NON-WIEFERICH PRIMES
IN ARITHMETIC PROGRESSIONS

YONG-GAO CHEN AND YU DING

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Abstract. Graves and Murty proved that for any integer \( a \geq 2 \) and any fixed integer \( k \geq 2 \), there are \( \gg \frac{\log x}{\log \log x} \) primes \( p \leq x \) such that \( a^{p-1} \not\equiv 1 \pmod{p^2} \) and \( p \equiv 1 \pmod{k} \), under the assumption of the abc conjecture. In this paper, for any fixed \( M \), the bound \( \frac{\log x}{\log \log x} \) is improved to \( \left( \frac{\log x}{\log \log x} \right) \left( \log \log \log x \right)^M \).

1. Introduction

In 1909, A. Wieferich [5] found that Fermat’s last theorem is related to the primes \( p \) with

\[
2^{p-1} \equiv 1 \pmod{p^2}.
\]

That is, for any odd prime \( p \), if the equation \( x^p + y^p + z^p = 0 \) has a solution in integers \( x, y, z \) with \( p \nmid xyz \), then (1.1) holds. Since then, such primes have been called Wieferich primes. For any integer \( a \geq 2 \) and any prime \( p \), if

\[
a^{p-1} \equiv 1 \pmod{p^2},
\]

then \( p \) is said to be a Wieferich prime for base \( a \). Otherwise, \( p \) is said to be a non-Wieferich prime for base \( a \). Currently, the only known Wieferich primes are 1093 and 3511. It is unknown whether there are infinitely many Wieferich primes and also unknown whether there are infinitely many non-Wieferich primes.

The abc conjecture says that, if \( a, b \) and \( c \) are positive integers with \( a + b = c \) and \( (a, b) = 1 \), then, for any \( \varepsilon > 0 \),

\[
c \ll \varepsilon (\text{rad}(abc))^{1+\varepsilon},
\]

where \( \text{rad}(abc) \) is the product of all distinct prime factors of \( abc \).

Silverman [4] proved that there are \( \gg \log x \) non-Wieferich primes under the assumption of the abc conjecture. DeKoninck and Doyon [1] proved the same result under the weaker assumption. In 2013, Graves and Murty [2] proved that for any integer \( a \geq 2 \) and any fixed integer \( k \geq 2 \), there are

\[
\gg \frac{\log x}{\log \log x}
\]

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primes $p \leq x$ such that
$$a^{p-1} \not\equiv 1 \pmod{p^2}, \quad p \equiv 1 \pmod{k},$$
under the assumption of the abc conjecture.

In this paper, the bound is improved.

**Theorem 1.1.** Let $a$ and $k$ be fixed integers with $a \geq 2$ and $k \geq 2$ and let $\mathcal{P}$ be the set of all primes. Suppose that the abc conjecture is true. Then, for any positive integer $M$, we have
$$\left| \{ p : p \leq x, p \in \mathcal{P}, p \equiv 1 \pmod{k}, a^{p-1} \not\equiv 1 \pmod{p^2} \} \right| \gg \frac{(\log x)(\log \log x)^M}{\log \log x}.$$

2. **Proof of Theorem 1.1**

In the following, we fix integers $a, k$ and $M$ with $a \geq 2$, $k \geq 2$ and $M \geq 1$. Let $p_i$ be the $i$th prime. Let
$$\delta_M = \prod_{i=1}^{M+1} \left( 1 - \frac{1}{p_i} \right)$$
and let $\mathcal{T}_M$ be the set of all square-free integers with exactly $M + 1$ prime factors. We follow the proof of Graves and Murty [2]. For any positive integer $n$, let $a^n - 1 = q_1^{\alpha_1} \cdots q_r^{\alpha_r}$ be the standard factorization of $a^n - 1$. Define
$$C_n = \prod_{i=1}^{\alpha_i} q_i, \quad D_n = \prod_{\alpha_i > 1} q_i^{\alpha_i}.$$ 

Let $\phi$ be the Euler totient function and $\Phi_n(a)$ be the $n$th cyclotomic polynomial. Let
$$C'_n = (C_n, \Phi_n(a)), \quad D'_n = (D_n, \Phi_n(a)).$$

Since $a^n - 1 = C_n D_n$, $(C_n, D_n) = 1$ and $\Phi_n(a) \mid a^n - 1$, it follows that
$$\Phi_n(a) = (a^n - 1, \Phi_n(a)) = (C_n D_n, \Phi_n(a)) = C'_n D'_n.$$

We need the following lemmas.

**Lemma 2.1** ([2, Lemma 2.3]). If $p$ is a prime with $p \mid \Phi_n(a)$, then either $p \mid n$ or $p \equiv 1 \pmod{n}$.

**Lemma 2.2** ([2, Lemma 2.4]). If $p$ is a prime with $p \mid C_n$, then
$$a^{p-1} \not\equiv 1 \pmod{p^2}.$$

**Lemma 2.3** ([3, Theorem 437]). Let $\pi_m(x)$ denote the number of square-free integers which do not exceed $x$ and have exactly $m$ prime factors. Then
$$\pi_m(x) \sim \frac{x(\log \log x)^{m-1}}{(m-1)! \log x}.$$

**Lemma 2.4.** Let $\varepsilon$ be a (small) positive number. Suppose that the abc conjecture is true. Then
$$C'_n \gg a^{\phi(n) - \varepsilon n}.$$

**Proof.** A proof is similar to that of [2, Theorem 3.1]. We omit the details here. □

The following lemma is one of the key lemmas in this paper.

**Lemma 2.5.** If $m < n$, then $(C'_m, C'_n) = 1$. 

Proof. Suppose that \((C'_m, C'_n) > 1\). Let \(p\) be a prime such that \(p \mid C'_m\) and \(p \mid C'_n\). By the definitions of \(C'_m\) and \(C'_n\), we have \(p \mid \Phi_m(a)\) and \(p \mid \Phi_n(a)\). So
\[
p \mid a^m - 1, \quad p \mid a^n - 1.
\]
Thus \(p \mid a^{(m,n)} - 1\). By \(m < n\), we have \((m,n) < n\). Since
\[
a^n - 1 = \frac{a^n - 1}{a^{(m,n)} - 1} \left( a^{(m,n)} - 1 \right),
\]
we have \(p \mid \Phi_n(a) \frac{a^n - 1}{a^{(m,n)} - 1}\). So
\[
p \mid a^m - 1, p \mid a^n - 1.
\]
Thus \(p \mid a^{(m,n)} - 1\). By \(m < n\), we have \((m,n) < n\). Since
\[
a^n - 1 = \frac{a^n - 1}{a^{(m,n)} - 1} \left( a^{(m,n)} - 1 \right),
\]
it follows that \(p^2 \mid a^n - 1\), a contradiction with \(p \mid C'_n\). Therefore,
\[
(C'_m, C'_n) = 1.
\]
\(\Box\)

Lemma 2.6. Suppose that the abc conjecture is true. Then there exists an integer \(n_0\) depending only on \(a, k, M\) such that, if \(n \in T_M\) with \(n \geq n_0\), then \(C'_{nk} > nk\).

Proof. Let
\[
\varepsilon = \frac{\delta_M \phi(k)}{3k}.
\]
By Lemma 2.4 we have
\[
C'_{nk} \gg a^{\phi(nk) - \varepsilon nk}.
\]
Since
\[
\phi(m) = m \prod_{p \mid m} \left( 1 - \frac{1}{p} \right)
\]
and
\[
\prod_{p \mid nk} \left( 1 - \frac{1}{p} \right) \geq \prod_{p \mid n} \left( 1 - \frac{1}{p} \right) \prod_{p \mid k} \left( 1 - \frac{1}{p} \right),
\]
it follows that \(\phi(nk) \geq \phi(n)\phi(k)\). If \(n \in T_M\), then, by (2.2), we have
\[
\phi(nk) - \varepsilon nk = \phi(n)\phi(k) - \varepsilon k = \delta_M n\phi(k) - \varepsilon nk = 2\varepsilon nk.
\]
It follows from (2.1) that if \(n \in T_M\), then
\[
C'_{nk} \gg a^{2\varepsilon nk} \gg a^{2\varepsilon nk - \log(nk)/\log a nk}.
\]
Therefore, there exists an integer \(n_0\) depending only on \(a, k, M\) such that, if \(n \in T_M\) with \(n \geq n_0\), then \(C'_{nk} > nk\).
\(\Box\)

Lemma 2.7. Let \(n_0\) be as in Lemma 2.6. If \(n \in T_M\) with \(n \geq n_0\), then there exists a prime \(q_n\) such that
\[
q_n \mid C'_{nk}, \quad q_n \equiv 1 \pmod{nk}, \quad a^{q_n - 1} \not\equiv 1 \pmod{q_n^2}.
\]

Proof. Let \(n \in T_M\) with \(n \geq n_0\). By Lemma 2.6 and \(C'_{nk}\) being square-free, there is a prime \(q_n\) such that \(q_n \mid C'_{nk}\) and \(q_n \nmid nk\). Since \(C'_{nk} \mid \Phi_{nk}(a)\) and \(q_n \nmid nk\), it follows from Lemma 2.4 that \(q_n \equiv 1 \pmod{nk}\). By \(q_n \mid C'_{nk}\), \(C'_{nk} \mid C_{nk}\) and Lemma 2.2 we have \(a^{q_n - 1} \not\equiv 1 \pmod{q_n^2}\).
\(\Box\)
Proof of Theorem 1.1 Let \( n_0 \) and \( q_n \) be as in Lemma 2.7. By Lemma 2.5, the primes \( q_n(n \in \mathcal{T}_M, n \geq n_0) \) are distinct. It is clear that \( a^{nk} - 1 \leq x \) if and only if
\[
n \leq \frac{\log(x + 1)}{k \log a}.
\]
Thus, \( a^{nk} - 1 \leq x \) with \( n \in \mathcal{T}_M \) if and only if
\[
n \leq \frac{\log(x + 1)}{k \log a}, \quad n \in \mathcal{T}_M.
\]
It follows from Lemma 2.3 that the number of integers \( n \) with \( a^{nk} - 1 \leq x \), \( n \in \mathcal{T}_M \) and \( n \geq n_0 \) is
\[
\gg \frac{(\log x)(\log \log x)^{M+1}}{\log \log x}.
\]
Since \( q_n \leq C'_nk \leq a^{nk} - 1 \), it follows that the number of \( q_n \) with \( q_n \leq x \), \( n \in \mathcal{T}_M \) and \( n \geq n_0 \) is
\[
\gg \frac{(\log x)(\log \log x)^{M+1}}{\log \log x}.
\]
By Lemma 2.7 we have
\[
q_n \equiv 1 \pmod{nk}, \quad a^{q_n - 1} \not\equiv 1 \pmod{q_n^2}.
\]
Therefore,
\[
\left| \left\{ p : p \leq x, p \in \mathcal{P}, p \equiv 1 \pmod{k}, a^{p - 1} \not\equiv 1 \pmod{p^2} \right\} \right| \gg \frac{(\log x)(\log \log x)^{M+1}}{\log \log x}.
\]
This completes the proof of Theorem 1.1. \( \square \)

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References


School of Mathematical Sciences and Institute of Mathematics, Nanjing Normal University, Nanjing 210023, People’s Republic of China

E-mail address: ygchen@njnu.edu.cn

School of Mathematical Sciences and Institute of Mathematics, Nanjing Normal University, Nanjing 210023, People’s Republic of China

E-mail address: 840172236@qq.com