PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 145, Number 4, April 2017, Pages 1567–1580
http://dx.doi.org/10.1090/proc/13416
Article electronically published on December 30, 2016

THE BESSEL DIFFERENCE EQUATION

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(Communicated by Mourad Ismail)

Abstract. We define a new difference equation analogue of the Bessel differential equation and investigate the properties of its solution, which we express using a ${}_2F_1$ hypergeometric function. We find analogous formulas for Bessel function recurrence relations, a summation transformation which is identical to the Laplace transform of classical Bessel functions, and oscillation.

1. Introduction

The classical Bessel functions are heavily studied special functions (see [2,3,11, 15,17]) that are defined via Bessel's differential equation

$$t^2y''(t)+ty'(t)+(t^2-n^2)y(t)=0, \tag{1}$$

for some (possibly complex) parameter n called the order of the equation. In this paper, we propose the following discrete analogue of (1): the second-order delay difference equation

$$t(t-1)\Delta^2 y(t-2)+t\Delta y(t-1)+t(t-1)y(t-2)-n^2y(t)=0 \tag{2}$$

is called the *Bessel difference equation*.

The solutions of Bessel's differential equation, $\mathcal{J}_n$, are called Bessel functions (of the first kind) and have series representation

$$\mathcal{J}_n(t)=\sum_{k=0}^{\infty}\frac{(-1)^k t^{2k+n}}{k!\Gamma(k+n+1)2^{2k+n}}. \tag{3}$$

We will achieve analogues for (2) of the following well-known facts about (1):

$$t\mathcal{J}_n'(t)=n\mathcal{J}_n(t)-t\mathcal{J}_{n+1}(t), \tag{4}$$

$$t\mathcal{J}_n'(t)=-n\mathcal{J}_n(t)+t\mathcal{J}_{n-1}(t), \tag{5}$$

$$2n\mathcal{J}_n(t)=t\mathcal{J}_{n-1}(t)+t\mathcal{J}_{n+1}(t), \tag{6}$$

$$2\mathcal{J}_n'(t)=\mathcal{J}_{n-1}(t)-\mathcal{J}_{n+1}(t), \tag{7}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[t^n\mathcal{J}_n(t)\right]=t^n\mathcal{J}_{n-1}(t). \tag{8}$$

Received by the editors March 14, 2016 and, in revised form, April 27, 2016.
2010 *Mathematics Subject Classification.* Primary 33C05, 39A12, 39A10; Secondary 39A21.
Key words and phrases. Discrete Bessel functions, discrete oscillation, delay difference equations, hypergeometric series, contiguous relation.

The Laplace transform of the classical Bessel function $\mathcal{J}_n$ is

$$\mathscr{L}\{\mathcal{J}_n\}(z) = \frac{[\sqrt{z^2+1}+z]^{-n}}{\sqrt{z^2+1}}. \tag{9}$$

This fact is deduced in [16] from the differential equation

$$(z^2+1)y''(z) + 3zy'(z) + (1-n^2)y(z) = 0 \tag{10}$$

using $y(z) = \mathscr{L}\{\mathcal{J}_n\}(z)$.

Let $u(t) = \mathcal{J}_n(t)$. The function u obeys the self-adjoint equation

$$(tu')' = -\left(t - \frac{n^2}{t}\right)u. \tag{11}$$

Let $u(t) = \sqrt{t}\mathcal{J}_n(t)$. For $t \geq 0$, the function u obeys the self-adjoint equation

$$u'' = -\left(\frac{1-4n^2+4t^2}{4t^2}\right)u. \tag{12}$$

We use the notation $(a)_n = a(a+1)\dots(a+n-1) = \dfrac{\Gamma(a+n)}{\Gamma(a)}$. The well-known hypergeometric series ${}_2F_1$ is given by the power series

$${}_2F_1\left(a,b;c;z\right) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!} z^k.$$

If a or b is zero or a negative integer, then ${}_2F_1$ is a finite series and converges everywhere. Otherwise, the series converges for $a, b \in \mathbb{C}$, $c \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$, and $|z| < 1$, and extended by analytic continuation for $|z| > 1$.

2. Difference equations

Let $D \subset \mathbb{C}$ and $f\colon D \to \mathbb{C}$. The forward difference operator Δ is defined by the formula

$$\Delta f(t) = f(t+1) - f(t).$$

We define the "nth order discrete monomial"

$$(-1)^n(-t)_n = t(t-1)\dots(t-n+1).$$

Directly from the definition, we see that $\Delta[(-1)^n(-t)_n] = n(-1)^{n-1}(-t)_{n-1}$ is a discrete analogue of the power rule. We say that a function $f\colon \mathbb{Z} \to \mathbb{R}$ is oscillatory if it changes sign or equals zero at infinitely many points. The self-adjoint form for homogeneous second-order difference equations is

$$\Delta(p(t)\Delta y(t)) + q(t)y(t+1) = 0, \tag{13}$$

and provided that $p(t) \neq 0$, expanding (13) yields

$$\Delta^2 y(t) + \frac{\Delta p(t) + q(t)}{p(t+1)}\Delta y(t) + \frac{q(t)}{p(t+1)}y(t) = 0. \tag{14}$$

3. The discrete Bessel equation

We will now investigate the Bessel difference equation (2). Expanding this equation out by the definition of the difference operator yields the equation

$$(t^2 - n^2)y(t) - t(2t-1)y(t-1) + 2t(t-1)y(t-2) = 0, \tag{15}$$

showing we are guaranteed uniqueness for $t > n$. Define

$$J_n(t) = \frac{(-1)^n(-t)_n}{2^n n!} {}_2F_1\left(\frac{n-t}{2}, \frac{n+1-t}{2}; n+1; -1\right),$$

which is valid for all $t, n \in \mathbb{C}$. Throughout the rest of the paper, we will understand t to be a discrete variable, i.e., $t \in \mathbb{Z}$. For $t \in \mathbb{N}_0 \cup \{-1\}$, we see that this definition reduces to

$$J_n(t) = \sum_{k=0}^{\infty} \frac{(-1)^{k+n}(-t)_{2k+n}}{k!(n+k)!2^{2k+n}}$$

which appears to be very similar to the series (3). Figure 1 contains a table of values for $J_n(t)$ for various n, and the graphs of J_0, J_1, and J_2 appear in Figures 2, 3, and 4 respectively. Also note that $J_n(t)$ is a polynomial for $t \geq 0$ and $n \in \mathbb{N}_0$. However, this series representation is not optimal, because it diverges for $t < -1$. At $t = -1$, we see that

$$J_n(-1) = \sum_{k=0}^{\infty} \frac{(-1)^{k+n}(1)_{2k+n}}{k!(n+k)!2^{2k+n}} = \frac{(-1)^n}{2^n} \sum_{k=0}^{\infty} \frac{(-1)^k(2k+n)!}{k!(k+n)!2^{2k}},$$

which converges to $\dfrac{(-1)^n}{(\sqrt{2}+1)^n\sqrt{2}}$ by [9, p. 203]. We now show that J_n is a solution of (2).

Theorem 1. *The function*

$$J_n(t) = \frac{(-1)^n(-t)_n}{2^n n!} {}_2F_1\left(\frac{n-t}{2}, \frac{n+1-t}{2}; n+1; -1\right)$$

solves (2).

Proof. From [8, (36), p. 103], we know that the hypergeometric function ${}_2F_1$ obeys the contiguous relation

$$(c-a-b)F(a,b;c;z) - (c-a)F(a-1,b;c;z) + b(1-z)F(a,b+1;c;z) = 0.$$

Taking $a = \dfrac{n-t}{2} + 1$, $b = \dfrac{n-t}{2} + \dfrac{1}{2}$, $c = n+1$, $z = -1$, and the definition of J_n yields

$$\left(t - \frac{1}{2}\right)\frac{J_n(t-1)}{(-1)^n(1-t)_n} - \left(\frac{n}{2} + \frac{t}{2}\right)\frac{J_n(t)}{(-1)^n(-t)_n} + (n-t+1)\frac{J_n(t-2)}{(-1)^n(2-t)_n} = 0.$$

Multiplying by $-2t$ yields

$$t(1-2t)\frac{J_n(t-1)}{(-1)^n(1-t)_n} + (n+t)\frac{J_n(t)}{(-1)^{n-1}(1-t)_{n-1}} - 2t(n-t+1)\frac{J_n(t-2)}{(-1)^n(2-t)_n} = 0.$$

t	$J_0(t)$	$J_1(t)$	$J_2(t)$
-1	$0.7071\dots$	$-0.2928\dots$	$0.1213\dots$
0	1.000	0.000	0.000
1	1.000	0.5000	0.0000
2	0.5000	1.0000	0.2500
3	-0.5000	1.1250	0.7500
4	-1.6250	0.5000	1.2500
5	-2.1250	-0.9375	1.2500
6	-1.1875	-2.6250	0.2343
7	1.4375	-3.3359	-1.8593
8	4.7734	-1.6875	-4.1562
9	6.4609	2.7773	-4.7812
10	3.6835	8.3984	-1.6699
20	-197.6987	76.6649	186.6956
30	-2175.4364	-5144.4537	1717.9433
40	144243.0781	-57784.8851	-140126.0213
50	1724667.0284	4111518.8945	-1500198.3390

FIGURE 1. Table of values for various J_n.

Now multiply by $(-1)^n(1-t)_n$ to see

$$t(1-2t)J_n(t-1)+(n+t)\frac{(-1)^n(1-t)_n}{(-1)^{n-1}(1-t)_{n-1}}J_n(t) - 2t(n-t+1)\frac{(-1)^n(1-t)_n}{(-1)^n(2-t)_n}J_n(t-2)=0.$$

Now notice that

$$\frac{(-1)^n(1-t)_n}{(-1)^{n-1}(1-t)_{n-1}}=t-n$$

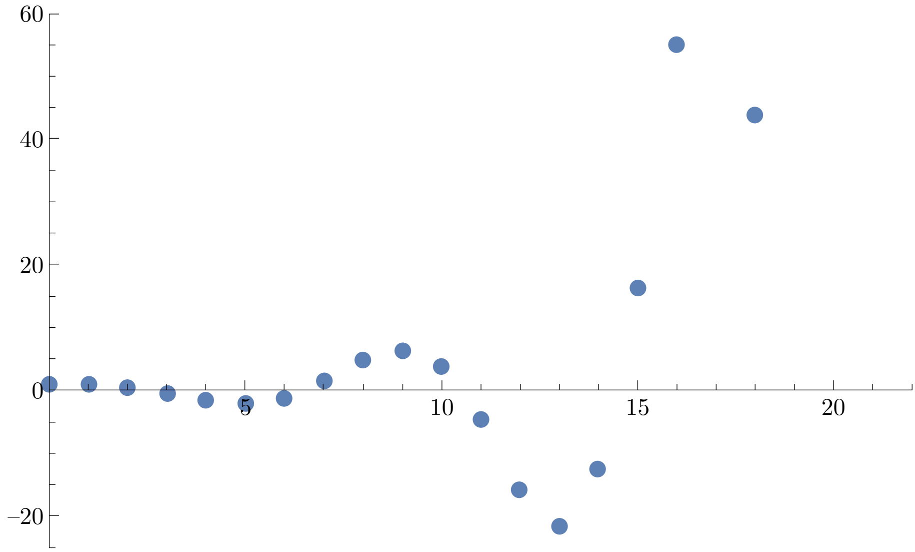

FIGURE 2. Discrete Bessel function J_0.

and

$$\begin{aligned}(n-t+1)\frac{(-1)^n(1-t)_n}{(-1)^n(2-t)_n} &= (n-t+1)(t-1)\frac{(-1)^{n-1}(2-t)_{n-1}}{(-1)^n(2-t)_n}\\ &= \frac{(n-t+1)(t-1)}{(t-1-n)}\\ &= -(t-1),\end{aligned}$$

showing that J_n satisfies (15), as was to be shown. □

Analogues of Bessel functions are not an original idea — the theory of q-Bessel functions is well developed [10, 13]. It should be noted that our functions are not the same as those in [6], which studies the equation

$$\Delta^2 y(t) + \frac{1}{t-\frac{1}{2}}\Delta y(t) + \frac{4t\lambda}{t-\frac{1}{2}}y(t) = 0,$$

derived from discretizing Laplace's equation in cylindrical coordinates with difference operators. A similar system is studied in [14]. It is also different from those studied in [12], which investigates series solutions to the equation

$$(t+2n+2)\Delta^2 y(t) + (2n+1)\Delta y(t) + \lambda(t+1)y(t) = 0.$$

Coincidentally, this equation matches our discrete Bessel equation for $n = 0$ and $\lambda = 1$, but not for other values of n or λ. We now put the discrete Bessel equation in a standard form.

Lemma 2. *A function y solves (2) if and only if y solves*

$$\left[(t+2)^2 - n^2\right]\Delta^2 y(t) + \left[t+2-2n^2\right]\Delta y(t) + \left[(t+2)(t+1)-n^2\right]y(t) = 0. \tag{16}$$

Proof. In (2), we substitute t by $t+2$ to obtain

$$(t+2)(t+1)\Delta^2 y(t) + (t+2)\Delta y(t+1) + (t+2)(t+1)y(t) - n^2 y(t+2) = 0.$$

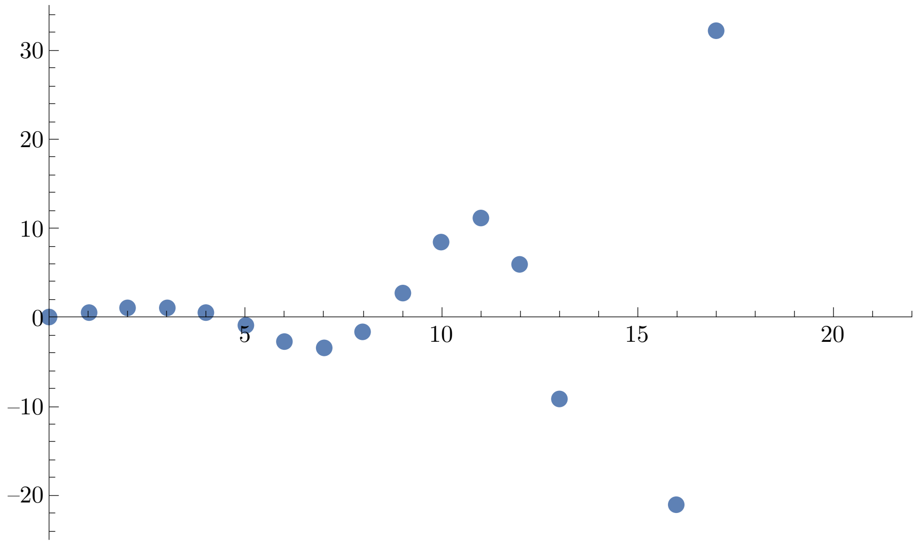

FIGURE 3. Discrete Bessel function J_1.

Now since $\Delta y(t+1) = \Delta^2 y(t) + \Delta y(t)$ and $y(t+2) = \Delta^2 y(t) + 2\Delta y(t) + y(t)$, we get

$$\left[(t+2)^2 - n^2\right] \Delta^2 y(t) + \left[(t+2) - 2n^2\right] \Delta y(t) + \left[(t+2)(t+1) - n^2\right] y(t) = 0,$$

as was to be shown. $\square$

Recall the binomial coefficient notation

$$\binom{n}{k} = \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)}.$$

The following theorem is an analogue of (11).

Theorem 3. *The equation* (2) *is equivalent to* (13) *for integers* $t \geq n$*, where*

$$p(t) = \prod_{k=n}^{t-1} \frac{(k+2)^2 - n^2}{2(k+2)(k+1)} = \frac{2^{n-t}(t+1)}{\binom{t+1}{n}\binom{t+1}{n+1}} \binom{1+n+t}{t-n}$$

and

$$q(t) = \frac{(t+2)(t+1) - n^2}{2(t+2)(t+1)} p(t).$$

Proof. Note that $p(t) > 0$ for all $t \geq n$. First we compute

$$\Delta p(t) = -\frac{1}{2}\left[\frac{t(t+2) + n^2}{(t+2)(t+1)}\right] p(t), \quad t \in \mathbb{N} \setminus \{1, 2, \ldots, n-1\},$$

and

$$p(t+1) = p(t) \frac{(t+2)^2 - n^2}{2(t+2)(t+1)}.$$

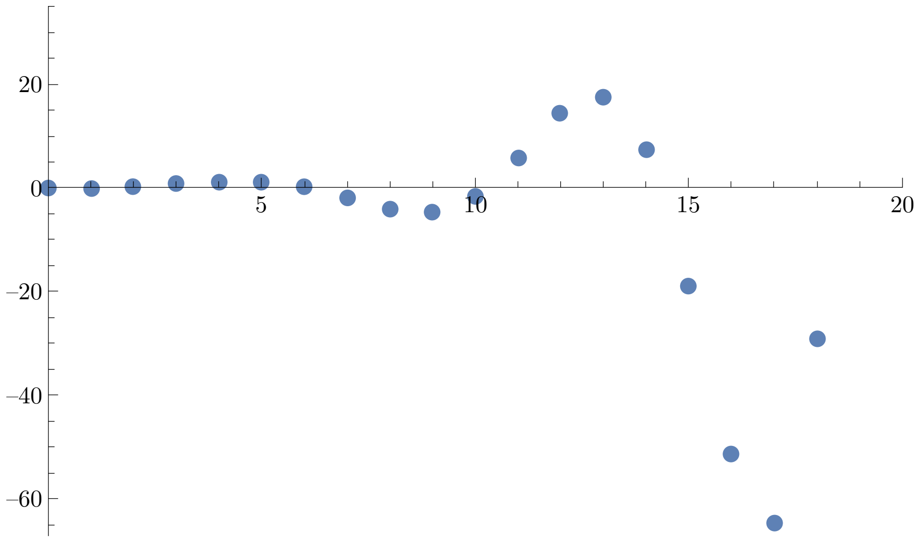

FIGURE 4. Discrete Bessel function J_2.

We start with (13) and expand the first term to get

$$\begin{aligned}0 &= p(t+1)\Delta^2 y(t) + \Delta p(t)\Delta y(t) + q(t)y(t+1)\\ &= \frac{(t+2)^2-n^2}{2(t+2)(t+1)}p(t)\Delta^2 y(t) - \frac{1}{2}\left[\frac{t(t+2)+n^2}{(t+2)(t+1)}\right]p(t)\Delta y(t)\\ &+\frac{(t+2)(t+1)-n^2}{2(t+2)(t+1)}p(t)y(t+1).\end{aligned}$$

Multiply this by $\dfrac{2(t+2)(t+1)}{p(t)}$ to get

$$0 = [(t+2)^2-n^2]\Delta^2 y(t) - [t(t+2)+n^2]\Delta y(t) + [(t+2)(t+1)-n^2]y(t+1).$$

Since $y(t+1) = \Delta y(t) + y(t)$ and $(t+2)(t+1) - t(t+2) = (t+2)$, we see

$$0 = [(t+2)^2-n^2]\Delta^2 y(t) + [t+2-2n^2]\Delta y(t) + [(t+2)(t+1)-n^2]y(t),$$

but this is (16) which Lemma 2 shows to be equivalent to (2). □

Theorem 4. *The equation* (2) *is equivalent to*

$$\begin{aligned}(17)\quad 2(t+1)(t+2)\Delta^2 y(t) - \left[t(t+2)+n^2\right]\Delta y(t+1)&\\ + \left[(t+1)(t+2)-n^2\right]y(t+1) &= 0.\end{aligned}$$

Proof. In (2) replace t by $t+2$ to get

$$(t+2)(t+1)\Delta^2 y(t) + (t+2)\Delta y(t+1) + (t+2)(t+1)y(t) - n^2 y(t+2) = 0.$$

Now note as a consequence of the definition of Δ that

$$y(t) = y(t+1) - \Delta y(t+1) + \Delta^2 y(t)$$

and

$$y(t+2) = \Delta y(t+1) + y(t+1).$$

Substituting in these formulas and algebraically simplifying yields the result. $\square$

4. Properties of J_n

We now derive various recurrence relations that J_n obey. First, we present an analogue of (4).

Theorem 5. *For all $n \in \mathbb{C} \setminus \{-1, -2, \ldots\}$, we have*

$$t\Delta J_n(t-1) = nJ_n(t) - tJ_{n+1}(t-1).$$

Proof. First calculate

$$\begin{aligned}\Delta J_n(t) &= \frac{n(-1)^{n-1}(-t)_{n-1}}{2^n n!} {}_2F_1\left(\frac{n-(t+1)}{2}, \frac{n-(t+1)}{2} + \frac{1}{2}; n+1; -1\right)\\ &\quad + \frac{(-1)^n(-t)_n}{2^n n!} \Delta {}_2F_1\left(\frac{n-t}{2}, \frac{n-t}{2} + \frac{1}{2}; n+1; -1\right).\end{aligned}$$

Hence

$$\begin{aligned}t\Delta J_n(t-1) &= nJ_n(t) + \frac{t(-1)^n(1-t)_n}{2^n n!} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{n-t}{2} + \frac{1}{2}\right)_k}{k!(n+1)_k} \left[\left(\frac{n-t}{2}\right)_k - \left(\frac{n-t}{2}+1\right)_k\right]\\ &= nJ_n(t) - \frac{(-1)^n(-t)_{n+1}}{2^n n!} \sum_{k=1}^{\infty} \frac{(-1)^k \left(\frac{n-t}{2} + \frac{1}{2}\right)_k \left(\frac{n-t}{2}+1\right)_{k-1}}{(k-1)!(n+1)_k}\\ &= nJ_n(t) + \frac{(-1)^n(-t)_{n+1}}{2^n n!} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{n-t}{2} + \frac{1}{2}\right)_{k+1} \left(\frac{n-t}{2}+1\right)_{k}}{k!(n+1)_{k+1}}\\ &= nJ_n(t) - \frac{(-1)^{n+1}(-t)_{n+2}}{2^{n+1}(n+1)!} {}_2F_1\left(\frac{n-t}{2}+1, \frac{n-t}{2} + \frac{1}{2} + 1; n+1; -1\right).\end{aligned}$$

Observing that

$$tJ_{n+1}(t-1) = \frac{(-1)^{n+2}(-t)_{n+2}}{2^{n+1}(n+1)!} {}_2F_1\left(\frac{n-t}{2}+1; \frac{n-t}{2} + \frac{1}{2} + 1; n+1; -1\right)$$

completes the proof. $\square$

Now we prove an analogue of (5).

Theorem 6. *For all $n \in \mathbb{C} \setminus \{-1, -2, \ldots\}$, we have*

$$t\Delta J_n(t-1) = -nJ_n(t) + tJ_{n-1}(t-1).$$

Proof. Compute

$$\begin{aligned}t\Delta J_n(t-1) &= nJ_n(t) + \frac{(-1)^{n+1}(-t)_{n+1}}{2^n n!} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{n-t}{2} + \frac{1}{2}\right)_k}{k!(n+1)_k} \left[\left(\frac{n-t}{2}\right)_k - \left(\frac{n-t}{2}+1\right)_k\right]\\ &= nJ_n(t) + \frac{(-1)^n(-t)_n(t-n)}{2^n n!} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{n-t}{2} + \frac{1}{2}\right)_k \left(\frac{n-t}{2}\right)_k}{k!(n+1)_k} \left[\frac{2k}{t-n}\right]\\ &= -nJ_n(t) + \frac{(-1)^n(-t)_n}{2^n(n-1)!} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{n-t}{2} + \frac{1}{2}\right)_k \left(\frac{n-t}{2}\right)_k}{k!(n)_{k+1}} [2k+2n]\\ &= -nJ_n(t) + \frac{(-1)^n(-t)_n}{2^{n-1}(n-1)!} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{n-t}{2} + \frac{1}{2}\right)_k \left(\frac{n-t}{2}\right)_k}{k!(n)_k}\\ &= -nJ_n(t) + tJ_{n-1}(t-1),\end{aligned}$$

as was to be shown. $\square$

The formulas in the following corollary are analogues of (6) and (7) and come directly from the two previously derived recurrence relations. For the first one, we subtract the formulas, and for the second one, we add the formulas, divide by t, and replace $t-1$ with t.

Corollary 7. *For all $n \in \mathbb{C} \setminus \{-1,-2,\ldots\}$, we have*

$$2nJ_n(t) = t[J_{n-1}(t-1) + J_{n+1}(t-1)]$$

and

$$2\Delta J_n(t) = J_{n-1}(t) - J_{n+1}(t).$$

Now we prove an analogue of (8).

Theorem 8. *For all $n \in \mathbb{C} \setminus \{-1,-2\ldots\}$, we have*

$$\Delta[(-t)_n J_n(t-n)] = (-t)_n J_{n-1}(t-n).$$

Proof. Compute

$$\begin{aligned}
\Delta\left[(-t)_n J_n(t-n)\right] &= (-1-t)_n J_n(t+1-n) - (-t)_n J_n(t-n) \\
&= \frac{(-1-t)_n(-1)^n(n-t-1)_n}{2^n n!} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{2n-t}{2} - \frac{1}{2}\right)_k \left(\frac{2n-t}{2}\right)_k}{k!(n+1)_k} \\
&\quad - \frac{(-t)_n(-1)^n(n-t)_n}{2^n n!} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{2n-t}{2}\right)_k \left(\frac{2n-t}{2} + \frac{1}{2}\right)_k}{k!(n+1)_k} \\
&= \frac{(-t)_n(-1)^{n-1}(n-t)_{n-1}}{2^n (n-1)!} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{2n-t}{2}\right)_k \left(\frac{2n-t}{2} - \frac{1}{2}\right)_k (2n+2k)}{k!(n)_{k+1}} \\
&= \frac{(-t)_n(-1)^{n-1}(n-t)_{n-1}}{2^{n-1} (n-1)!} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{2n-t}{2}\right)_k \left(\frac{2n-t}{2} - \frac{1}{2}\right)_k}{k!(n)_k} \\
&= (-t)_n J_{n-1}(t-n),
\end{aligned}$$

as was to be shown. □

5. The $\tilde{\mathcal{Z}}$-transform of discrete Bessel functions

The well-known $\mathcal{Z}$-transform of a function f is defined by

$$\mathcal{Z}\{f\}(z) := \sum_{k=0}^{\infty} \frac{f(k)}{z^k}.$$

We will make use of a modified $\mathcal{Z}$-transform which we will call the $\tilde{\mathcal{Z}}$-transform (see [4, 5]), given by the formula

$$\tilde{\mathcal{Z}}\{f\}(z) = \frac{\mathcal{Z}\{f\}(z+1)}{z+1} = \sum_{k=0}^{\infty} \frac{f(k)}{(z+1)^{k+1}}.$$

It is known [7, Exercise 13, p. 281] that

$$\mathcal{Z}\{\Delta^n f\}(z) = (z-1)^n \mathcal{Z}\{f\}(z) - z\sum_{j=0}^{n-1}(z-1)^{n-j-1}\Delta^j f(0),$$

and so by definition, we see

$$\begin{aligned}\tilde{\mathcal{Z}}\{\Delta^n f\}(z) &= \frac{z^n \mathcal{Z}\{f\}(z+1)}{z+1} - \frac{(z+1)\displaystyle\sum_{j=0}^{n-1} z^{n-j-1}\Delta^j f(0)}{z+1} \\ &= z^n \tilde{\mathcal{Z}}\{f\}(z) - \sum_{j=0}^{n-1} z^{n-j-1}\Delta^j f(0).\end{aligned}$$

To get a formula for the $\tilde{\mathcal{Z}}$-transform of J_n, we want to know how to relate the $\tilde{\mathcal{Z}}$-transform of $(-1)^k(-t)_k f(t-k)$ to derivatives with respect to z of the function $\tilde{\mathcal{Z}}\{f\}(z)$.

Lemma 9. *If* $f(t) = \displaystyle\sum_{j=0}^{\infty} a_k(-1)^k(-t)_k$, *then*

$$(\tilde{\mathcal{Z}}\{f\})^{(n)}(z) = (-1)^n \tilde{\mathcal{Z}}\{f_n\}(z),$$

where $f_n(t) = (-1)^n(-t)_n f(t-n)$.

Proof. The computation

$$\begin{aligned}(\tilde{\mathcal{Z}}\{f\})^{(n)}(z) &= \frac{\mathrm{d}^n}{\mathrm{d}z^n}\sum_{k=0}^{\infty}\frac{f(k)}{(z+1)^{k+1}} \\ &= (-1)^n \sum_{k=0}^{\infty}\frac{(k+1)(k+2)\dots(k+n)f(k)}{(z+1)^{k+n+1}} \\ &= (-1)^n \sum_{k=n}^{\infty}\frac{(k-n+1)(k-n+2)\dots(k-1)kf(k-n)}{(z+1)^{k+1}} \\ &= (-1)^n \sum_{k=0}^{\infty}\frac{(-1)^n(-k)_n f(k-n)}{(z+1)^{k+1}} \\ &= (-1)^n \tilde{\mathcal{Z}}\{f_n\}(z)\end{aligned}$$

proves the claim. ☐

We now demonstrate an analogue of (9) by showing that (10) holds for the function $y(z) = \tilde{\mathcal{Z}}\{J_n\}(z)$.

Theorem 10. *For all* $n \in \mathbb{C}\setminus\{0,-1,\dots\}$, *we have*

$$\tilde{\mathcal{Z}}\{J_n\}(z) = \frac{[\sqrt{z^2+1}+z]^{-n}}{\sqrt{z^2+1}}.$$

Proof. Using Lemma 9 in (2), we get

$$(\tilde{\mathcal{Z}}\{\Delta^2 J_n\})''(z) - (\tilde{\mathcal{Z}}\{\Delta J_n\})'(z) + (\tilde{\mathcal{Z}}\{J_n\})''(z) - n^2\tilde{\mathcal{Z}}\{J_n\}(z) = 0.$$

Now we apply the formula for the $\tilde{\mathcal{Z}}$-transform of differences to get

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\Big[z^2\tilde{\mathcal{Z}}\{J_n\}(z) - zJ_n(0) - \Delta J_n(0)\Big] &- \frac{\mathrm{d}}{\mathrm{d}z}\Big[z\tilde{\mathcal{Z}}\{J_n\}(z) - J_n(0)\Big] \\ &+ \frac{\mathrm{d}^2}{\mathrm{d}z^2}\tilde{\mathcal{Z}}\{J_n\}(z) - n^2\tilde{\mathcal{Z}}\{J_n\}(z) = 0.\end{aligned}$$

We compute the derivatives and simplify to obtain

$$(z^2+1)(\tilde{\mathcal{Z}}\{J_n\})''(z)+3z(\tilde{\mathcal{Z}}\{J_n\})'(z)+(1-n^2)\tilde{\mathcal{Z}}\{J_n\}(z).$$

This is simply (10) with $y(z)=\tilde{\mathcal{Z}}\{J_n\}(z)$. Therefore

$$\tilde{\mathcal{Z}}\{y\}(z)=\frac{[\sqrt{z^2+1}+z]^{-n}}{\sqrt{z^2+1}},$$

as claimed. □

6. Oscillatory behavior of J_n

From the self-adjoint form (13), the oscillation of y can be deduced using the Leighton–Wintner theorem [5, Theorem 4.64] which says if there exists $t_0\in\mathbb{Z}$ such that

$$\sum_{k=t_0}^{\infty}\frac{1}{p(k)}=\sum_{k=t_0}^{\infty}q(k)=\infty, \tag{18}$$

then the solution y is oscillatory. First we present a lemma that will help us prove that discrete Bessel functions are oscillatory.

Lemma 11. *Let $n\in\mathbb{N}_0$ be fixed. The function $v\colon\{n,n+1,n+2,\dots\}\to\mathbb{R}$ defined by*

$$v(t)=\frac{2^{\frac{t-n}{2}}}{\binom{t}{\frac{t-n}{2}}}$$

satisfies the recurrence relation

$$v(t+2)=\frac{(t+2)^2-n^2}{2(t+1)(t+2)}v(t).$$

Moreover,

$$\lim_{t\to\infty}\frac{v(t)}{v(t+1)}=\sqrt{2}.$$

Proof. Calculate

$$\begin{aligned}
v(t+2) &= \frac{2^{\frac{t+2-n}{2}}}{\binom{t+2}{\frac{t+2-n}{2}}}\\
&= \frac{2^{1+\frac{t-n}{2}}}{\frac{\Gamma(t+3)}{\Gamma(2+\frac{t-n}{2})\Gamma(2+\frac{t+n}{2})}}\\
&= \frac{2}{\frac{(t+2)(t+1)}{(1+\frac{t-n}{2})(1+\frac{t+n}{2})}}\,\frac{2^{\frac{t-n}{2}}}{\frac{\Gamma(t+1)}{\Gamma(1+\frac{t-n}{2})\Gamma(1+\frac{t+n}{2})}}\\
&= \frac{2}{\frac{(t+2)(t+1)}{(1+\frac{t-n}{2})(1+\frac{t+n}{2})}}\,\frac{2^{\frac{t-n}{2}}}{\binom{t}{\frac{t-n}{2}}}\\
&= \frac{(t+2)^2-n^2}{2(t+1)(t+2)}v(t),
\end{aligned}$$

as was to be shown. Now calculate

$$\begin{aligned}\frac{v(t)}{v(t+1)} &= \frac{t+1}{2}\frac{\Gamma(\frac{t-n}{2}+1)\Gamma(\frac{t+n}{2}+1)\Gamma(t+2)}{\Gamma(\frac{t-n}{2}+1+\frac{1}{2})\Gamma(\frac{t+n}{2}+1+\frac{1}{2})\Gamma(t+1)} \\ &= \frac{\sqrt{2}(t+1)}{\sqrt{(t-n+2)(t+n+2)}}\left(\frac{\Gamma\left(\frac{t-n}{2}+1\right)\sqrt{\frac{t-n}{2}+1}}{\Gamma\left(\frac{t-n}{2}+1+\frac{1}{2}\right)}\right)\left(\frac{\Gamma\left(\frac{t+n}{2}+1\right)\sqrt{\frac{t+n}{2}+1}}{\Gamma\left(\frac{t+n}{2}+1+\frac{1}{2}\right)}\right).\end{aligned} \tag{19}$$

Notice that the first factor on the right-hand side of (19) tends to $\sqrt{2}$ as $t \to \infty$. By [1, (6.1.47), p. 257], we see that

$$\lim_{z\to\infty}\frac{\Gamma(z+a)}{z^a\Gamma(z)} = \lim_{z\to\infty}\frac{z^a\Gamma(z)}{\Gamma(z+a)} = 1,$$

and so the second and third factors on the right-hand side of (19) tend to 1 as $t \to \infty$, as was to be shown. □

Theorem 12. *For all $n \in \mathbb{R}\setminus\{0,-1,\dots\}$, J_n is oscillatory.*

Proof. Let y solve (17) so that

$$\Delta^2 y(t) = \frac{t(t+2)+n^2}{2(t+1)(t+2)}\Delta y(t+1) - \frac{(t+1)(t+2)-n^2}{2(t+1)(t+2)}y(t+1). \tag{20}$$

By putting $u = vy$, it follows that

$$\Delta u(t) = y(t+1)\Delta v(t) + v(t)\Delta y(t)$$

and

$$\Delta^2 u(t) = y(t+1)\Delta^2 v(t) + (\Delta v(t+1) + \Delta v(t))\Delta y(t+1) + v(t)\Delta^2 y(t).$$

Using (20), we see

$$\begin{aligned}\Delta^2 u(t) = &\left[\frac{\Delta^2 v(t)}{v(t+1)} - \frac{v(t)}{v(t+1)}\frac{(t+1)(t+2)-n^2}{2(t+1)(t+2)}\right]u(t+1) \\ &+\left[\Delta v(t+1) + \Delta v(t) + v(t)\frac{t(t+2)+n^2}{2(t+1)(t+2)}\right]\Delta y(t+1).\end{aligned} \tag{21}$$

We now define the function v to be a solution of the difference equation

$$\Delta v(t+1) + \Delta v(t) + v(t)\frac{t(t+2)+n^2}{2(t+1)(t+2)} = 0,$$

i.e.,

$$v(t+2) = \frac{(t+2)^2-n^2}{2(t+1)(t+2)}v(t). \tag{22}$$

We have freedom to pick the initial conditions, and so we pick them to match the function v from Lemma 11. Consequently, (21) implies the self-adjoint difference equation (an analogue of (12)),

$$\Delta^2 u(t) + \left[\frac{v(t)}{v(t+1)}\frac{(t+1)(t+2)-n^2}{2(t+1)(t+2)} - \frac{\Delta^2 v(t)}{v(t+1)}\right]u(t+1) = 0,$$

where the functions p and q in (13) obey the formulas

$$p(t) = 1$$

and

$$q(t) = \frac{v(t)}{v(t+1)}\frac{(t+1)(t+2)-n^2}{2(t+1)(t+2)} - \frac{\Delta^2 v(t)}{v(t+1)}.$$

Using the difference operator identity $\Delta^2 f(t) = f(t+2) - 2f(t+1) + f(t)$ and (22), we may compute

$$
\begin{aligned}
q(t) &= \frac{v(t)}{v(t+1)} \frac{(t+1)(t+2)-n^2}{2(t+1)(t+2)} - \frac{v(t+2)-2v(t+1)+v(t)}{v(t+1)} \\
&= \frac{v(t)}{v(t+1)} \left[\frac{(t+1)(t+2)-n^2}{2(t+1)(t+2)} - 1 \right] - \frac{(t+2)^2-n^2}{2(t+1)(t+2)} \frac{v(t)}{v(t+1)} + 2 \\
&= \frac{v(t)}{v(t+1)} \left[\frac{-(t+2)(t+1)-n^2-(t+2)^2+n^2}{2(t+1)(t+2)} \right] + 2 \\
&= 2 - \frac{v(t)}{v(t+1)} \frac{2t+3}{2t+2}.
\end{aligned} \tag{23}
$$

From (23) and Lemma 11, we see

$$\lim_{t\to\infty} q(t) = 2 - \sqrt{2} > 0,$$

showing that q is eventually bounded below by $\frac{1}{2}$. Therefore $u = vy$ is oscillatory, and since v is positive, $y = J_n$ is oscillatory. □

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