a mass at the joint vertex. We have also seen in Remark 3.3 that the damping coefficient $\alpha$ is uniquely determined by the spectrum. Finally, it follows from (3.2) that

$$\frac{\phi_N(\lambda^2)}{\phi_D(\lambda^2)} = i\alpha \frac{\Phi(\lambda) - \Phi(-\lambda)}{\Phi(\lambda) + \Phi(-\lambda)}. $$

Hence we conclude from (2.18) and (2.19) that

$$M = \lim_{\lambda \to \infty} \frac{\phi_N(\lambda^2)}{\lambda^2 \phi_D(\lambda^2)} = i\alpha \lim_{\lambda \to \infty} \frac{\Phi(\lambda) - \Phi(-\lambda)}{\lambda(\Phi(\lambda) + \Phi(-\lambda))} $$

which means that $M$ is uniquely determined by the spectrum of the damped problem.

**ACKNOWLEDGMENT**

This work was partially supported by a grant of the NRF of South Africa, grant no. 80956.

**REFERENCES**


