

## A NON-RESTRICTED COUNTEREXAMPLE TO THE FIRST KAC–WEISFEILER CONJECTURE

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ABSTRACT. In 1971 Kac and Weisfeiler made two important conjectures regarding the representation theory of restricted Lie algebras over fields of positive characteristic. The first of these predicts the maximal dimension of the simple modules, and can be stated without the hypothesis that the Lie algebra is restricted. In this short article we construct the first example of a non-restricted Lie algebra for which the prediction of the first Kac–Weisfeiler conjecture fails. Our method is to present pairs of Lie algebras which have isomorphic enveloping algebras but distinct indexes.

### 1. INTRODUCTION

Let  $\mathbb{k}$  be an algebraically closed field of positive characteristic  $p$  and  $L$  be a finite dimensional Lie algebra over  $\mathbb{k}$ . It is well known that all simple modules have finite dimension and that the dimensions are uniformly bounded above by some integer. We denote by  $M(L)$  the least upper bound of dimensions of simple  $L$ -modules. The coadjoint module is denoted  $L^*$  and the stabiliser of  $\chi \in L^*$  is denoted  $L_\chi := \{x \in L : \chi[x, L] = 0\}$ . In recent years it has become apparent that when  $L$  is restricted, the number  $\dim L_\chi$  often plays a role in bounding the dimensions of  $L$ -modules with  $p$ -character  $\chi$ . Therefore we may expect the index of  $L$  to be an important global invariant of  $L$ -mod, where index is defined

$$\text{ind } L := \min_{\chi \in L^*} \dim L_\chi.$$

For each  $\chi \in L^*$  the form  $X, Y \mapsto \chi[X, Y]$  is non-degenerate and skew on  $L/L_\chi$  and so the number  $\dim L - \text{ind } L$  is always even. The first Kac–Weisfeiler conjecture (KW1) predicts that the index of a restricted Lie algebra  $L$  is involved in the representation theory in the following way:

$$(1.1) \quad M(L) = p^{\frac{1}{2}(\dim L - \text{ind } L)}$$

[5, §1.2]. The conjecture is striking for both its simplicity and its generality, and has attracted much attention over the past 45 years. Since the statement may be phrased without the hypothesis that  $L$  is restricted, there has been some small hope that it may hold in general. In this paper we shall show that for certain non-restricted Lie algebras (1.1) fails. These are the very first examples of this kind in the literature.

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For a given Lie algebra  $L$  the problem of calculating  $\text{ind } L$  belongs to the realm of elementary linear algebra and the meat of the KW1 conjecture lies in computing  $M(L)$ . There is no procedure for determining this invariant in general, and practically nothing is known about representations of Lie algebras which are not restricted, which is undoubtedly why it has taken so long for (1.1) to be refuted for non-restricted algebras. The most general result appears in [4, Thm. 4.4] where it is shown that if a restricted Lie algebra  $L$  admits a  $\chi \in L^*$  such that  $L_\chi$  is a torus, then KW1 holds for  $L$ .

Over the past 10 years, various authors have been studying the isomorphism problem for enveloping algebras (see [2] for example). In its most general form, the question is: *can two non-isomorphic Lie algebras admit isomorphic enveloping algebras?* For finite dimensional Lie algebras over fields of characteristic zero there are no known examples of this pathological behaviour, however in characteristic  $p$  such algebras are not hard to construct (we shall see new examples of this phenomenon in Proposition 2). Several weaker variants of the isomorphism problem have been considered, asking which properties are shared by Lie algebras  $L$  and  $L'$  such that  $U(L) \cong U(L')$ , for instance nilpotence, solvability, derived length. The key observation of this article is that (1.1) implies a weak variant of the isomorphism problem: if (1.1) holds for all  $\mathbb{k}$ -Lie algebras and  $U(L) \cong U(L')$ , then  $\text{ind } L = \text{ind } L'$ . This is simply because both  $M(L)$  and  $\dim L$  depend only upon the isomorphism class of  $U(L)$ ; in the language of [2] we would say that  $\text{ind } L$  is determined by  $U(L)$ . Our method is to disprove this corollary of (1.1) by exhibiting two Lie algebras with isomorphic enveloping algebras but distinct indexes.

For any set  $X$  we use the notation  $\langle X \rangle$  to denote the vector space spanned by  $X$ . We now describe a family of examples for which (1.1) fails. Let  $k \geq 3$  and let  $L$  be the Lie algebra  $\langle x_1, x_2, \dots, x_k, D_0, D \rangle$  such that

$$(1.2) \quad \begin{aligned} D_0 &\text{ is central,} \\ \langle x_1, \dots, x_k \rangle &\text{ is abelian,} \\ [D, x_i] &= x_i \text{ for } i = 1, \dots, k-2, \\ [D, x_{k-1}] &= x_k, \\ [D, x_k] &= 0. \end{aligned}$$

In this article we shall prove the following result:

**Theorem.** *Let  $L$  denote the Lie algebra with generators  $x_1, \dots, x_k, D_0, D$  and relations (1.2). We have  $p^2 | M(L)$  and  $\text{ind } L = k$  so that  $M(L) \neq p^{\frac{1}{2}(\dim L - \text{ind } L)}$ .*

Note that the example above is not restrictable, since  $\text{ad}(D)^p \notin \text{ad}(L)$ .

**Question.** i) Does there exist a restricted Lie algebra for which KW1 fails?  
ii) Do there exist two restricted Lie algebras  $L$  and  $L'$  with  $U(L) \cong U(L')$  and  $\text{ind } L \neq \text{ind } L'$ ?

A positive answer to question (ii) would imply a positive answer to question (i), whilst a negative answer to (ii) would offer supporting evidence for the KW1 conjecture, as well as having independent value in the context of the isomorphism problem.

## 2. LIE ALGEBRAS WITH ISOMORPHIC ENVELOPING ALGEBRAS

In this section we prove a basic result which allows us to construct families of Lie algebras which have isomorphic enveloping algebras. For any Lie algebra  $L$  we

consider the restricted closure  $\bar{L}$  of  $\text{ad}(L)$  inside  $\text{Der}(L)$ , i.e. the smallest restricted subalgebra of  $\text{Der}(L)$  containing  $\text{ad}(L)$ .

**Lemma 1.** *Every element of  $\bar{L}$  is of the form  $\sum_{i=0}^k \text{ad}(X_i)^{p^i}$  for some  $k \geq 0$  and elements  $X_1, \dots, X_k \in L$ .*

*Proof.* We start by showing that for each  $k \geq 0$  the sum  $\sum_{i=0}^k \text{ad}(L)^{p^i}$  is a vector space. The case  $k = 0$  is obvious so we may proceed by induction. Using the formulas derived in Chapter 2 of [1] we have  $\text{ad}(X + Y)^{p^k} = \text{ad}(X)^{p^k} + \text{ad}(Y)^{p^k} \pmod{\text{ad}(L)}$  and so

$$\sum_{i=0}^k (\text{ad}(X_i)^{p^i} + \text{ad}(Y_i)^{p^i}) = \sum_{i=0}^{k-1} (\text{ad}(X_i)^{p^i} + \text{ad}(Y_i)^{p^i}) + \text{ad}(X_k + Y_k)^{p^k} \pmod{\text{ad}(L)}$$

and so by induction  $\sum_{i \geq 0} \text{ad}(L)^{p^i}$  is a vector space.

If  $X_1 = \text{ad}(X)$  and  $Y_1 = \text{ad}(Y)$ , then  $[X_1^{p^k}, Y_1^{p^j}] = -\text{ad}(X_1)^{p^{k-1}}(\text{ad}(Y_1)^{p^j} X_1)$  and so  $\sum_{i \geq 0} \text{ad}(L)^{p^i}$  is closed under the bracket. Using the same formulas mentioned in the first paragraph of the proof it is clear that  $\sum_{i \geq 0} \text{ad}(L)^{p^i}$  is closed under taking  $p$ th powers, and so it is a restricted algebra containing  $\text{ad}(L)$ . It is easy to see that it is the smallest such algebra.  $\square$

For  $D \in \text{Der}(L)$  we write  $L_D$  for the semidirect product  $L \rtimes \mathbb{k}D$ .

**Proposition 2.** *For every  $D, D' \in \bar{L}$  we have  $U(L_D) \cong U(L_{D'})$ .*

*Proof.* Write  $L_{D_0} := L \rtimes \mathbb{k}D_0$  where  $D_0$  is central in  $L_{D_0}$ . We shall show that  $U(L_D) \cong U(L_{D_0})$  for every  $D \in \bar{L}$ . According to the previous lemma we can write  $D = \sum_{i=0}^k \text{ad}(X_i)^{p^i}$  for some  $k \geq 0$  and elements  $X_1, \dots, X_k \in L$ .

We define a linear map

$$\begin{aligned} \phi & : L_D \hookrightarrow U(L_{D_0}); \\ L & \xrightarrow{\text{Id}} L; \\ D & \longmapsto \sum_{i=0}^k X_i^{p^i} + D_0. \end{aligned}$$

By construction  $\phi[X, Y] = \phi(X)\phi(Y) - \phi(Y)\phi(X)$  for all  $X, Y \in L_D$ . Furthermore, every element of  $L_{D_0} \subseteq U(L_{D_0})$  lies in the algebra generated by the image and so, by the universal property of the enveloping algebra there is a surjective algebra homomorphism  $\Phi : U(L_D) \rightarrow U(L_{D_0})$ .

To see that the map is injective we appeal to the graded algebra, as follows. Suppose that  $I = \text{Ker}\Phi$  is a non-zero ideal of  $U(L_D)$ . Then  $U(L_D)/I \cong U(L_{D_0})$  and, in particular, their Gelfand–Kirillov dimensions coincide. By [3, Prop. 8.1.15(iii)] we have

$$\dim L = \text{GKdim } U(L_{D_0}) = \text{GKdim } (U(L_D)/I).$$

The PBW filtration on  $U(L_D)$  induces a filtration on  $U(L_D)/I$  and, according to Proposition 7.6.13 of op. cit. we have

$$\text{gr}(U(L_D)/I) \cong \text{gr } U(L_D)/\text{gr } I \cong S(L_D)/\text{gr } I.$$

Now Proposition 8.1.14 in op. cit. tells us that

$$\text{GKdim } (U(L_D)/I) = \text{GKdim } (S(L_D)/\text{gr } I).$$

Since  $S(L_D)/\text{gr } I$  is a commutative affine algebra, Theorem 8.2.14(i) in that same book tells us that  $\text{GKdim}(S(L_D)/\text{gr } I)$  is equal to the Krull dimension of  $S(L_D)/\text{gr } I$ , which is necessarily less than

$$\text{GKdim } S(L_D) = \dim L_D = \dim L_{D_0},$$

since  $S(L_D)/\text{gr } I$  is a proper quotient. This contradiction tells us that  $I = 0$  as desired. □

### 3. CALCULATING INDEXES

We continue to let  $\mathbb{k}$  have characteristic  $p > 0$  and pick  $k \geq 3$ . Let  $\mathcal{A} = \langle x_1, \dots, x_k \rangle$  be an abelian Lie algebra and define a derivation of  $\mathcal{A}$  by  $D(x_i) = x_i$  for all  $i = 1, \dots, k - 2$ ,  $D(x_{k-1}) = x_k$ ,  $D(x_k) = 0$ . We consider the semidirect product  $\mathcal{A}_D$ . Now denote by  $D_0$  the zero derivation of  $\mathcal{A}_D$ , and observe that  $D^p$  is a derivation of  $\mathcal{A}_D$ . Define

$$(3.1) \quad \begin{aligned} L &:= (\mathcal{A}_D)_{D_0}, \\ L' &:= (\mathcal{A}_D)_{D^p}. \end{aligned}$$

According to Proposition 2 we have  $U(L) \cong U(L')$ .

**Lemma 3.** *We have*

$$\begin{aligned} \text{ind } L &= k; \\ \text{ind } L' &\leq k - 2. \end{aligned}$$

*Proof.* Pick  $\chi \in L^*$  and observe that  $[L, L] = \langle x_1x_2, \dots, x_{k-2}, x_k \rangle$ , which implies that  $L_\chi$  is completely determined by  $(\chi(x_1), \chi(x_2), \dots, \chi(x_{k-2}), \chi(x_k)) \in \mathbb{k}^{k-1}$ . Choosing scalars  $a_i, b_i, n_j, m_j \in \mathbb{k}$  with  $i = 1, \dots, k$  and  $j = 1, 2$  determines two elements of  $L$ :

$$\begin{aligned} X &= \sum_i a_i x_i + n_1 D + n_2 D_0; \\ Y &= \sum_i b_i x_i + m_1 D + m_2 D_0. \end{aligned}$$

Observe that

$$(3.2) \quad \chi[X, Y] = \sum_{i=1}^{k-2} (n_1 b_i - m_1 a_i) \chi(x_i) + (n_1 b_{k-1} - m_1 a_{k-1}) \chi(x_k).$$

The assertion  $X \in L_\chi$  is equivalent to saying that the right hand side of (3.2) vanishes for every choice of  $b_i, m_j$  for  $i = 1, \dots, k$  and  $j = 1, 2$ . We shall use this observation to show that  $\dim L_\chi \geq k$  for all  $\chi \in L^*$ . If  $\chi(x_i) \neq 0$  for some  $i \in \{1, \dots, k - 2\}$ , then we may pick  $Y$  by setting scalars  $b_j = \delta_{i,j}$  and  $m_1 = m_2 = 0$ . Now the vanishing of (3.2) ensures  $n_1 = 0$ . If  $\chi(x_k) \neq 0$ , then we may pick  $b_j = \delta_{j,k-1}$  and  $m_1 = m_2 = 0$  to arrive at the conclusion  $n_1 = 0$  similarly.

In either case, the assertion  $X \in L_\chi$  is now equivalent to  $m_1(\sum_{i=1}^{k-2} a_i \chi(x_i)) + m_1 a_{k-1} \chi(x_k) = 0$ . This final condition on  $L_\chi$  is a single linear dependence between the scalars  $a_1, \dots, a_k$ , and we conclude that  $\dim L_\chi \geq k$  for all  $\chi \in L^*$ , and that this lower bound is attained whenever

$$(\chi(x_1), \chi(x_2), \dots, \chi(x_{k-2}), \chi(x_k)) \neq (0, 0, \dots, 0).$$

This proves  $\text{ind } L = k$ .

Next observe that  $[L', L'] = \langle x_1, x_2, \dots, x_{k-2}, x_k \rangle$ , and so  $\chi' \in L'^*$  is determined by  $(\chi'(x_1), \chi'(x_2), \dots, \chi'(x_{k-2}), \chi'(x_k)) \in \mathbb{k}^{k-1}$ . We pick scalars as before so that

$$X = \sum_i a_i x_i + n_1 D + n_2 D^p,$$

$$Y = \sum_i b_i x_i + m_1 D + m_2 D^p,$$

are arbitrary elements of  $L'$ , and we have

$$\chi'[X, Y] = \sum_{i=1}^{k-2} (n_1 b_i - m_1 a_i + n_2 b_i - m_2 a_i) \chi'(x_i) + (n_1 b_{k-1} - m_1 a_{k-1}) \chi'(x_k).$$

Now  $X \in L'_{\chi'}$  is equivalent to the vanishing of the right hand side for every choice of  $b_i, m_j$ . It will suffice to exhibit  $\chi'$  such that  $\dim L'_{\chi'} \leq k - 2$ . To this end we take  $\chi'(x_1) = \chi'(x_2) = \dots = \chi'(x_{k-2}) = \chi'(x_k) = 1$ . Setting  $b_j = \delta_{j,k-1}$  and  $m_1 = m_2 = 0$  we obtain  $n_1 = 0$ , whilst taking  $b_j = \delta_{j,1}$  with  $m_1 = m_2 = 0$  subtends  $n_2 = 0$ . Now take  $b_i = 0$  for all  $i$  and  $m_1 = 0, m_2 = 1$  to get  $\sum_{i=1}^{k-2} a_i = 0$ . Finally set  $b_i = 0$  for all  $i$  and  $m_1 = 1, m_2 = 0$  to get  $a_k = 0$ . This shows that  $\dim L'_{\chi'} \leq k - 2$ , and we are done.  $\square$

#### 4. PROOF OF THE THEOREM

We let  $L$  and  $L'$  be the Lie algebras discussed in (3.1). Notice that  $L$  is also the algebra described in (1.2) and in the statement of the main theorem. Our goal is to show that (1.1) fails for  $L$ . Since the conjecture asserts that  $M(L) = p^{\frac{1}{2}(\dim L - \text{ind } L)} = p$  it will suffice to show that  $p^2 | M(L)$ . By Proposition 2 we know that  $U(L) \cong U(L')$  which implies that  $M(L) = M(L')$ .

We claim that  $L'$  is a restricted Lie algebra. It is easy to check from the relations that  $\text{ad}(x_i)^p = 0$  for all  $i = 1, \dots, k$ , that  $\text{ad}(D)^p = \text{ad}(D^p)$  and that  $\text{ad}(D^p)^p = \text{ad}(D)^{p^2} = \text{ad}(D)^p = \text{ad}(D^p)$  as derivations of  $L'$ . Now we may apply Jacobson's theorem [1, Theorem 2.2.3] to conclude that  $L'$  admits a restricted structure. By Lemma 3 and [4, §5.4, Remark 1] there is a non-negative integer  $s$  such that

$$p^{\frac{1}{2}(\dim L' - \text{ind } L')} = p^{2+s} | M(L').$$

$\square$

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