

COCYCLE CONJUGACY CLASSES OF BINARY SHIFTS

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ABSTRACT. We show that every binary shift on the hyperfinite II_1 factor R is cocycle conjugate to at least countably many non-conjugate binary shifts. This holds in particular for binary shifts of infinite commutant index.

1. PRELIMINARIES

Let $(a) = a_0, a_1, a_2, \dots$ be a *bitstream*, i.e., a sequence of 0's and 1's in $GF(2)$. Let v_0, v_1, v_2, \dots be a sequence of self-adjoint unitary operators (which we shall call generators) that satisfy the translation-invariant commutation relations

$$v_i v_{i+j} = (-1)^{a_j} v_{i+j} v_i$$

for all non-negative integers i and j . The commutation relations imply that $a_0 = 0$. The algebra, \mathfrak{A} , obtained as the norm closure of the set of all linear combinations of the words in the v_i 's is isomorphic to the CAR algebra if and only if the mirror sequence $\dots, a_2, a_1, a_0, a_1, a_2, \dots$ is not periodic [8, Theorem 2.3]. Here we restrict our attention to the CAR algebra case. Using the GNS representation for the unique trace on \mathfrak{A} , [5], which sends all non-trivial words in the v_i 's to 0, the hyperfinite II_1 factor R is obtained as the strong closure of the set of all linear combinations of the words in the v_i 's. Then the mapping sending v_i to v_{i+1} gives rise to a unital $*$ -homomorphism α on R , called a *binary shift*. The image $\alpha(R)$ has subfactor index 2, i.e., $[R : \alpha(R)] = 2$. Note that R is generated by $\alpha(R)$ and v_0 .

Definition 1 (cf. [6]). Two unital $*$ -endomorphisms α and β on R are said to be *conjugate* if there is a $*$ -automorphism γ of R such that

$$\alpha(v) = \gamma(\beta(\gamma^{-1}(v))), \text{ for all } v \in R.$$

Two unital $*$ -endomorphisms α and β are said to be *cocycle conjugate* if there is a unitary $u \in R$ so that $\text{Ad}(u) \circ \alpha$ and β are conjugate.

Theorem 1 ([6, Theorem 3.6]). *Binary shifts α and β on R are conjugate if and only if they have the same bitstream.*

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For $n \in \mathbb{N}$ let \mathcal{A}_n be the $n \times n$ Toeplitz matrix over $GF(2)$, given by

$$\mathcal{A}_n = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \cdots & a_{n-1} \\ a_1 & a_0 & a_1 & a_2 & \cdots & a_{n-2} \\ a_2 & a_1 & a_0 & a_1 & \cdots & a_{n-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & \cdots & a_0 \end{bmatrix}.$$

Note that \mathcal{A}_n is symmetric with main diagonal all 0's. For each $n \in \mathbb{N}$ let $\nu_n = \text{null}(\mathcal{A}_n)$, i.e., the dimension of $\ker(\mathcal{A}_n)$. The importance of the nullity sequence $\{\nu_n : n \in \mathbb{N}\}$ is that it gives information about the centers of the algebras \mathfrak{A}_n generated by v_0, \dots, v_{n-1} (see Theorem 3 and Theorem 4 below).

Theorem 2 ([7, Theorem 5.4], [3, Theorem 2.7]). *The sequence $\{\nu_n : n \in \mathbb{N}\}$ is the concatenation of finite length strings of the form $1, 0$ or $1, 2, \dots, r-1, r, r-1, \dots, 0$ for some $r > 1$, where the length $2r$ of the string may vary.*

Remark 1. If the mirror bitstream is periodic there is a non-negative integer n such that $\nu_{n+j+1} = \nu_{n+j} + 1$ for all $j \in \mathbb{N}$, [10], and in this case the C^* -algebra $C^*(v_0, v_1, \dots)$ is isomorphic to $\mathcal{T} \otimes C(X)$, where X is the Cantor set, and \mathcal{T} is either $\mathbb{C}I$ or the tensor product of finitely many copies of $M_2(\mathbb{C})$ [1, Theorem 3.1].

Theorem 3 ([7, Corollary 5.5], [10, Lemma 3.2]). *Let α be a binary shift on R with bitstream (a) . For any positive integer n there exists a one-to-one correspondence between vectors $\mathbf{c} = [c_0, c_1, \dots, c_{n-1}]^t$ in $\ker(\mathcal{A}_n)$ and ordered words $v_0^{c_0} v_1^{c_1} \cdots v_{n-1}^{c_{n-1}}$ in the center \mathfrak{Z}_n of \mathfrak{A}_n . Therefore, as \mathfrak{Z}_n consists of linear combinations of the words it contains, its dimension as an algebra over \mathbb{C} is 2^{ν_n} . In particular, if $\nu_n = 0$, then $\mathfrak{Z}_n = \mathbb{C}I$ and \mathfrak{A}_n is a full matrix algebra over \mathbb{C} .*

The following result about the centers \mathfrak{Z}_n of the algebras \mathfrak{A}_n , along with the correspondence between \mathfrak{Z}_n and $\ker(\mathcal{A}_n)$ discussed above, shows that to understand the structure of $\ker(\mathcal{A}_n)$ for all n it suffices to know the structure of $\ker(\mathcal{A}_n)$, where $\nu_{n-1} = 0$ and (therefore necessarily, by Theorem 2) $\nu_n = 1$.

Theorem 4 ([7, Lemma 6.5], [10, Theorem 3.4]). *Let $n \in \mathbb{N}$ and $d \in \mathbb{N}$ be such that the nullities ν_n through ν_{n+d-1} are 1 through d , respectively, and ν_{n+d} through ν_{n+2d-1} are $d-1$ through 0, respectively. Let $z = v_0^{c_0} v_1^{c_1} \cdots v_{n-1}^{c_{n-1}}$ be the word generating \mathfrak{Z}_n ; then the exponents c_0, c_1, \dots, c_{n-1} form a palindrome, i.e., $[c_0, c_1, \dots, c_{n-1}] = [c_{n-1}, c_{n-2}, \dots, c_0]$, with $c_0 = c_{n-1} = 1$. For $m = 0, 1, \dots, d-1$, the center \mathfrak{Z}_{n+m} is generated by the words $z, \alpha(z), \dots, \alpha^m(z)$. \mathfrak{Z}_{n+d} is generated by $\alpha(z), \dots, \alpha^{d-1}(z)$, \mathfrak{Z}_{n+d+1} by $\alpha^2(z), \dots, \alpha^{d-1}(z)$, and so on, and \mathfrak{Z}_{n+2d-1} is trivial.*

2. MAIN RESULT

We now construct countably many binary shifts that are cocycle conjugate to a given binary shift α . Fix $n \in \mathbb{N}$ such that $\nu_{n-1} = 0$ and $\nu_n = 1$. As above, let $z = v_0^{c_0} v_1^{c_1} \cdots v_{n-1}^{c_{n-1}}$ be the word generating \mathfrak{Z}_n , the center of the algebra \mathfrak{A}_n generated by v_0, v_1, \dots, v_{n-1} . By Theorem 4, the exponents of z form a palindrome. Define a unitary operator u by

$$u = \begin{cases} \frac{I+z}{\sqrt{2}}, & \text{if } z^* = -z, \\ \frac{I+iz}{\sqrt{2}}, & \text{if } z^* = z. \end{cases}$$

Note for any word w in the generators,

$$\text{Ad}(u)w = \begin{cases} w, & \text{if } zw = wz, \\ -zw, & \text{if } zw = -wz \text{ and } z^* = -z, \\ -izw, & \text{if } zw = -wz \text{ and } z^* = z. \end{cases}$$

Set $\beta = \text{Ad}(u) \circ \alpha, u_0 = v_0$ and for $i \in \mathbb{N}$, let $u_i = \beta^i(v_0)$. Since z commutes with v_0, v_1, \dots, v_{n-1} it follows that $u_i = v_i$ for $i = 0, 1, \dots, n - 1$.

First suppose $\nu_{n+1} = 0$. Then \mathfrak{Z}_{n+1} is trivial. Since $z \in \mathfrak{Z}_n$, and therefore commutes with v_0 through v_{n-1} , we conclude that z anticommutes with v_n (otherwise z would be in \mathfrak{Z}_{n+1} , which is false). So we have

$$(1) \quad u_n = \beta^n(v_0) = \beta(\beta^{n-1}(v_0)) = \beta(v_{n-1}) = u^*v_nu = \lambda_0zv_n$$

(where λ_0 is -1 or $-i$) and so $u_{n+1} = \beta(u_n) = u^*(\lambda_0\alpha(z)v_{n+1})u$ is either $\lambda_1z\alpha(z)v_{n+1}$ or $\lambda_1\alpha(z)v_{n+1}$ with λ_1 a fourth root of unity. Write u_{n+1} as $\lambda_1z^{r_0}\alpha(z)v_{n+1}$. Using the symmetry arising from the fact that the exponent pattern of z forms a palindrome, we conclude that since z anticommutes with v_n , then $\alpha(z)$ anticommutes with v_0 . To see this note that

$$\begin{aligned} v_nz &= v_nv_0^{c_0}v_1^{c_1} \cdots v_{n-1}^{c_{n-1}} \\ &= (-1)^{(a_n c_0)}v_0^{c_0}v_nv_1^{c_1} \cdots v_{n-1}^{c_{n-1}} \\ &= (-1)^{(a_n c_0 + a_{n-1} c_1)}v_0^{c_0}v_1^{c_1}v_nv_2^{c_2} \cdots v_{n-1}^{c_{n-1}} \\ &\vdots \\ &= (-1)^{(a_n c_0 + a_{n-1} c_1 + \cdots + a_1 c_{n-1})}v_0^{c_0}v_1^{c_1} \cdots v_{n-1}^{c_{n-1}}v_n \\ &= (-1)^{(a_n c_0 + a_{n-1} c_1 + \cdots + a_1 c_{n-1})}zv_n \end{aligned}$$

whereas, using the palindrome property for the exponents of z ,

$$\begin{aligned} v_0\alpha(z) &= v_0\alpha(v_0^{c_0}v_1^{c_1} \cdots v_{n-1}^{c_{n-1}}) \\ &= v_0\alpha(v_0^{c_n-1}v_1^{c_{n-2}} \cdots v_{n-1}^{c_0}) \\ &= v_0v_1^{c_{n-1}}v_2^{c_{n-2}} \cdots v_n^{c_0} \\ &= (-1)^{(a_1 c_{n-1} + a_2 c_{n-2} + \cdots + a_n c_0)}(v_1^{c_{n-1}}v_2^{c_{n-2}} \cdots v_n^{c_0})v_0 \\ &= (-1)^{(a_1 c_{n-1} + a_2 c_{n-2} + \cdots + a_n c_0)}\alpha(z)v_0. \end{aligned}$$

Also v_0 commutes with z . Since $u_0 = v_0$ and since $u_{n+1} = \lambda_1z^{r_0}\alpha(z)v_{n+1}$, we conclude that $b_{n+1} \neq a_{n+1}$. Therefore α and β have different bitstreams.

Finally observe that the operators $u_0, u_1, \dots, u_{n-1}, u_n, u_{n+1}, u_{n+2}, \dots$ are the operators $v_0, v_1, \dots, v_{n-1}, \lambda_0zv_n, \lambda_1z^{r_0}\alpha(z)v_{n+1}, \lambda_2z^{r_1}\alpha(z)^{r_0}\alpha^2(z)v_{n+2}, \dots$ for some scalars λ_i of modulus 1. Since $z \in \mathfrak{A}_n$ it is clear that for all $m \in \mathbb{N}$, $\mathfrak{B}_m = \mathfrak{A}_m$, where \mathfrak{B}_m is the algebra generated by u_0, \dots, u_{m-1} . So β is a binary shift (with unitary generators u_0, u_1, \dots and bitstream $(b) = b_0, b_1, \dots$) which is cocycle conjugate to α . Since α and β have different bitstreams they are not conjugate, by Theorem 1.

Next suppose the nullity sequence for α satisfies the hypotheses of Theorem 4 with $d \geq 2$. Then by the conclusion of the theorem, z , the generator of \mathfrak{Z}_n , is in \mathfrak{Z}_{n+d-1} but not \mathfrak{Z}_{n+d} . So z commutes with generators v_0 through v_{n+d-2} and anticommutes with v_{n+d-1} . Setting u as above and $\beta = \text{Ad}(u) \circ \alpha$ we see that u_0 through

u_{n+d-2} coincide with v_0 through v_{n+d-2} , but $u_{n+d-1} = \beta(u_{n+d-2}) = \beta(v_{n+d-2}) = u^*v_{n+d-1}u = \lambda_0zv_{n+d-1}$, $u_{n+d} = u^*\alpha(\lambda_0zv_{n+d-1})u = \lambda_1z^{r_0}\alpha(z)v_{n+d}$, $u_{n+d+1} = \lambda_2z^{r_1}\alpha(z)^{r_0}\alpha^2(z)v_{n+d+1}$, and so on, for scalars λ_i of modulus 1.

As $\alpha^{d-1}(z)$ is in \mathfrak{J}_{n+2d-2} , and therefore commutes with v_0 through v_{n+2d-3} , it follows that $\alpha^d(z)$ commutes with v_1 through v_{n+2d-2} . As \mathfrak{J}_{n+2d-1} is trivial, however, and as \mathfrak{A}_{n+2d-1} is generated by v_0 through v_{n+2d-2} we conclude that $\alpha^d(z)$ anticommutes with v_0 . Therefore the following statements are true:

- $u_0, u_1, \dots, u_{n+d-2}$ agree with $v_0, v_1, \dots, v_{n+d-2}$, respectively,
- u_{n+d-1} is a scalar multiple of zv_{n+d-1} ,
- u_{n+d} is a scalar multiple of $z^{s_0}\alpha(z)v_{n+d}$, for some exponent s_0 ,
- u_{n+d+1} is a scalar multiple of $z^{s_1}\alpha(z^{s_0})\alpha^2(z)v_{n+d+1}$, for some exponent s_1 ,
- \vdots
- u_{n+2d-2} is a scalar multiple of $z^{s_{d-2}} \dots \alpha^{d-2}(z)^{s_0}\alpha^{d-1}(z)v_{n+2d-2}$, and
- u_{n+2d-1} is a scalar multiple of $z^{s_{d-1}} \dots \alpha^{d-1}(z)^{s_0}\alpha^d(z)v_{n+2d-1}$.

It follows that the bitstream entries b_0 through b_{n+2d-2} coincide with a_0 through a_{n+2d-2} . Next we show that $b_{n+2d-1} \neq a_{n+2d-1}$. This follows from the fact that $u_0 (= v_0)$ commutes with $z, \alpha(z), \dots, \alpha^{d-1}(z)$ and anticommutes with $\alpha^d(z)$, by Theorem 4, along with the fact that u_{n+2d-1} is a scalar multiple of $z^{s_{d-1}} \dots \alpha^{d-2}(z^{s_1})\alpha^{d-1}(z^{s_0})\alpha^d(z)v_{n+2d-1}$. So $b_{n+2d-1} \neq a_{n+2d-1}$.

An argument similar to the one given in the original case shows that $\mathfrak{A}_m = \mathfrak{B}_m$ for all m , so β is a binary shift on R cocycle conjugate, but not conjugate, to α . So we have nearly completed the proof of the following.

Theorem 5. *Any binary shift on the hyperfinite II_1 factor R is cocycle conjugate to at least countably many others.*

End of proof. Let β_1 be the binary shift β constructed above. To construct β_2 from α , choose $n_2 \in \mathbb{N}$ such that $n_2 > n_1 = n$ and $\nu_{n_2-1} = 0, \nu_{n_2} = 1$, and mimic the construction already made for β . Note that the first place where the bitstreams for β_2 and α differ will occur past the first place where the bitstream for β_1 and α differ, so β_1 and β_2 are not conjugate. Continuing this process we can construct countably many binary shifts which are mutually non-conjugate but which are all cocycle conjugate to α .

Definition 2. For $k \in \mathbb{N}$, a binary shift α on R is said to have commutant index k if the relative commutant algebra $\alpha^k(R)' \cap R$ is non-trivial and k is the first non-negative integer for which this is the case. If $\alpha^k(R)' \cap R = \mathbb{C}I$ for all $k \in \mathbb{N}$, then we say that α has infinite commutant index.

Here are a few facts about the commutant index of a binary shift. The minimal commutant index is 2, [4], and for every $k \in \{\infty, 2, 3, \dots\}$ there are binary shifts of commutant index k [10, Section 5]. A binary shift has finite commutant index if and only if its bitstream is eventually periodic [2, Theorem 5.8]. The commutant index is a cocycle conjugacy invariant [2], [10, Theorem 5.5].

Remark 2. By [9, Corollary 4.10], all binary shifts of commutant index 2 are cocycle conjugate, so it makes sense to ask whether, given a binary shift α of index 2, the list $\{\alpha, \beta_k, k \in \mathbb{N}\}$ includes all binary shifts of commutant index 2. To see that this

is not the case, note from the construction of the β_k 's in the proof of the theorem that for every $m \in \mathbb{N}$ the algebra generated by the first m generators for α coincides with the algebra generated by the first m generators for β_k . Hence they have the same centers and therefore, by Theorem 3, the same nullity sequence. Suppose $n \in \mathbb{N}$ is the first positive integer for which $\nu_n = \text{null}(\mathcal{A}_n) = 0$. Set $c_j = 0$ for $0 \leq j \leq n-1$ and $c_n = 1$. Then by the proof of Corollary 2.13 of [10] c_0, \dots, c_n may be completed to form a bitstream which corresponds to a binary shift γ of commutant index 2. Note, however, that the corresponding Toeplitz matrix C_n is the zero matrix, with nullity n . Since $\text{null}(C_n) \neq \text{null}(A_n) = 0$, γ is not in the list $\alpha, \beta_k, k \in \mathbb{N}$.

Remark 3. Suppose α is a binary shift of finite commutant index k ; then there is a word w in the generators of α which generates the two-dimensional relative commutant algebra, $\alpha^k(R)' \cap R$, [10, Theorem 5.6]. Let R_∞ be the subfactor of R generated by $\{w, \alpha(w), \alpha^2(w), \dots\}$. Then α restricts to a binary shift on R_∞ , called the derived shift α_∞ ; see [2]. In [10] it was shown that if the nullity sequences for α and α_∞ eventually coincide, then α and α_∞ are cocycle conjugate [10, Theorem 7.12].

Problems. (1) A binary shift α is said to be of infinite commutant index if $\alpha^k(R)' \cap R = \mathbb{C}I$ for all k . Is there an α of infinite commutant index which is cocycle conjugate to uncountably many binary shifts?

(2) Are any two binary shifts of infinite commutant index cocycle conjugate?

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