

## DISTORTION OF IMBEDDINGS OF GROUPS OF INTERMEDIATE GROWTH INTO METRIC SPACES

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ABSTRACT. We show that groups of subexponential growth can have arbitrarily bad distortion for their imbeddings into Hilbert space.

More generally, consider a metric space  $\mathcal{X}$ , and assume that it admits a sequence of finite groups of bounded-size generating set which does not imbed coarsely in  $\mathcal{X}$ . Then, for every unbounded increasing function  $\rho$ , we produce a group of subexponential word growth all of whose imbeddings in  $\mathcal{X}$  have distortion worse than  $\rho$ .

This implies that Liouville groups may have arbitrarily bad distortion for their imbeddings into Hilbert space, precluding a converse to the result by Naor and Peres that groups with distortion much better than  $\sqrt{t}$  are Liouville.

### 1. INTRODUCTION

Let  $G$  be a finitely generated group, and let  $(\mathcal{X}, d)$  be a metric space. The extent to which  $G$ , with its word metric, may be imbedded in  $\mathcal{X}$  with not-too-distorted metric is an asymptotic invariant of  $G$ , introduced by Gromov in [10, §7.E]. The general definition of distortion is as follows:

**Definition 1.1.** Consider a Lipschitz map  $\Phi : (\mathcal{Y}, d) \rightarrow (\mathcal{X}, d)$  between metric spaces, with Lipschitz norm  $\|\Phi\| = \sup_{y \neq y'} d(\Phi(y), \Phi(y'))/d(y, y')$ . The *distortion*<sup>1</sup> of  $\Phi$  is the function

$$\rho_\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+, \quad \rho_\Phi(t) = \inf_{d(y, y') \geq t} \|\Phi\|^{-1} d(\Phi(y), \Phi(y')).$$

It is the largest increasing function  $\rho : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that

$$\rho(d(y, y')) \leq \|\Phi\|^{-1} d(\Phi(y), \Phi(y')) \leq d(y, y') \text{ for all } y \neq y' \in \mathcal{Y}.$$

If  $\mathcal{Y}$  has bounded diameter, then  $\rho_\Phi(t) = +\infty$  for all  $t > \text{diam } \mathcal{Y}$ . We say that  $\Phi$  has distortion *better than*  $\rho$  if  $\rho_\Phi(t) > \rho(t)$  for all  $t \in \mathbb{R}_+$  large enough, *worse than*  $\rho$  if  $\rho_\Phi(t) < \rho(t)$  for all  $t \in \mathbb{R}_+$  large enough, and that  $\Phi$  is a *coarse imbedding* if its distortion is unbounded.<sup>2</sup>

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<sup>1</sup>This is sometimes called *compression* in the literature; we avoid that terminology, since “compression better than” is ambiguous: does it imply “more compressed” or “less compressed”? In comparison, “distortion better than” avoids this ambiguity.

<sup>2</sup>An alternative terminology is *uniform imbedding*.

More generally, if  $(\mathcal{Y}_i)_{i \in I}$  is a family of metric spaces, a *coarse imbedding* is a sequence  $(\Phi_i: \mathcal{Y}_i \rightarrow \mathcal{X})$  of Lipschitz imbeddings with bounded Lipschitz norm and  $\inf_{i \in I} \rho_{\Phi_i}(t)$  an unbounded function of  $t$ .

Our main result is:

**Theorem A** (= Theorem 5.2). *Let  $\mathcal{X}$  be a metric space, and let  $(G_i)_{i \in \mathbb{N}}$  be a sequence of  $d$ -generated finite groups with fixed generating sets, such that  $(G_i)$ , with their word metric, do not imbed coarsely in  $\mathcal{X}$ .*

*Then, for every unbounded increasing function  $\rho: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , there exists a finitely generated group  $W$  of subexponential growth such that every imbedding of  $W$  in  $\mathcal{X}$  has distortion worse than  $\rho$ .*

*Furthermore, the group  $W$  contains an infinite subsequence of the  $G_i$ 's.*

The heart of the proof is a control of the distortion of the imbedding of  $G_i$  in  $W$ : it will depend on  $G_1, \dots, G_{i-1}$  but not on  $G_i$  itself. See §1.1 for a more detailed sketch.

See §6 for the definition of B-convex Banach spaces; briefly said, they are spaces in which  $(\mathbb{C}^n, \|\cdot\|_1)$  does not imbed almost-isometrically for some  $n$ . Hilbert space is an example of B-convex space. Note that, by scaling, Lipschitz imbeddings into vector spaces can always be assumed to have norm 1.

**Corollary B** (= Corollary 6.1). *For every unbounded increasing function  $\rho: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  and for every B-convex Banach space  $\mathcal{X}$  there exists a finitely generated group  $W$  of subexponential word growth such that every imbedding of  $W$  in  $\mathcal{X}$  has distortion worse than  $\rho$ .*

Furthermore, the group  $W$  depends on  $\rho$  but only mildly on  $\mathcal{X}$ : for every unbounded increasing function  $\rho$  there exists a group  $W$  of subexponential growth, such that every imbedding of  $W$  in a B-convex Banach space  $\mathcal{X}$  has distortion worse than  $c\rho$  for some constant  $c$ ; and moreover  $c$  depends only on the convexity parameters  $n, \epsilon$  of  $\mathcal{X}$ ; see Remark 6.4.

Recall that a finitely generated group  $G$  has *subexponential growth* if, for every  $\lambda > 1$ , the number of group elements that are products of at most  $n$  generators grows more slowly than the exponential function  $\lambda^n$ . A group has *locally subexponential growth* if all its finitely generated subgroups have subexponential growth.

Arzhantseva, Druţu and Sapir construct in [1, Theorem 1.5], for every unbounded increasing function  $\rho$ , a group which imbeds coarsely into Hilbert space, and such that all of its imbeddings into a uniformly convex Banach space have distortion worse than  $\rho$ . Olshanskii and Osin construct, moreover, amenable groups with the same property in [18, Corollary 1.4]. Note that these references use a slightly weaker definition of “worse” than ours, namely by ‘ $f \ll g$ ’ they mean  $f(x) \leq ag(bx) + c$  for some constants  $a, b, c$ .

These examples all have exponential growth. In contrast, the main point of our construction is to produce such groups having subexponential growth.

A finitely generated group is *Liouville* if all bounded harmonic functions on its Cayley graphs are constant; equivalently, if all simple random walks on it have trivial Poisson boundary. Groups of subexponential growth are Liouville (via e.g. the entropy criterion; see [2]). We deduce:

**Corollary C.** *For every unbounded increasing function  $\rho: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , there exists a finitely generated Liouville group all of whose imbeddings of into Hilbert space have distortion worse than  $\rho$ .*

It follows from [17, Theorem 1.1] by Naor and Peres that, if an amenable group admits an imbedding into Hilbert space with distortion better than  $t^{1/2+\epsilon}$  for some  $\epsilon > 0$ , then it is Liouville. Corollary C shows therefore that no converse of their result exists.

Note that groups of subexponential growth are amenable, and Bekka, Cherix and Valette show in [6] that amenable groups imbed coarsely into Hilbert space. In fact, their imbeddings can be shown to have distortion better than an unbounded function which depends only on the Følner function. Tessera [19, Theorem 10] gives such formulæ in terms of the isoperimetric profile inside balls.

For example, consider the Grigorchuk groups  $G_\omega$  of intermediate growth, introduced in [9]. They admit “self-similar random walks”  $\mu_\omega$  in the sense of [5, §6] and [12], with asymptotic entropy  $H(\mu_\omega^n) \lesssim n^{1/2}$  by [5, Corollary 6.3]; so their probability of return satisfies  $\mu_\omega^n(1) \asymp \exp(-n^{1/2})$  using the general estimate  $\mu^n(1) \geq \exp(-2H(\mu^n))$ , their Følner function satisfies  $F(n) \lesssim \exp(n^2)$  using Nash inequalities (see [20, Corollary 14.5(b)], and therefore they admit imbeddings into Hilbert space of distortion better than  $t^{1/2-\epsilon}$  for every  $\epsilon > 0$ , by a result of Gournay [8].

On the other hand, groups of subexponential growth can have arbitrarily large Følner function [7]. Our result can therefore be seen as a strengthening of this fact.

**1.1. Sketch of the proof.** We showed in [4] that, for every countable group  $B$  all of whose finitely generated subgroups have subexponential growth, there exists a finitely generated group  $W$  of subexponential growth in which  $B$  imbeds. The goal of this paper is to construct such imbeddings with a good control of the restriction of the word metric of  $W$  to  $B$ . More precisely, let us consider for  $B$  the restricted direct product of finite  $d$ -generated groups  $H_i$ , and scale the word metrics on the groups  $H_i$  in such a manner that the resulting  $\ell_1$  metric on  $B$  has subexponential growth. We may then ensure that the metric on  $B$  is bi-Lipschitz equivalent to the restriction of the word metric from  $W$ .

The group  $W$  will be a subgroup of the unrestricted permutational wreath product of  $W$  with a group of intermediate growth such as the first Grigorchuk group.

In more detail, our argument proceeds in three steps: first, given a sequence of finite groups  $G_1, G_2, \dots$  with bounded number of generators, we imbed each  $G_i$  as the derived subgroup of a finite group  $H_i$  with one more generator, in such a manner that the metric on  $G_i$  is at universally bounded distance from a particular metric on  $[H_i, H_i]$ , the *perfect metric*. See §2.

We then show in §4 that an arbitrary subset of the  $H_i$ 's imbeds in a finitely generated group  $W$  of subexponential growth with controlled distortion; more precisely, the distortion constants of an imbedded  $H_i$  in  $W$  depends only on the previous  $H_j$ 's, but not on  $H_i$ .

Finally, in §5 we assume that the  $H_i$  do not imbed coarsely in a metric space  $\mathcal{X}$ . Given any unbounded increasing function  $\rho$ , we select the subset of  $H_i$ 's appropriately so that the distortion of  $W$  in  $\mathcal{X}$  is worse than  $\rho$  on arbitrarily large subintervals. We construct two such groups  $W, W'$  such that the intervals on which their respective distortions are worse than  $\rho$  overlap. The desired group with distortion worse than  $\rho$  is the direct product  $W \times W'$ .

The idea of using finite groups  $H_i$  whose Cayley graphs are expanders was already used by Arzhantseva, Druţu and Sapir [1]; that of using a restricted direct product of the  $H_i$ 's was used by Olshanskii and Osin [18].

To prove the corollary in §6, we choose for the  $G_i$ , or even directly for the  $H_i$ , a family of superexpanders such as those constructed by Lafforgue in [15].

2. IMBEDDING A GROUP IN A DERIVED SUBGROUP

Let  $G = \langle S \rangle$  be a group with fixed generating set  $S$ . We denote by  $\| \cdot \|_S$  the word norm on  $G$ , or  $\| \cdot \|_G$  if the generating set is clear. Let us now define another norm on  $[G, G]$ . For this, let us say that a word  $w$  in the free group  $F_S$  is *balanced* if it belongs to  $[F_S, F_S]$ ; namely, if it contains as many  $s$ 's as  $s^{-1}$ 's for every letter  $s \in S$ . The *perfect norm* on  $[G, G]$  is

$$\|g\|_{\text{perfect}} = \min\{\|w\|_S : w \in F_S \text{ is a balanced word representing } g\}.$$

We denote by  $d_S(x, y) = d_G(x, y) = \|xy^{-1}\|_S$  and  $d_{\text{perfect}}(x, y) = \|xy^{-1}\|_{\text{perfect}}$  the corresponding distances.

**Proposition 2.1.** *Let  $G = \langle S \rangle$  be a finite group with fixed generating set of cardinality  $d$ . Then there exists a finite group  $H = \langle T \rangle$  with generating set of cardinality  $d + 1$  and an imbedding  $\iota : G \rightarrow [H, H]$  such that*

$$2\|g\|_G \leq \|\iota(g)\|_{\text{perfect}} \leq 4\|g\|_G \text{ for all } g \in G.$$

*Proof.* Let  $m$  denote the cardinality of  $G$ , and let  $x$  denote a generator of  $\mathbb{Z}$ . Since  $G * \mathbb{Z}$  is residually finite, there exists a finite quotient  $Q = \langle S \cup \{x\} \rangle$  of  $G * \mathbb{Z}$  such that the balls of radius  $m$  in  $G * \mathbb{Z}$  and in  $Q$  coincide.

In the standard wreath product  $Q \wr C_{2m}$ , consider the following elements: for every  $s \in S$ , the function  $t_s : C_{2m} \rightarrow Q$  with values  $(1, s, s^2, \dots, s^{m-1}, 1, s^x, s^{2x}, \dots, s^{(m-1)x})$ ; and the generator  $r$  of  $C_{2m}$ . Set

$$T = \{t_s : s \in S\} \cup \{r\} \quad \text{and} \quad H = \langle T \rangle \leq Q \wr C_{2m}.$$

Define  $\iota : G \rightarrow Q \wr C_{2m}$  by

$$\iota(g) : C_{2m} \rightarrow Q \text{ taking values } (g, \dots, g, g^x, \dots, g^x).$$

Note first that, for  $s \in S$ , we have  $\iota(s) = [t_s, r]$ . This immediately implies  $\iota(G) \leq [H, H]$  and gives the inequality  $\|\iota(g)\|_{\text{perfect}} \leq 4\|g\|_G$  for all  $g \in G$ .

Note then that if a word of length  $\leq m/2$  in  $T$  is not of the form  $q_0 r^{-1} q_1 r \cdots q_{2n-2} r^{-1} q_{2n-1} r q_{2n}$  with all  $q_i \in \langle t_s : s \in S \rangle$ , then it cannot belong to the image of  $\iota$ . On the other hand, if it is of that form, then write it as  $f : C_{2m} \rightarrow Q$ , and note that  $f(1)$  depends only on  $q_1, \dots, q_{2n-1}$  and has length at most the sum of their lengths. This gives the other inequality  $2\|g\|_G \leq \|\iota(g)\|_{\text{perfect}}$ . □

3. PERMUTATIONAL WREATH PRODUCTS

We recall briefly some notions introduced in [4, §4]. Let  $G = \langle S \rangle$  be a finitely generated group acting transitively on the right on a set  $X$ . We consider  $X$  as the vertex set of a graph still denoted  $X$ , with for all  $x \in X, s \in S$  an edge labelled  $s$  from  $x$  to  $xs$ . We denote by  $d$  the path metric on this graph.

For an arbitrary group  $H$ , we denote by  $H \wr_X G$  the *permutational wreath product*, defined as

$$H \wr_X G = \{f : X \rightarrow H \text{ finitely supported}\} \rtimes G;$$

the right action of  $g \in G$  on a function  $f : X \rightarrow H$  is  $f^g : x \mapsto f(xg^{-1})$ , and coincides with conjugation  $g^{-1}fg$  in  $H \wr_X G$ .

**Definition 3.1.** A sequence  $(x_0, x_1, \dots)$  in  $X$  is *spreading* if for all  $R$  there exists  $N$  such that if  $i, j \geq N$  and  $i \neq j$ , then  $d(x_i, x_j) \geq R$ .

**Definition 3.2.** A sequence  $(x_i)$  in  $X$  *locally stabilises* if for all  $R$  there exists  $N$  such that if  $i, j \geq N$ , then the  $S$ -labelled radius- $R$  balls centered at  $x_i$  and  $x_j$  in  $X$  are equal.

The following definition adapts to the context of group actions the notion of a *parallelogram-free* sequence that was introduced in [18] for regular actions (see [4, §4] for more details):

**Definition 3.3.** A sequence of points  $(x_i)$  in  $X$  is *rectifiable* if for all  $i, j$  there exists  $g \in G$  with  $x_i g = x_j$  and  $x_k g \neq x_\ell$  for all  $k \notin \{i, \ell\}$ .

There exist examples of the following: a group  $G$  of subexponential growth, an action of  $G$  on  $X$ , a spreading, locally stabilising, rectifiable sequence  $(x_i)$  in  $X$ , with the additional property that, for every group  $H$  of subexponential growth, the restricted wreath product  $H \wr_X G$  also has subexponential growth.

A concrete such example is described in [4, Lemma 4.11] and [3]; let us recall it briefly. Let  $G = G_{012} = \langle a, b, c, d \rangle$  denote the first Grigorchuk group. It acts on the set of infinite sequences  $\{0, 1\}^\infty$  over a two-letter alphabet, which is naturally the boundary of a binary rooted tree. Denote by  $X = \mathbf{1}^\infty G_{012}$  the orbit of the rightmost ray in that tree. Then the sequence  $(x_i)$  defined by  $x_i = \mathbf{0}^i \mathbf{1}^\infty$  is spreading, locally stabilising and rectifiable, and by [4, Lemma 5.2] the wreath product  $H \wr_X G$  has intermediate growth as soon as  $H$  has subexponential growth.

#### 4. IMBEDDING A SEQUENCE OF GROUPS IN A GROUP OF SUBEXPONENTIAL GROWTH

We assume that the following data have been fixed: a group  $G$ , a  $G$ -set  $X$ , and an infinite spreading, locally stabilising and rectifiable sequence  $(x_i)$  in  $X$ . We assume that, for every finitely generated group  $H$  of subexponential growth, the restricted wreath product  $H \wr_X G$  has subexponential growth.<sup>3</sup> See the previous section for such an example.

Let  $B$  be a group, let  $\bar{b} = (b_1, b_2, \dots)$  be a sequence in  $B$ , and let  $\bar{n} = (0 \leq n(1) < n(2) < \dots)$  be an increasing integer sequence. Assume that  $\bar{b}$  and  $\bar{n}$  have the same length, finite or infinite. Consider the function  $f: X \rightarrow B$  defined by

$$f(x_{n(1)}) = b_1, \quad f(x_{n(2)}) = b_2, \quad \dots, \quad f(x) = 1 \text{ for other } x,$$

and denote by  $W_{\bar{b}, \bar{n}}$  the subgroup  $\langle G, f \rangle$  of the unrestricted wreath product  $B^X \rtimes G$ .

**Lemma 4.1** ([4, Lemma 6.1]). *Let  $\bar{b} = (b_1, b_2, \dots)$  be a sequence in  $B$  that generates  $B$ . Then, for all choices of  $\bar{n}$ , the group  $W_{\bar{b}, \bar{n}}$  contains  $[B, B]$  as a subgroup.  $\square$*

We now apply this construction to the special case of  $B$  a restricted direct product of finite groups. The heart of the argument is the following variant of [4, Corollary 6.3]. Recall that a map  $\Phi: (\mathcal{Y}, d) \rightarrow (\mathcal{X}, d)$  is  $(K, L)$ -bi-Lipschitz if  $Kd(y, y') \leq d(\Phi(y), \Phi(y')) \leq Ld(y, y')$  for all  $y, y' \in \mathcal{Y}$ .

**Proposition 4.2.** *Let  $(H_i)_{i \in \mathbb{N}}$  be a sequence of  $d$ -generated finite groups, with fixed generating sets.*

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<sup>3</sup>This is called the “subexponential wreathing property” in [4].

Then there exists a family of groups  $(W_S)_{S \subset \mathbb{N}}$  indexed by subsets  $S$  of  $\mathbb{N}$ , each of subexponential growth, with the following property: for all  $s \in S$ , there is an imbedding  $\Psi_s: [H_s, H_s] \rightarrow W_S$  that is  $(K, L)$ -bi-Lipschitz with respect to the perfect metric on  $[H_s, H_s]$  and the word metric on  $W_S$ , and such that the constants  $K, L$  depend only on  $\{i \in S: i < s\}$ .

If  $S = \{s(1), s(2), \dots\}$ , then the sequence  $(W_{\{s(1), \dots, s(i)\}})_{i \in \mathbb{N}}$  converges to  $W_S$  in the space of marked groups (also known as the Cayley and Chabauty-Grigorchuk topology).

Furthermore, each  $W_S$  will be of the form  $W_{\bar{b}, \bar{n}}$ , with  $\bar{n}$  finite whenever  $S$  is finite. Writing  $W_{\{s(1), \dots, s(i)\}} = \langle G, f_i \rangle$ , the functions  $f_i$  and  $f_{i+1}$  agree on the support of  $f_i$ , so  $W_S$  may be constructed iteratively given an oracle that determines  $s(i+1)$  in terms of  $W_{\{s(1), \dots, s(i)\}}$ .

*Proof.* Up to replacing  $(H_i)_{i \in \mathbb{N}}$  by  $(H_i)_{i \in S}$ , we lighten notation and suppose  $S = \mathbb{N}$  or a prefix  $\{1, 2, \dots, n\}$  thereof. We also fix a finite generating set  $U$  for  $G$ .

Let us write  $T_i = \{t_{i,1}, \dots, t_{i,d}\}$  the generating set of  $H_i$ . We consider the restricted direct product  $B = \prod_{i \geq 1} H_i$ . It is a countable, locally finite group, generated by  $\{t_{1,1}, \dots, t_{1,d}, t_{2,1}, \dots\} = \{b_1, b_2, \dots\} =: \bar{b}$ . By [4, Corollary 6.3], a sequence  $\bar{n} = (n(1), n(2), \dots)$  may be chosen in such a manner that the corresponding group  $W_{\bar{b}, \bar{n}}$  has subexponential growth; and, furthermore, the term  $n(i)$  depends only on  $b_1, \dots, b_{i-1}, n(1), \dots, n(i-1)$ . We define  $W_S$  as  $W_{\bar{b}, \bar{n}}$ , and denote by  $U \cup \{f\}$  the generating set of  $W_S$ .

Let us now turn to the imbedding of  $[H_i, H_i]$  in  $W_S$  and its bi-Lipschitz constants. Consider  $[H_i, H_i]$  as a subgroup of  $W_S$ , imbedded as the functions  $X \rightarrow H_i \subset B$  supported only at  $x_{n(di)}$ . This is an imbedding by Lemma 4.1. Denote by  $\Psi_i: [H_i, H_i] \rightarrow W_S$  this imbedding.

Assume that  $n(1), \dots, n(di)$  have already been chosen; and note that their choice relies only on  $H_1, \dots, H_{i-1}$ . Recall also that  $f(x_{n(di-d+j)}) = t_{i,j}$  in the construction of  $W_S$ . We now show that there exist constants  $K, L$  independent of  $H_i$  such that the imbedding  $\Psi_i: [H_i, H_i] \rightarrow W_S$  is  $(K, L)$ -bi-Lipschitz. In other words, by scaling the metric on  $[H_i, H_i]$  by a factor of  $L$ , we obtain that the distortion of  $[H_i, H_i]$  in  $W_S$  is at worst  $\rho(t) = tK/L$ .

Let  $g_{i,1}, \dots, g_{i,d} \in G$  be such that  $x_{n(di-d+j)}g_{i,j} = x_{n(di)}$  and the only  $x_k$  mapped to another  $x_\ell$  under  $g_{i,j}g_{i,j}^{-1}$  are  $x_{n(di-d+j)}g_{i,j}g_{i,j}^{-1} = x_{n(di-d+j)}$ ; such elements exist because  $(x_i)$  is rectifiable. This condition ensures that the functions  $f^{g_{i,1}}, \dots, f^{g_{i,d}}$  have disjoint support except at  $x_{n(di)}$  or where they take identical values. Let  $L'$  be an upper bound for the lengths over  $U$  of all  $g_{i,1}, \dots, g_{i,d}$ , and note that  $L'$  depends only on  $n(1), \dots, n(i-1)$ .

Here is an explicit way of computing the imbedding  $\Psi_i: [H_i, H_i] \rightarrow W_S$ : for  $h \in [H_i, H_i]$ , write it as a minimal-length balanced word in  $T_i$ , and in it replace each letter  $t_{i,j}$  by  $f^{g_{i,j}}$  to obtain a word representing  $\Psi_i(h)$ .

On the one hand,  $\|\Psi_i(h)\|_{W_S} \leq (2L' + 1)\|h\|_{\text{perfect}}$  because each letter  $t_{i,j} \in T_i$  gets mapped to a word over  $U \cup \{f\}$  of length  $2\|g_{i,j}\|_U + 1 \leq 2L' + 1$ ; on the other hand,  $\|\Psi_i(h)\|_{W_S} \geq \|h\|_{\text{perfect}}$ , because at most one element of  $T_i$  is contributed by each generator of  $W_S$ . We may therefore take as bi-Lipschitz constants  $(K, L) = (1, 2L' + 1)$ .

Since the sequence  $(x_i)$  is spreading and locally stabilising, the balls in the groups  $W_{\{s(1), \dots, s(i)\}}$  stabilise as  $i \rightarrow \infty$ . □

**Corollary 4.3.** *Let  $(G_i)_{i \in \mathbb{N}}$  be a sequence of  $d$ -generated finite groups.*

*Then there exists a family of groups  $(W_S)_{S \subset \mathbb{N}}$  indexed by subsets  $S$  of  $\mathbb{N}$ , each of subexponential growth, with the following property: for all  $s \in S$ , there is an imbedding  $\Psi_s: G_s \rightarrow W_S$  that is  $(K, L)$ -bi-Lipschitz with respect to the word metrics, and such that the constants  $K, L$  depend only on  $\{i \in S: i < s\}$ .*

*Proof.* Using Proposition 2.1, imbed each  $G_i$  in  $[H_i, H_i]$  for a  $(d + 1)$ -generated group  $H_i$  in such a manner that the inclusion map  $\iota_i: (G_i, d_{G_i}) \rightarrow (H_i, d_{\text{perfect}})$  is  $(2, 4)$ -bi-Lipschitz. Apply then Proposition 4.2 to the family  $(H_i)_{i \in \mathbb{N}}$ .  $\square$

### 5. IMBEDDINGS INTO METRIC SPACES

Let  $\mathcal{X}$  be a metric space. Given a sequence of metric spaces such as  $((G_i, d_{G_i}))_{i \in \mathbb{N}}$ , we say that it *imbeds coarsely* in  $\mathcal{X}$  if there exists an unbounded increasing function  $\rho$  and a sequence of Lipschitz imbeddings  $(\Phi_i: G_i \rightarrow \mathcal{X})$ , each with distortion better than  $\rho$ . We are interested in the opposite property:

**Definition 5.1.** Let  $\mathcal{X}$  be a metric space. A sequence of metric spaces  $((G_i, d_{G_i}))_{i \in \mathbb{N}}$  *does not imbed coarsely* in  $\mathcal{X}$  if the following holds: there exists a constant  $M$  such that, if  $(\Phi_i: G_i \rightarrow \mathcal{X})$  is a sequence of  $L$ -Lipschitz imbeddings, then, for all  $t \in \mathbb{R}$ , there are  $i \in \mathbb{N}$  and  $y, y' \in G_i$  with  $d(y, y') \geq t$  and  $d(\Phi_i(y), \Phi_i(y')) \leq LM$ .

**Theorem 5.2.** *Let  $(G_i)_{i \in \mathbb{N}}$  be a sequence of  $d$ -generated finite groups. Assume furthermore that  $(G_i)$  does not imbed coarsely in a metric space  $\mathcal{X}$ . Let  $\rho$  be any unbounded increasing function  $\mathbb{R}_+ \rightarrow \mathbb{R}_+$ . Then there exists a finitely generated group  $W$  of subexponential growth such that every imbedding of  $W$  in  $\mathcal{X}$  has distortion worse than  $\rho$ .*

*Proof.* Using Proposition 2.1, imbed each  $G_i$  in  $[H_i, H_i]$  for a group  $H_i$  in such a manner that the inclusion map  $(G_i, d_{G_i}) \rightarrow (H_i, d_{\text{perfect}})$  is bi-Lipschitz. We identify  $G_i$  with its image in  $H_i$ . Let  $(W_S)_{S \subset \mathbb{N}}$  be the family of groups given by Proposition 4.2 applied to  $(H_i)_{i \in \mathbb{N}}$ .

The group  $W$  will be of the form  $W = W_S \times W_{S'}$ , for sequences  $S = \{s(1), s(3), s(5), \dots\}$  and  $S' = \{s(2), s(4), \dots\}$  that we construct iteratively as follows. Assume that  $s(1), \dots, s(i - 1)$  have been constructed; we shall now determine  $s := s(i)$  and  $s' := s(i + 1)$ . Let  $K_i, L_i, K_{i+1}, L_{i+1}$  be the constants such that the imbedding of  $G_s$  into  $W_S$  or  $W_{S'}$  is  $(K_i, L_i)$ -bi-Lipschitz and the imbedding of  $G_{s'}$  into  $W_{S'}$  or  $W_S$  is  $(K_{i+1}, L_{i+1})$ -bi-Lipschitz.

Since the groups  $G_j$  do not imbed coarsely in  $\mathcal{X}$ , there exists a constant  $M$  such that, if  $(\Phi_j: G_j \rightarrow \mathcal{X})$  is a sequence of  $L$ -Lipschitz imbeddings, then, for all  $t \in \mathbb{R}$ , there are  $j \in \mathbb{N}$  and  $y, y' \in G_j$  with  $d(y, y') \geq t$  and  $d(\Phi_j(y), \Phi_j(y')) \leq LM$ .

We now make use of the unbounded function  $\rho$ . Let  $t_i \in \mathbb{R}$  be large enough so that  $\rho(t_i) > L_{i+1}M$ . We choose  $s$  large enough so that there exist  $y, y' \in G_s$  with  $d(y, y') \geq t_i/K_i$  and  $d(\Phi_s(x), \Phi_s(y)) \leq LM$  in any  $L$ -Lipschitz imbedding  $\Phi_s$  of  $G_s$  into  $\mathcal{X}$ . Without loss of generality, the sequences  $(t_i)$  and  $(s(i))$  are strictly increasing. This determines  $s = s(i)$ , and finishes the inductive construction of  $S$  and  $S'$ .

Let  $\Phi: W \rightarrow \mathcal{X}$  be an  $L$ -Lipschitz imbedding, and let us prove that  $\Phi$  has distortion worse than  $\rho$ . By precomposing  $\Phi$  with the imbedding  $\Psi_s$  of  $G_s$  in  $W_S$  or  $W_{S'}$ , we get for every  $s = s(i) \in S \cup S'$  an  $L_i L$ -imbedding  $\Phi_s = \Phi \circ \Psi_s$  of  $G_s$  into  $\mathcal{X}$ .

Consider  $t \in \mathbb{R}_+$ , and suppose without loss of generality  $t \geq t_1$ . Let  $i$  be such that  $t_{i-1} \leq t < t_i$ . Set  $s = s(i)$ . Since the imbedding  $\Phi_s$  is  $L_i L$ -Lipschitz, there are  $y, y' \in G_s$  with  $d(y, y') \geq t_i/K_i$  and  $d(\Phi_s(y), \Phi_s(y')) \leq L_i LM$ . On the other hand,  $\Psi_s$  is  $(K_i, L_i)$ -bi-Lipschitz so  $d(\Psi_s(y), \Psi_s(y')) \geq t_i$ . This proves that the distortion  $\rho_\Phi$  of  $\Phi$  satisfies

$$\rho_\Phi(t) \leq \rho_\Phi(t_i) \leq \|\Phi\|^{-1} L_i LM = L_i M < \rho(t_{i-1}) \leq \rho(t),$$

so the distortion of  $\Phi$  is worse than  $\rho$ . □

We note from the proof that the distortion of a single copy  $W_S$  is worse than  $\rho$  along an unbounded sequence.

### 6. APPLICATIONS

We now consider some metric spaces  $\mathcal{X}$  to which Theorem 5.2 applies. For example, let  $\mathcal{X}$  be Hilbert space. Take for  $(H_i)$  a sequence of expanders; recall that there exist residually finite groups with Kazhdan’s property (T) such as  $\mathbf{SL}_3(\mathbb{Z})$ , see [13], and that any family of finite quotients of a group with property (T) forms a family of expander graphs, as shown by Margulis [16]. It is well known that expanders do not imbed coarsely into Hilbert space; see e.g. [11, page 158].

Here is a more general class of spaces  $\mathcal{X}$ . We recall the following definition from [15]. A Banach space  $\mathcal{X}$  is called *of type  $> 1$* , or *B-convex*, if there are  $n \in \mathbb{N}$  and  $\epsilon > 0$  such that no Lipschitz imbedding  $(\mathbb{C}^n, \|\cdot\|_1) \rightarrow \mathcal{X}$  with distortion better than  $t \mapsto t/(1 + \epsilon)$  exists. For example, Hilbert space is B-convex with  $n = 2$  and any  $\epsilon > \sqrt{2} - 1$ .

**Corollary 6.1.** *Let  $\mathcal{X}$  be a B-convex Banach space, and let  $\rho$  be an unbounded increasing function  $\mathbb{R}_+ \rightarrow \mathbb{R}_+$ . Then there exists a finitely generated group  $W$  of subexponential growth such that every imbedding of  $W$  in  $\mathcal{X}$  has distortion worse than  $\rho$ .*

*In particular, let  $\rho$  be any unbounded increasing function  $\mathbb{R}_+ \rightarrow \mathbb{R}_+$ . Then there exists a finitely generated group  $W$  of subexponential growth such that every imbedding of  $W$  into Hilbert space has distortion worse than  $\rho$ .*

The proof uses the following construction of “superexpanders”:

**Proposition 6.2** (Lafforgue, [15, Corollaire 0.5]). *There exists a sequence  $(Q_i)_{i \in \mathbb{N}}$  of finite quotients of a finitely generated group  $H$  such that the sequence of quotient Cayley graphs does not imbed coarsely into a B-convex Banach space.*

*Proof.* What Lafforgue shows, actually, is that there exists a constant  $C$  such that, for every  $i \in \mathbb{N}$  and every 1-Lipschitz map  $\Phi: Q_i \rightarrow \mathcal{X}$  with 0 mean, one has  $\mathbb{E}_{x \in Q_i} \|\Phi(x)\|^2 \leq C$ . A classical argument (see, e.g., [14, page 600]) implies that there are two points  $x, y \in Q_i$  with  $d(x, y) \geq c \log(\#Q_i)$  and  $d(\Phi(x), \Phi(y)) \leq \sqrt{2C}$ , for a constant  $c$  depending only on the number of generators of  $H$ . □

*Proof of Corollary 6.1.* Consider the sequence of superexpanders  $(Q_i)_{i \in \mathbb{N}}$  given by Proposition 6.2. Using Proposition 2.1, imbed them as derived subgroups in finite groups  $H_i$ . Apply then Theorem 5.2. □

*Remark 6.3.* The construction of the groups  $H_i$  in the proof of Corollary 6.1 is explicit. Indeed, Lafforgue gives an explicit construction of the groups  $Q_i$  in Proposition 6.2. Consider a prime power  $q$ , the group  $H = \mathbf{SL}_3(\mathbb{F}_q[t])$ , and its images  $Q_i$  in  $\mathbf{SL}_3(\mathbb{F}_q[t]/(t^i))$ .



Since  $\mathbb{F}_q[t]$  is a Euclidean domain,  $H$  is generated by elementary matrices. Furthermore, the classical identities  $X_{i,j}(P+Q) = X_{i,j}(P)X_{i,j}(Q)$  and  $X_{i,j}(PQ) = [X_{i,k}(P), X_{k,j}(Q)]$  between elementary matrices, when  $\{i, j, k\} = \{1, 2, 3\}$ , imply that  $H$  is generated by  $A = \mathbf{SL}_3(\mathbb{F}_q)$  and  $B = \langle X_{1,2}(t) \rangle$ . Since  $A$  is perfect and  $B^A = \{1 + tM : M \in M_{3 \times 3}(\mathbb{F}_q) \text{ and } \text{tr}(M) = 0\}$  admits no non-trivial  $A$ -invariant functional,  $H$  is also perfect.

We fix as generating set  $T = A \cup B$ , and denote by  $T_i$  its natural image in  $Q_i$ . Then the groups  $Q_i$  are perfect, all generated by the same number of elements, and the identity map  $(Q_i, d_{\text{perfect}}) \rightarrow (Q_i, d_{Q_i})$  is  $(J^{-1}, 1)$ -bi-Lipschitz for a constant  $J \geq 1$  independent of  $i$ : indeed, represent each  $t \in T$  as a balanced word, and let  $J$  be the maximal length of these balanced words. Then

$$\|g\|_{T_i} \leq \|g\|_{\text{perfect}} \leq J\|g\|_T \leq J\|g\|_{T_i} \text{ for all } g \in [Q_i, Q_i].$$

We may therefore take  $H_i = Q_i$ .

*Remark 6.4.* The construction of  $W$  in Corollary 6.1 can be made independent of the choice of space  $\mathcal{X}$ : fix  $n \in \mathbb{N}$  and  $\epsilon > 0$ ; then for every unbounded increasing function  $\mathbb{R}_+ \rightarrow \mathbb{R}_+$  there exists a group  $W$  of subexponential growth with the following property: if  $\mathcal{X}$  is any B-convex Banach space not  $(1 + \epsilon)$ -isometrically containing  $\ell_1(\mathbb{C}^n)$ , then every imbedding of  $W$  in  $\mathcal{X}$  has distortion worse than  $\rho$ . This follows formally from Corollary 6.1 because an  $\ell^2$ -sum of such Banach spaces is again of the same form.

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#### REFERENCES

- [1] Goul'nara N. Arzhantseva, Cornelia Druțu, and Mark V. Sapir, *Compression functions of uniform embeddings of groups into Hilbert and Banach spaces*, J. Reine Angew. Math. **633** (2009), 213–235, DOI 10.1515/CRELLE.2009.066. MR2561202 (2010m:20065)
- [2] André Avez, *Théorème de Choquet-Deny pour les groupes à croissance non exponentielle* (French), C. R. Acad. Sci. Paris Sér. A **279** (1974), 25–28. MR0353405
- [3] Laurent Bartholdi and Anna Erschler, *Growth of permutational extensions*, Invent. Math. **189** (2012), no. 2, 431–455, DOI 10.1007/s00222-011-0368-x. MR2947548
- [4] Laurent Bartholdi and Anna Erschler, *Imbeddings into groups of intermediate growth*, Groups Geom. Dyn. **8** (2014), no. 3, 605–620, DOI 10.4171/GGD/241, available at [arXiv:math/1403.5584](https://arxiv.org/abs/math/1403.5584). MR3267517
- [5] Laurent Bartholdi and Anna Erschler, *Poisson-Furstenberg boundary and growth of groups*, Prob. Theory Rel. Fields, posted on 2016, DOI 10.1007/s00440-016-0712-6, available at [arXiv:math/1107.5499](https://arxiv.org/abs/math/1107.5499).
- [6] M. E. B. Bekka, P.-A. Cherix, and A. Valette, *Proper affine isometric actions of amenable groups*, Novikov conjectures, index theorems and rigidity, Vol. 2 (Oberwolfach, 1993), London Math. Soc. Lecture Note Ser., vol. 227, Cambridge Univ. Press, Cambridge, 1995, pp. 1–4, DOI 10.1017/CBO9780511629365.003. MR1388307
- [7] Anna G. Erschler, *Piecewise automatic groups*, Duke Math. J. **134** (2006), no. 3, 591–613, DOI 10.1215/S0012-7094-06-13435-X. MR2254627 (2007k:20086)
- [8] Antoine Gournay, *The Liouville property via Hilbertian compression*, to appear in Ann. Inst. Fourier.
- [9] R. I. Grigorchuk, *Degrees of growth of finitely generated groups and the theory of invariant means* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **48** (1984), no. 5, 939–985. MR764305

- [10] M. Gromov, *Asymptotic invariants of infinite groups*, Geometric group theory, Vol. 2 (Sussex, 1991), London Math. Soc. Lecture Note Ser., vol. 182, Cambridge Univ. Press, Cambridge, 1993, pp. 1–295. MR1253544
- [11] Misha Gromov, *Spaces and questions*, Geom. Funct. Anal. **Special Volume** (2000), 118–161, DOI 10.1007/978-3-0346-0422-2.5. GAFA 2000 (Tel Aviv, 1999). MR1826251 (2002e:53056)
- [12] Vadim A. Kaimanovich, “*Münchhausen trick*” and amenability of self-similar groups, Internat. J. Algebra Comput. **15** (2005), no. 5–6, 907–937. MR2197814
- [13] David A. Každan, *On the connection of the dual space of a group with the structure of its closed subgroups* (Russian), Funkcional. Anal. i Priložen. **1** (1967), 71–74. MR0209390 (35 #288)
- [14] Vincent Lafforgue, *Un renforcement de la propriété (T)* (French, with English and French summaries), Duke Math. J. **143** (2008), no. 3, 559–602, DOI 10.1215/00127094-2008-029. MR2423763
- [15] Vincent Lafforgue, *Propriété (T) renforcée banachique et transformation de Fourier rapide* (French, with English summary), J. Topol. Anal. **1** (2009), no. 3, 191–206, DOI 10.1142/S1793525309000163. MR2574023
- [16] G. A. Margulis, *Explicit constructions of expanders* (Russian), Problemy Peredači Informacii **9** (1973), no. 4, 71–80. MR0484767
- [17] Assaf Naor and Yuval Peres, *Embeddings of discrete groups and the speed of random walks*, Int. Math. Res. Not. IMRN, posted on 2008, Art. ID rnn 076, 34, DOI 10.1093/imrn/rnn076. MR2439557 (2009m:20067)
- [18] Alexander Yu. Olshanskii and Denis V. Osin, *A quasi-isometric embedding theorem for groups*, Duke Math. J. **162** (2013), no. 9, 1621–1648, DOI 10.1215/00127094-2266251. MR3079257
- [19] Romain Tessera, *Asymptotic isoperimetry on groups and uniform embeddings into Banach spaces*, Comment. Math. Helv. **86** (2011), no. 3, 499–535, DOI 10.4171/CMH/232. MR2803851
- [20] Wolfgang Woess, *Random walks on infinite graphs and groups*, Cambridge Tracts in Mathematics, vol. 138, Cambridge University Press, Cambridge, 2000. MR1743100

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