PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 145, Number 5, May 2017, Pages 1917–1925
http://dx.doi.org/10.1090/proc/13356
Article electronically published on October 27, 2016

FINITE DIMENSIONAL HOPF ACTIONS ON DEFORMATION QUANTIZATIONS

PAVEL ETINGOF AND CHELSEA WALTON

(Communicated by Kailash C. Misra)

ABSTRACT. We study when a finite dimensional Hopf action on a quantum formal deformation A of a commutative domain A_0 (i.e., a deformation quantization) must factor through a group algebra. In particular, we show that this occurs when the Poisson center of the fraction field of A_0 is trivial.

1. Introduction

Throughout the paper, we will work over an algebraically closed field k of characteristic zero. Let us say that an associative algebra B has *No Finite Quantum Symmetry* (NFQS) if any action of a finite dimensional Hopf algebra H on B factors through a group algebra, and has *No Semisimple Finite Quantum Symmetry* (NSFQS) if this holds for semisimple Hopf actions. In previous papers ([CEW1, CEW2, EGMW, EW1]), we and coauthors established these properties for various classes of algebras. In particular, in [EW1] we proved the NSFQS property when $B =: A_0$ is a commutative domain.

The aim of this work is to investigate when these properties hold for Hopf actions on *quantum formal deformations* A of a commutative domain A_0. To do so, we use the Poisson structure on A_0 and on its fraction field $Q(A_0)$, which are induced by the multiplication of A. Namely, we show that if the Poisson center of $Q(A_0)$ is trivial, then the NFQS property holds. We summarize our main results in Table 1 below, along with recalling related results in the literature.

2. Preliminaries

In this section, we recall the basic terminology pertaining to deformations of k-algebras, including *quantum deformations* of commutative algebras. We also discuss localizations of such quantum deformations. The section ends with material on *inner-faithful* Hopf actions.

2.1. Deformations.

Let us introduce the following definitions.

Definition 2.1 (A, A_N). Let A_0 be an arbitrary k-algebra.

(a) A *(flat) formal deformation* of A_0 is a $k[[\hbar]]$-algebra A which is topologically free over $k[[\hbar]]$ (i.e., $A \cong A_0[[\hbar]]$ as $k[[\hbar]]$-modules) and equipped with an algebra isomorphism $A/\hbar A \cong A_0$.

Received by the editors February 21, 2016 and, in revised form, July 2, 2016.
2010 *Mathematics Subject Classification.* Primary 16T05, 16S80, 17B63, 16W70.
Key words and phrases. Deformation quantization, filtered deformation, Hopf algebra action, Poisson center.

TABLE 1. Various settings for No (Semisimple) Finite Quantum Symmetry, including our **main results** here

Property	module algebra B	$H \curvearrowright B$ preserves filtration of B?	Poisson center of $Q(A_0)$ triv.?	Reference
NSFQS	A_0 (commutative domain)	not required	not required	[EW1, Thm 1.3]
NSFQS	filtered deformation $\widetilde{A}$ of A_0	sufficient	not required	[EW1, Prop 5.4]
NFQS	$\mathbf{A}_n(k)$ (Weyl algebra)	not required	not required	[CEW2, Thm 1.1]
NSFQS	$\mathbf{A}_n(k[z_1, \ldots, z_s])$	not required	not required	[CEW1, Prop 4.3]
NFQS	$D(X)$ (algebra of diff'l ops)	not required	not required	[CEW2, Thm 1.2]
NSFQS	quantum deformation A of $A_0/_{k[[\hbar]]}$	not required	not required	**Proposition 3.1**
NFQS	quantum deformation A of $A_0/_{k[[\hbar]]}$	not required	sufficient	**Theorem 3.3**
NFQS	filtered deformation $\widetilde{A}$ of A_0	sufficient	sufficient	**Corollary 3.4**

(b) Given a nonnegative integer N, we say that a *(flat) N-th order deformation* of A_0 is a $k[\hbar]/(\hbar^{N+1})$-algebra A_N which is free as a $k[\hbar]/(\hbar^{N+1})$-module and equipped with an algebra isomorphism $A_N/\hbar A_N \cong A_0$.

(c) If, further, A_0 is a commutative k-algebra, then the not necessarily commutative algebras A and A_N above are referred to as *quantum deformations* of A_0.

Clearly, if A is a formal deformation of A_0, then $A/\hbar^{N+1}A$ is an N-th order deformation of A_0 for any $N \geq 0$, and $A = \varprojlim(A/\hbar^{N+1}A)$. Thus, formal deformations may be viewed as *deformations of infinite order*.

Given a Hopf algebra H_0, a *formal deformation* H and an *N-th order deformation* H_N of H_0 are defined similarly to Definition 2.1.

Definition 2.2 ($\widetilde{A}$)**.** Let A_0 be a graded k-algebra. A $\mathbb{Z}_+$-filtered algebra $\widetilde{A} = \bigcup_{n\geq 0} F^n\widetilde{A}$ is a *$\mathbb{Z}_+$-filtered deformation* of A_0 if we are given an isomorphism $\mathrm{gr}_F\widetilde{A} \cong A_0$ as graded k-algebras. (The algebra $\widetilde{A}$ is also called a *PBW deformation* of A_0.)

Any $\mathbb{Z}_+$-filtered deformation $\widetilde{A} = \bigcup_{n\geq 0} F^n\widetilde{A}$ of a graded algebra A_0 gives rise to its formal deformation via the Rees algebra construction.

Definition 2.3 ($R(\widetilde{A})$, $\widehat{R}(\widetilde{A})$)**.** With the notation above, the *Rees algebra* $R(\widetilde{A})$ is $\bigoplus_{n\geq 0} \hbar^n F^n\widetilde{A}$ and the *completed Rees algebra* $\widehat{R}(\widetilde{A})$ is $\prod_{n\geq 0} \hbar^n F^n\widetilde{A}$.

Clearly, $R(\widetilde{A})$ carries a grading, and is the span of the homogeneous elements of $\widehat{R}(\widetilde{A})$. Thus, $A := \widehat{R}(\widetilde{A})$ is a homogeneous formal deformation of A_0 with $\deg(\hbar) = 1$. Note also that $\widetilde{A}$ with its filtration can be recovered from $R(\widetilde{A})$ by the formula $\widetilde{A} = R(\widetilde{A})/(\hbar - 1)$. In fact, any homogeneous formal deformation A of A_0 gives rise to a $\mathbb{Z}_+$-filtered deformation via $\widetilde{A} = A_{\mathrm{hom}}/(\hbar - 1)$, where A_{hom} is the span of the homogeneous elements of A.

Now take A_0 to be a commutative k-algebra. Suppose A is a quantum N-th order deformation of A_0 for $1 \leq N \leq \infty$. Define the bilinear map $\{\ ,\ \} : A_0 \times A_0 \to A_0$ as follows: for any $a_0, b_0 \in A_0$, let $\{a_0, b_0\}$ be the image of $[a, b]$ in $\hbar A/\hbar^2 A \cong A_0$, where

a, b are any lifts of a_0, b_0 to A. (This map is well defined since A_0 is commutative.) It is well known that $\{\ ,\ \}$ is a derivation in each argument, which is a Lie bracket (i.e., a Poisson bracket) if $N \geq 2$.

Definition 2.4. Given A_0, a commutative k-algebra with Poisson structure as above, we say that the N-th order quantum deformation A of A_0 is an N-th order *deformation quantization* of the Poisson algebra $(A_0, \{\ ,\ \})$. (If we do not specify the order, then we mean that $N = \infty$.)

Example 2.5. (1) Take $A_0 = k[x, y]$ with Poisson bracket $\{y, x\} = 1$. Then, the Weyl algebra $\mathbf{A}_1(k) = k\langle x, y\rangle/(yx - xy - 1)$ is a filtered deformation of A_0 (with $\deg(x) = 0$, $\deg(y) = 1$), and gives rise to the quantum formal deformation $A = k[x, y][[\hbar]]$ of A_0 with multiplication defined by the Moyal formula

$$f * g = \sum_{i \geq 0} \frac{\hbar^i}{i!} \partial_y^i f \cdot \partial_x^i g.$$

(2) Take $A_0 = k[x_1, \ldots, x_n]$ with $\{x_i, x_j\} = \lambda_{ij} x_i x_j$, $\lambda_{ij} \in k$. Let $q_{ij} \in 1 + \hbar\lambda_{ij} + O(\hbar^2) \in k[[\hbar]]$, with $q_{ij} q_{ji} = 1$. Then, the $\hbar$-adically completed quantum polynomial algebra A generated by $x_1, \ldots, x_n$ with relations $x_i x_j = q_{ij} x_j x_i$ is a quantum formal deformation of A_0.
(3) Take a Lie algebra $\mathfrak{g}$ and let A_0 be the symmetric algebra $S(\mathfrak{g})$, with $\{x, y\} = [x, y]_{\mathfrak{g}}$ for $x, y \in \mathfrak{g}$. Then, the enveloping algebra $U(\mathfrak{g})$ is a $\mathbb{Z}_+$-filtered deformation of A_0.
(4) Let X be an abelian variety over k, $\mathcal{L}$ be an ample line bundle on X, and $\sigma \in \mathrm{Aut}(X(k[[\hbar]]))$ be such that $\sigma = \mathrm{id} \bmod \hbar$. Define the line bundles $\mathcal{L}_n := \mathcal{L} \otimes \mathcal{L}^{\sigma} \otimes \cdots \otimes \mathcal{L}^{\sigma^{n-1}}$ on X (with $\mathcal{L}_0 := \mathcal{O}_X$). Take $A := B(X, \mathcal{L}, \sigma) = \widehat{\bigoplus}_{n \geq 0} H^0(X, \mathcal{L}_n)$, the $\hbar$-adically completed twisted homogeneous coordinate ring of X ([ATV]). Given an ample line bundle $\mathcal{E}$ on X, we have that $\dim H^0(X, \mathcal{E})$ equals the Euler characteristic of $\mathcal{E}$, and hence is deformation-invariant. Therefore, A is a torsion-free, separated, and $\hbar$-adically complete $k[[\hbar]]$-module such that $A/\hbar A = A_0$, i.e., $A \cong A_0[[\hbar]]$ as a $k[[\hbar]]$-module (since a similar statement holds for every homogeneous component of A). Therefore, A is a quantum formal deformation of a homogeneous coordinate ring $A_0 := \bigoplus_{n \geq 0} H^0(X, \mathcal{L}^{\otimes n})$.

2.2. **Localization of quantum deformations.**

Lemma 2.6. *Let A_0 be a commutative domain, and let A_N be an N-th order quantum deformation of A_0, for $N < \infty$. Take S to be the set of all regular elements of A_N (i.e., $S = A_N \setminus \hbar A_N$). Then,*

1. *there exists the classical quotient ring $Q(A_N) = S^{-1} A_N$,*
2. *$Q(A_N)$ is an N-th order deformation of the quotient field $Q(A_0)$, and*
3. *$Q(A_N)$ is both left and right Artinian.*

Proof. To prove (1), we show that S satisfies both the right and left Ore conditions. Let $a \in A_N$ and $s \in S$. Note that $\mathrm{ad}(s)(a) \in \hbar A$, and so $\mathrm{ad}(s)^{N+1} a = 0$. Hence,

$$s^{N+1} a = \left(\sum_{j=0}^{N} s^{N-j} \mathrm{ad}(s)^j(a)\right) s,$$

and S satisfies the left Ore condition. The right Ore condition is proved similarly. Now (1) follows from Ore's theorem.

Part (2) follows easily from (1), and (3) follows immediately from (2). □

Now let A be a quantum formal deformation of A_0 (i.e., a deformation of infinite order). Define

$$Q(A) := \varprojlim Q(A/\hbar^{N+1}A).$$

Example 2.7. If A_0 is a field, then $Q(A) = A$ since all elements not in $\hbar A$ are already invertible. Therefore, $A[\hbar^{-1}]$ is a division algebra.

2.3. **Inner-faithful Hopf actions.** Recall that a Hopf algebra H *acts on an algebra* B (*from the left*) if B is a (left) H-module algebra, or equivalently, if B is an algebra object in the category of (left) H-modules.

Definition 2.8. We say that an action of a Hopf algebra H on an algebra B is *inner-faithful* if there does not exist a nonzero Hopf ideal of H that annihilates the H-module B.

One can always pass to an inner-faithful Hopf action by considering an action of a quotient Hopf algebra.

We will need the following auxiliary result; the standard proofs are omitted.

Lemma 2.9. *Let H be a finite dimensional Hopf algebra.*

(1) *Suppose that H acts on a $\mathbb{Z}_+$-filtered algebra $\widetilde{A} = \bigcup_{n\geq 0} F^n\widetilde{A}$ so that $F_n\widetilde{A}$ is H-stable for all $n \geq 0$. Then, there is an induced H-module algebra structure on $\mathrm{gr}_F\widetilde{A}$ given by $h \cdot \overline{a} = \overline{(h\cdot a)}_n$ where $a \in F^n\widetilde{A}$ is any lift of $\overline{a} \in F^n\widetilde{A}/F^{n-1}\widetilde{A}$. Also, there is an induced H-action on the Rees algebra $R(\widetilde{A})$ and the completed Rees algebra $\widehat{R}(\widetilde{A})$ so that $\hbar^n F^n\widetilde{A}$ is H-stable for all $n \geq 0$; this action is inner-faithful if and only if the given H-action on $\widetilde{A}$ is inner-faithful.*

(2) *Suppose H acts on a formal deformation A of an algebra A_0. If the action of H on A_0 is inner-faithful, then so is the H-action on A. The converse holds if H is semisimple.*

Proof. We will only prove (2). If $I \subset H$ is a Hopf ideal annihilating A, then it clearly annihilates A_0, implying the forward direction. The converse follows from the following standard fact: if H is a semisimple algebra and V a formal deformation of an H-module V_0, then V is isomorphic to $V_0[[\hbar]]$ as an H-module. □

Remark 2.10. The converse in Lemma 2.9(2) may fail if H is not semisimple, as shown by [CWWZ, Example 3.2(d)].

3. The main results

In this section we present the main results, including the results highlighted in Table 1, along with Theorem 3.2 which is needed for the proof of Theorem 3.3. The proof of Theorem 3.2 is postponed to the next section.

First, we obtain the following generalization of [EW1, Proposition 5.4].

Proposition 3.1. *If H_0 is a semisimple Hopf algebra and A_0 is a commutative domain, then the action of H_0 on a quantum formal deformation A of A_0 factors through a group action.*

Proof. Without loss of generality, we may assume that the H_0-action on A is inner-faithful. Since H_0 is semisimple, by Lemma 2.9(2) the induced action of H_0 on A_0 is inner-faithful. Hence, H_0 is a finite group algebra by [EW1, Theorem 1.3]. □

We would like to generalize this result to the case when H_0 is not necessarily semisimple and, still more generally, to the case when we have an action of a formal deformation H of a finite dimensional Hopf algebra H_0. In this case, *nontrivial* actions of H_0 on a commutative domain A_0 (that is, ones not factoring through a group action) are possible; see e.g., [EW2]. We want to see when these actions can lift to actions of H on A.

Recall that A_0 carries a Poisson bracket induced by the deformation A, and by virtue of being a biderivation, this bracket extends uniquely to the quotient field $Q(A_0)$. The following theorem shows that a nontrivial action of H_0 on A_0 cannot lift if the induced Poisson bracket on the fraction field $Q(A_0)$ has trivial center; the proof is presented in Section 4.

Theorem 3.2. *Let H be a formal deformation of a finite dimensional Hopf algebra H_0 which acts on a quantum formal deformation A of a commutative domain A_0. If the Poisson center of $Q(A_0)$ is trivial (i.e., $\{f, g\} = 0$ for all $g \in Q(A_0)$ implies $f \in k$), then the induced action of H_0 on A_0 factors through a group action.*

Using Theorem 3.2, we prove our main result, which is the following theorem.

Theorem 3.3. *Let H_0 be a finite dimensional Hopf algebra which acts on a quantum formal deformation A of a commutative domain A_0. If the Poisson center of $Q(A_0)$ is trivial, then the action of H_0 on A factors through a group action.*

Proof. Without loss of generality, we may assume that the action of H_0 on A is inner-faithful.

Let I be the annihilator of A_0 as an H_0-module, i.e., the set of $x \in H_0$ such that $xA \subset \hbar A$. The action of $H := H_0[[\hbar]]$ (the trivial deformation) on A satisfies the assumptions of Theorem 3.2. Thus, by Theorem 3.2, the action of H_0 on A_0 factors through a group algebra; in other words, $H_0/I = kG$ for some finite group G. In particular, I is a Hopf ideal. Then, $I^\infty := \bigcap_{m\geq 0} I^m$ is a Hopf ideal in H_0 acting trivially on A. So $I^\infty = 0$ by inner-faithfulness. Hence, there is $r > 0$ such that $I^r = 0$; let us take the smallest such r. Since I is a nilpotent ideal and H_0/I is semisimple, we get that $I = \mathrm{Rad}(H_0)$. So the radical of H_0 is a Hopf ideal.

Our job is to show that I acts by zero on A (then it would follow that $H_0 = kG$). Assume the contrary. Let s be the largest integer such that $IA \subset \hbar^s A$ (it exists since we have assumed that $IA \neq 0$). Consider $H' := \sum_{m=0}^{r-1} \hbar^{-ms} I^m[[\hbar]] \subset H[\hbar^{-1}]$ (where $I^0 = H_0$); it is the Rees algebra of H_0 with respect to the decreasing filtration by powers of I, with $\deg(I) = s$. Since I is a Hopf ideal, we have $\Delta(I) \subset H \otimes I + I \otimes H$. Hence

$$\Delta(\hbar^{-ms} I^m) \subset \sum_{p+q=m} (\hbar^{-mp} I^p) \otimes (\hbar^{-mq} I^q),$$

so $\Delta(H') \subset H' \otimes H'$, and we obtain that H' is a Hopf algebra. Furthermore, H' is a formal deformation of the Hopf algebra $\mathrm{gr}H_0 := \bigoplus_{m=0}^{r-1} I^m/I^{m+1}$, the associated graded algebra of H_0 under the radical filtration (which, in this case, is a Hopf algebra filtration, as $\mathrm{Rad}(H_0)$ is a Hopf ideal of H_0). Moreover, by definition H' acts on A. Hence $\mathrm{gr}H_0$ acts on A_0 by reducing modulo $\hbar$.

By Theorem 3.2, the action of $\mathrm{gr}H_0$ on A_0 must factor through a group algebra. In particular, the radical $\mathrm{gr}I$ (which is a Hopf ideal of $\mathrm{gr}H_0$) acts by zero on A_0.

On the other hand, by our assumption, there exists $x \in I$ and $a \in A$ such that $xa = \hbar^s b$, where b has a nonzero image b_0 in A_0. Then, $(\hbar^{-s}x)a = b$. So, denoting by x_0 the image of $\hbar^{-s}x \in \hbar^{-s}I \subset H'$ in $\mathrm{gr}I \subset \mathrm{gr}H_0$, and denoting by a_0 the image of a in A_0, we obtain $x_0 a_0 = b_0 \neq 0$. This means that $\mathrm{gr}I$ acts by nonzero on A_0, a contradiction. The theorem is proved. $\square$

Corollary 3.4. *Let $\widetilde{A}$ be a $\mathbb{Z}_+$-filtered algebra such that $A_0 = \mathrm{gr}\widetilde{A}$ is a commutative domain. Suppose that a finite dimensional Hopf algebra H acts on $\widetilde{A}$ preserving the filtration of $\widetilde{A}$. If the Poisson center of $Q(A_0)$ is trivial, then the action of H factors through a group action.*

Proof. Without loss of generality, we assume that H acts on $\widetilde{A}$ inner-faithfully. Since the H-action on $\widetilde{A}$ preserves the filtration of $\widetilde{A}$, it extends to an inner-faithful H-action on the completed Rees algebra $\widehat{R}(\widetilde{A})$ by Lemma 2.9(1). Now H is a finite group algebra by Theorem 3.3. $\square$

Remark 3.5. Suppose that A_0 is a finitely generated commutative domain, that is, $A_0 = \mathcal{O}(X)$, the algebra of regular functions on some irreducible affine variety X over k. Then, the condition that the Poisson center of $Q(A_0) = k(X)$ is trivial holds, in particular, when the induced Poisson bracket on X is generically symplectic (i.e., there exists a dense smooth affine open set $U \subset X$ and a closed nondegenerate 2-form ω on U such that $\{f, g\} = (df \otimes dg, \omega^{-1})$ for any $f, g \in \mathcal{O}(X)$). For example, one may take X to be any affine symplectic variety, and A a deformation quantization of $\mathcal{O}(X)$ (e.g., Fedosov's quantization); see [BK].

Example 3.6. The condition in Theorem 3.2 and Theorem 3.3 that the Poisson center of $Q(A_0)$ is trivial cannot be replaced by a weaker condition that the Poisson center of A_0 is trivial. For example, consider the quantum polynomial algebra A with generators x, y, z and relations $xy = qyx$, $xz = qzx$, $zy = qyz$, where $q = \exp(\hbar)$. Then, the induced Poisson bracket on $A_0 = k[x, y, z]$ is given by $\{x, y\} = xy$, $\{z, y\} = yz$, $\{x, z\} = xz$, and it is easy to see that the Poisson center of A_0 is trivial. On the other hand, the Poisson center of $Q(A_0)$ contains the element xy/z.

Let H_0 be the Sweedler Hopf algebra with grouplike generator g such that $g^2 = 1$ and $(1, g)$-skew-primitive generator a such that $ga = -ag$ and $a^2 = 0$. Define an action of H_0 on A by

$$g \cdot x = x, \quad g \cdot y = y, \quad g \cdot z = -z, \quad a \cdot x = 0, \quad a \cdot y = 0, \quad a \cdot z = xy.$$

It is easy to check that this action is well defined, and does not factor through a group algebra, even after reducing modulo $\hbar$.

4. Proof of Theorem 3.2

Since H acts on A, it acts on $A/\hbar^{N+1}A$ for any N. Hence, H acts on the classical quotient ring $Q(A/\hbar^{N+1}A)$ by [SV, Theorem 2.2], and by taking the inverse limit in N, we get an action of H on $Q(A)$. Thus, without loss of generality we may assume that A_0 is a field.

One of the main steps of the proof is to show that many invariants in $A_0^{H_0}$ lift to invariants in A^H. Namely, let us say that an element $a_0 \in A_0^{H_0}$ is a *liftable invariant* if there exists $a \in A^H$ equal to a_0 modulo $\hbar$.

Notation (*K*). Let $K \subset A_0$ be the subset (in fact, subfield) of liftable invariants under the action of H_0.

Lemma 4.1. *The field A_0 is an algebraic extension of K.*

Proof. Let $d := \dim H_0 = \dim_{k((\hbar))} H[\hbar^{-1}]$. Let $D := A[\hbar^{-1}]$, which is a division algebra over $k((\hbar))$ by Example 2.7. Further, $H[\hbar^{-1}]$ acts $k((\hbar))$-linearly on D. Thus, by [BCF, Corollary 2.3], D has dimension $\le d$ over $D^{H[\hbar^{-1}]}$ as a left vector space. Now let $x_0 \in A_0$ and $x \in A$ be its lift to A. As $[D : D^{H[\hbar^{-1}]}] \le d$, we have that x satisfies an equation

$$b_0x^n + b_1x^{n-1} + \cdots + b_n = 0, \tag{1}$$

where $b_0 = 1$, $b_i \in D^{H[\hbar^{-1}]}$ and $n \le d$. Let m be the smallest value of the $\hbar$-adic valuation of b_i in D (over all i); clearly, $m \le 0$. Projecting (1) to $\hbar^m A/\hbar^{m+1}A$, we get a nontrivial equation

$$c_0x_0^s + c_1x_0^{s-1} + \cdots + c_s = 0 \tag{2}$$

of possibly lower degree $s \le n$. Note that $c_i \in K$ by definition, so x_0 is algebraic over K. □

Now we proceed with the proof of Theorem 3.2. Consider the Galois map

$$\beta : A_0 \otimes A_0 \to A_0 \otimes H_0^*, \quad f \otimes g \mapsto (f \otimes 1)\rho(g),$$

where $\rho : A_0 \to A_0 \otimes H_0^*$ is the coaction map. Then,

$$B := \mathrm{Im}\beta$$

is a commutative coideal subalgebra in the Hopf algebra $A_0 \otimes H_0^*$ (regarded as a finite dimensional Hopf algebra over A_0); the commutativity is clear and the coideal subalgebra condition follows from an argument similar to [EW1, Lemma 3.2]. Moreover, by [CEW2, Lemma 3.3] it suffices to show that

(†) B is defined over k, that is, $B = A_0 \otimes B_0$, where B_0 is a subalgebra of H_0^*.

Let $\{h_i\}$ be a basis of H_0, and let $\{h_i^*\}$ be the dual basis of H_0^*. Then for $f \in A_0$

$$\rho(f) = \textstyle\sum_{i=1}^d \rho_i(f) \otimes h_i^*,$$

where $\rho_i : A_0 \to A_0$.

Lemma 4.2. *Suppose $a_0 \in K$ is a liftable invariant. Then for any $f_0 \in A_0$ and all i, one has*

$$\rho_i(\{a_0, f_0\}) = \{a_0, \rho_i(f_0)\}.$$

Proof. Let us fix an isomorphism $H \cong H_0[[\hbar]]$ as $k[[\hbar]]$-modules, and by abusing notation, denote the coaction of H^* on A also by ρ and its components by ρ_i. Let a be a lift of a_0 to A^H, and let f be a lift of f_0 to A. We have

$$\rho_i([a, f]) = [a, \rho_i(f)].$$

Projecting this equation to $\hbar A/\hbar^2 A \cong A_0$, we obtain the desired statement. □

Introduce the following notation. Let $r := \dim B$, and $v_1, \ldots, v_r$ be elements of A_0 such that $\rho(v_1), \ldots, \rho(v_r)$ are linearly independent, and hence form a basis of B over A_0. Let $h_1, \ldots, h_d$ be a basis of H_0, and let $\mathbf{B} := (b_{ij})$ be the matrix representing B in the Grassmannian $\mathrm{Gr}_r(A_0 \otimes H_0^*) =: \mathrm{Gr}_r(d)$ of r-dimensional subspaces in a d-dimensional space with respect to these bases. Namely, $\rho(v_i) = \sum_j b_{ij} \otimes h_j^*$ where $b_{ij} = \rho_j(v_i) \in A_0$.

Recall that the homogeneous coordinate ring of $\mathrm{Gr}_r(d)$ under the Plücker embedding is generated by the minors Δ_I of an r-by-d matrix attached to subsets $I \subset \{1, \dots, d\}$ with $|I| = r$. Pick I so that $\Delta_I(\mathbf{B}) \neq 0$. Let $J \subset \{1, \dots, d\}$ with $|J| = r$ be such that $|J \cap I| = r - 1$. Then, the Plücker coordinates $p_{IJ} := \Delta_J / \Delta_I$ are rational functions on $\mathrm{Gr}_r(d)$ which form a local coordinate system near $\mathbf{B}$.

Note that B is defined over k precisely when $B \in \mathrm{Gr}_r(H_0^*) \subset \mathrm{Gr}_r(A_0 \otimes H_0^*)$. So property (†) is equivalent to the property that for all J, the ratios $p_{IJ}(\mathbf{B})$ lie in k, which is what remains to be shown.

To this end, let $a_0 \in K$ be a liftable invariant. Since the vectors $\rho(v_i)$ form a basis of B, there exists an r-by-r matrix $\mathbf{C} = (c_{im})$ with $c_{im} \in A_0$, such that

$$\rho(\{a_0, v_i\}) = \textstyle\sum_m c_{im} \rho(v_m).$$

By Lemma 4.2,

$$\sum_j \{a_0, \rho_j(v_i)\} \otimes h_j^* = \sum_{m,j} c_{im} \rho_j(v_m) \otimes h_j^*.$$

So,

$$\{a_0, b_{ij}\} = \textstyle\sum_m c_{im} b_{mj}.$$

This implies that $\{a_0, \Delta_I(\mathbf{B})\} = \mathrm{Tr}(\mathbf{C}) \Delta_I(\mathbf{B})$, and thus

$$\{a_0, p_{IJ}(\mathbf{B})\} = \frac{1}{\Delta_I(\mathbf{B})^2} \Big(\Delta_I(\mathbf{B}) \{a_0, \Delta_J(\mathbf{B})\} - \Delta_J(\mathbf{B}) \{a_0, \Delta_I(\mathbf{B})\} \Big) = 0. \tag{3}$$

Now by Lemma 4.1, any $f \in A_0$ satisfies an equation $c_0 f^s + c_1 f^{s-1} + \cdots + c_0 = 0$ for some $c_i \in K$, with s minimal. Since the Poisson bracket is a biderivation, we have

$$0 = \{\textstyle\sum_{i=0}^s c_{s-i} f^i, p_{IJ}(\mathbf{B})\} \overset{(3)}{=} \left(\textstyle\sum_{i=1}^s i c_{s-i} f^{i-1}\right) \{f, p_{IJ}(\mathbf{B})\}.$$

Since s is minimal, $\sum_{i=1}^s i c_{s-i} f^{i-1} \neq 0$. This implies that $\{f, p_{IJ}(\mathbf{B})\} = 0$ for any $f \in A_0$. Finally, since the Poisson center of A_0 is trivial, we obtain that $p_{IJ}(\mathbf{B}) \in k$. Theorem 3.2 is proved.

Remark 4.3. One can generalize the main results of this article by replacing the induced Poisson bracket on A_0 with the *induced Poisson bracket of depth* m as follows.

Let A be a noncommutative formal deformation of A_0, and let m be the largest integer such that $[a, b] \in \hbar^m A$ for all $a, b \in A$. Given $a_0, b_0 \in A_0$, pick lifts a, b of a_0, b_0 to A, and consider the projection $\{a_0, b_0\}$ of $[a, b]$ to $\hbar^m A / \hbar^{m+1} A$. Then, it is well known that $\{\ ,\}$ is a nonzero Poisson bracket for A_0; let us call it the *induced Poisson bracket of depth* m. The same construction applies to filtered deformations, by passing to the completed Rees algebra.

This generalizes the above setting, in which $m = 1$. More precisely, the usual induced Poisson bracket is the bracket of depth 1. If it turns out to be zero, then we can define the Poisson bracket of depth 2. If it also turns out to be zero, then we can define a Poisson bracket of depth 3, and so on, until we reach some depth m where the bracket is nonzero (which will necessarily happen if A is noncommutative).

Now Theorem 3.2, Theorem 3.3, and Corollary 3.4 generalize to this setting in a straightforward fashion, with the same proofs. In other words, if the Poisson center of $Q(A_0)$ with respect to a Poisson bracket of any depth m is trivial, then the appropriate Hopf action must factor through a group action.

Acknowledgments

The authors were supported by the National Science Foundation: NSF-grants DMS-1502244 and DMS-1550306.

References

[ATV] M. Artin, J. Tate, and M. Van den Bergh, *Some algebras associated to automorphisms of elliptic curves*, The Grothendieck Festschrift, Vol. I, Progr. Math., vol. 86, Birkhäuser Boston, Boston, MA, 1990, pp. 33–85. MR1086882

[BK] R. Bezrukavnikov and D. Kaledin, *Fedosov quantization in algebraic context* (English, with English and Russian summaries), Mosc. Math. J. **4** (2004), no. 3, 559–592, 782. MR2119140

[BCF] Jeffrey Bergen, Miriam Cohen, and Davida Fischman, *Irreducible actions and faithful actions of Hopf algebras*, Israel J. Math. **72** (1990), no. 1-2, 5–18, DOI 10.1007/BF02764609. Hopf algebras. MR1098978

[CWWZ] K. Chan, C. Walton, Y. H. Wang, and J. J. Zhang, *Hopf actions on filtered regular algebras*, J. Algebra **397** (2014), 68–90, DOI 10.1016/j.jalgebra.2013.09.002. MR3119216

[CEW1] Juan Cuadra, Pavel Etingof, and Chelsea Walton, *Semisimple Hopf actions on Weyl algebras*, Adv. Math. **282** (2015), 47–55, DOI 10.1016/j.aim.2015.05.014. MR3374522

[CEW2] J. Cuadra, P. Etingof, and C. Walton, *Finite dimensional Hopf actions on Weyl algebras*, arXiv:1509.01165.

[EGMW] P. Etingof, D. Goswami, A. Mandal, and C. Walton, *Hopf coactions on commutative algebras generated by a quadratically independent comodule*, arXiv:1507.08486.

[EW1] Pavel Etingof and Chelsea Walton, *Semisimple Hopf actions on commutative domains*, Adv. Math. **251** (2014), 47–61, DOI 10.1016/j.aim.2013.10.008. MR3130334

[EW2] Pavel Etingof and Chelsea Walton, *Pointed Hopf actions on fields, I*, Transform. Groups **20** (2015), no. 4, 985–1013, DOI 10.1007/s00031-015-9328-7. MR3416436

[SV] Serge Skryabin and Freddy Van Oystaeyen, *The Goldie Theorem for H-semiprime algebras*, J. Algebra **305** (2006), no. 1, 292–320, DOI 10.1016/j.jalgebra.2006.06.030. MR2264132

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
E-mail address: etingof@math.mit.edu

Department of Mathematics, Temple University, Philadelphia, Pennsylvania 19122
E-mail address: notlaw@temple.edu