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BRAUER CHARACTERS OF q'- DEGREE

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ABSTRACT. We show that if p is a prime and G is a finite p-solvable group satisfying the condition that a prime q divides the degree of no irreducible p-Brauer character of G, then the normalizer of some Sylow q-subgroup of G meets all the conjugacy classes of p-regular elements of G.

1. INTRODUCTION

Throughout this paper, G will be a finite group and p will be a prime. Let $\mathrm{Irr}(G)$ be the set of all complex irreducible characters of G and let $\mathrm{IBr}(G)$ be the set of irreducible p-Brauer characters of G. The celebrated Itô-Michler theorem says that p does not divide $\chi(1)$ for all $\chi \in \mathrm{Irr}(G)$ if and only if G has a normal abelian Sylow p-subgroup.

Many variations of this theorem have been proposed and studied in the literature. See the recent survey paper by G. Navarro on this topic in [9]. One might ask whether there is any version of the Itô-Michler theorem for Brauer characters of finite groups. Now let q be a prime and assume that q divides the degree of no irreducible p-Brauer character of G. Indeed, it is known that if $q = p$, then G has a normal Sylow q-subgroup. (See Theorem 3.1 of [9].)

This raises the question of whether there exists a similar result when $q \neq p$. In particular, in Problem 3.2 of [9], Navarro asks when G is a p-solvable group and q divides the degree of no irreducible p-Brauer character, is it true that every p-regular conjugacy class of G intersects the normalizer of a Sylow q-subgroup of G? In this paper, we prove that this is true.

Theorem A. *Let p be a prime and let G be a finite p-solvable group. Let q be a prime and suppose that q divides the degree of no irreducible p-Brauer character of G. Then every p-regular conjugacy class of G meets $\mathbf{N}_G(Q)$, where Q is a Sylow q-subgroup of G.*

The conclusion of Theorem A can be restated in the language of permutation group theory as follows. Let H be a proper subgroup of a finite group G. Following [4], we say that an element $x \in G$ is an *H-derangement* in G if the conjugacy class x^G containing x does not meet H. We write $\Delta_H(G)$ for the set of all H-derangements of G. If H is core-free in G, then G is a permutation group acting on the right coset space $\Omega = G/H$ with point stabilizer H and $\Delta(G) = \Delta_H(G)$ is the set of all derangements or fixed-point-free elements of G on Ω.

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A classical theorem due to Jordan [5] says that the set $\Delta_H(G)$ is non-empty. Notice that

$$\Delta_H(G) = G \setminus \cup_{g\in G} H^g.$$

Derangements have many applications in topology and number theory (see [11]). Derangements have been used in studying zeros of ordinary character theory. Now with this concept, Theorem A can be restated as follows.

Let p and q be primes and let G be a finite p-solvable group and Q a Sylow q-subgroup of G. If q divides the degree of no irreducible p-Brauer character of G, then all $\mathbf{N}_G(Q)$-derangements of G have order divisible by p.

As already mentioned in [9], using the Itô-Michler theorem for Brauer characters and a result in [2] which states that every finite permutation group of degree > 1 contains a derangement of prime power order, it is easy to see that for a finite group G, every p-Brauer character of G has p'-degree if and only if every $\mathbf{N}_G(P)$-derangement of G, for some Sylow p-subgroup P of G, has order divisible by p. That is, Theorem A holds for all finite groups when $q = p$. Unfortunately, this does not hold true when q is different from p. We will provide examples to show that the p-solvable assumption on G in Theorem A is necessary.

Returning to Problem 3.2 of [9], we note that Navarro asks for a characterization of p-solvable groups where $q \neq p$ and q does not divide the degree of any irreducible p-Brauer character of G. Manz and Wolf studied these groups in [8]. Many of the results of that paper can be found in Section 13 of [7]. In Corollary 13.15 of [7], they prove that $\mathbf{O}^{q'}(G)$ is solvable. When G has an abelian Sylow q-subgroup, it turns out that this additional condition is sufficient. However, it is not difficult to find examples where the Sylow q-subgroups are not abelian (see Section 3). We actually obtain a full characterization for p-solvable groups G that have $q \nmid \varphi(1)$ for all $\varphi \in \mathrm{IBr}(G)$ for primes $p \neq q$, however the conditions are so technical that it is not worth including them in the current paper. We refer the reader to an earlier version of this paper (arXiv:1601.05373) for the aforementioned results.

2. Proof of Theorem A

Throughout this section, p and q are distinct primes and G is a finite group. Suppose that q divides the degree of no irreducible p-Brauer character of G and let N be a normal subgroup of G. Since $\mathrm{IBr}(G/N) \subseteq \mathrm{IBr}(G)$ and by [10, Corollary 8.7], we see that both N and G/N satisfy this property. In particular, all p-Brauer characters of $\mathbf{O}^{q'}(G)$ have q'-degree. We show the converse of this holds when G/N is p-solvable.

Lemma 2.1. *Let p and q be distinct primes and suppose that $G/\mathbf{O}^{q'}(G)$ is p-solvable. Then $q \nmid \varphi(1)$ for all $\varphi \in \mathrm{IBr}(G)$ if and only if $q \nmid \beta(1)$ for all $\beta \in \mathrm{IBr}(\mathbf{O}^{q'}(G))$.*

Proof. By the discussion above, it suffices to show that if all irreducible p-Brauer characters of $L := \mathbf{O}^{q'}(G)$ have q'-degree, then $q \nmid \varphi(1)$ for all $\varphi \in \mathrm{IBr}(G)$. Let $\varphi \in \mathrm{IBr}(G)$ and let $\theta \in \mathrm{IBr}(L)$ be an irreducible constituent of φ_L. By [10, Theorem 8.30], we have $\varphi(1)/\theta(1)$ divides $|G/L|$. Since G/L is a q'-group and $q \nmid \theta(1)$ by our assumption, we deduce that $q \nmid \varphi(1)$. □

As we mentioned in the introduction, Manz and Wolf studied these groups in [8], and the results from that paper can be found in Section 13 of [7]. We next formally state their results which can be found as Theorem 13.8 and Corollary 13.15 of [7].

Lemma 2.2. *Let p and q be distinct primes and let G be a finite p-solvable group. Suppose that $q \nmid \varphi(1)$ for all $\varphi \in \mathrm{IBr}(G)$. Then the following hold:*

(i) $\mathbf{O}^{q'}(G)$ *is solvable and so G is q-solvable.*
(ii) In each q-series of G, the q-factors are abelian.
(iii) A Sylow q-subgroup of G is metabelian.
(iv) $G/\mathbf{O}_{p,q}(G)$ *has q-length at most* 1.

Fix a prime p. Let G be a finite group and let H be a proper subgroup of G. For brevity, we say that the pair (G,H) has property $\mathcal{D}_p$ if $x^G \cap H$ is not empty for all p-regular elements $x \in G$ or equivalently all H-derangements of G have order divisible by p.

The following slightly generalizes Lemma 4.2 in [1].

Lemma 2.3. *Let H be a proper subgroup of a finite group G and L be a normal subgroup of G such that $G = HL$. If T is a proper subgroup of L containing $H \cap L$, then $\Delta_T(L) \subseteq \Delta_H(G)$.*

Proof. Let $x \in \Delta_T(L)$ and assume that $x \notin \Delta_H(G)$. Then $x^g \in H$ for some $g \in G$. Since $x \in L \trianglelefteq G$, we have $x^g \in L$, so $x^g \in H \cap L \leq T$. As $g \in G = HL = LH$, we can write $g = lh$ with $h \in H$ and $l \in L$. Then $x^g = x^{lh} = (x^l)^h \in H$ which implies that $x^l \in H$ and since both x and l are in L, we obtain that $x^l \in H \cap L \leq T$, which is a contradiction. □

We collect in the next lemma some properties of finite groups satisfying $\mathcal{D}_p$.

Lemma 2.4. *Let p be a prime, G be a finite group and H be a subgroup of G. Let $L \trianglelefteq G$.*

(1) *If $G = HL$ and (G,H) satisfies $\mathcal{D}_p$, then so does $(L, H \cap L)$.*
(2) *If L is a p-group or p'-group, (G,H) satisfies $\mathcal{D}_p$ and $G \neq HL$, then $(G/L, HL/L)$ also satisfies $\mathcal{D}_p$.*
(3) *If $H \leq K < G$ and (G,H) satisfies $\mathcal{D}_p$, then (G,K) satisfies $\mathcal{D}_p$.*
(4) *If $L \trianglelefteq G$ such that $L \leq H$ and $(G/L, H/L)$ satisfies $\mathcal{D}_p$, then (G,H) satisfies $\mathcal{D}_p$.*

Proof. (1) This follows immediately since $\Delta_{H\cap L}(L) \subseteq \Delta_H(G)$ by Lemma 2.3

(2) Let $\overline{G} = G/L$. Since $G \neq HL$, we see that $\overline{H}$ is a proper subgroup of $\overline{G}$. Let $\overline{x} \in \Delta_{\overline{H}}(\overline{G})$. Then $\overline{x}^{\overline{G}} \cap \overline{H} = \emptyset$ which implies that $x^G \cap HL = \emptyset$ and thus $x^G \cap H = \emptyset$ or $x \in \Delta_H(G)$ so p divides $|x|$, the order of x.

If L is a p'-group, then the order of $\overline{x}$ must be divisible by p and we are done. So, assume that L is a p-group and that the order of $\overline{x}$, say n, is indivisible by p. As L is a p-group and $p \nmid n$, we see that $|x| = p^a n$ for some integer a. There exist integers u, v such that $1 = up^a + vn$. Hence $x = (x^{p^a})^u (x^n)^v$ where $x^n \in L$. Thus $\overline{x} = \overline{y}$ with $y = x^{up^a}$ and $|y| = n$. We now have that $\overline{y}^{\overline{G}} \cap \overline{H} = \emptyset$ and thus $y^G \cap HL = \emptyset$ so $y \in \Delta_H(G)$ and hence p divides $|y|$, which is a contradiction.

(3) is obvious since $\Delta_K(G) \subseteq \Delta_H(G)$.

(4) Let $x \in \Delta_H(G)$. Then $x^G \cap H = \emptyset$ and thus $\overline{x}^{\overline{G}} \cap \overline{H} = \emptyset$ since $\overline{H} = H/L$. So p divides $|x|$ as $(G/L, H/L)$ satisfies $\mathcal{D}_p$. Clearly p must divide $|x|$ and we are done. □

We now apply Lemma 2.4 for $H = \mathbf{N}_G(Q)$, where Q is a Sylow q-subgroup of G. The next lemma asserts that the condition '$\mathbf{N}_G(Q)$ meets every p-regular class of a finite group G' is inherited to normal subgroups.

Lemma 2.5. *Let p and q be distinct primes. Let Q be a Sylow q-subgroup of G and let $L \trianglelefteq G$. Suppose that $x^G \cap \mathbf{N}_G(Q) \neq \emptyset$ for all p-regular elements x of G. Then $x^L \cap \mathbf{N}_L(Q \cap L) \neq \emptyset$ for all p-regular elements x of L. In particular, if $Q \leq L$, then $x^L \cap \mathbf{N}_L(Q) \neq \emptyset$ for all p-regular elements x of L.*

Proof. Let $H = \mathbf{N}_G(Q)$ and $U = Q \cap L$. Then $U \trianglelefteq Q \trianglelefteq H \leq \mathbf{N}_G(U)$ and $U \in \mathrm{Syl}_q(L)$. Since $L \trianglelefteq G$, we have $G = \mathbf{N}_G(U)L$ by Frattini's argument.

If $U \trianglelefteq L$, then the conclusion is trivially true. So, we may assume that $\mathbf{N}_L(U)$ is a proper subgroup of L which implies that both H and $\mathbf{N}_G(U)$ are proper subgroups of G. It suffices to show that the pair $(L, \mathbf{N}_L(U))$ satisfies $\mathcal{D}_p$.

Clearly, (G, H) satisfies $\mathcal{D}_p$ by the hypothesis, so $(G, \mathbf{N}_G(U))$ satisfies $\mathcal{D}_p$ by Lemma 2.4(3). Now part (1) of Lemma 2.4 implies that $(L, L \cap \mathbf{N}_G(U))$ satisfies $\mathcal{D}_p$ or $(L, \mathbf{N}_L(U))$ satisfies $\mathcal{D}_p$ as wanted. □

We next prove Theorem A under the additional hypothesis that $G = Q\mathbf{O}_{q'}(G)$ where Q is a Sylow q-subgroup of G.

Lemma 2.6. *Let p and q be distinct primes and let $Q \in \mathrm{Syl}_q(G)$. Suppose that $G = Q\mathbf{O}_{q'}(G)$ and that $q \nmid \varphi(1)$ for all $\varphi \in \mathrm{IBr}(G)$. Let $K = \mathbf{O}_{q'}(G)$ and $H = \mathbf{N}_G(Q)$. Then*

(i) Q is abelian and $x^K \cap \mathbf{C}_K(Q) \neq \emptyset$ for all p-regular elements $x \in K$;
(ii) $x^G \cap \mathbf{N}_G(Q)$ is non-empty for all p-regular elements $x \in G$.

Proof. Assume first that $q \nmid \varphi(1)$ for all $\varphi \in \mathrm{IBr}(G)$. Since $G = QK$, with $K \trianglelefteq G$ and $Q \cap K = 1$, we deduce that $H = Q\mathbf{C}_K(Q)$.

(0) Q is abelian. This follows from Lemma 2.2(ii).

(1) Every $\theta \in \mathrm{IBr}(K)$ is Q-invariant. Let $\theta \in \mathrm{IBr}(K)$ and let $T = I_G(\theta)$. By Clifford correspondence [10, Theorem 8.9], if $\psi \in \mathrm{IBr}(T|\theta)$, then $\psi^G \in \mathrm{IBr}(G)$ and so $\psi^G(1) = |G : T|\psi(1)$. As $K \trianglelefteq T \leq G$ and $q \nmid \psi^G(1)$, we must have that $T = G$ or equivalently θ is G-invariant and hence it is Q-invariant.

(2) Q stabilizes all p-regular conjugacy classes of K. From (1), Q stabilizes all irreducible p-Brauer characters of K and hence it stabilizes all the projective indecomposable characters Φ_μ associated with $\mu \in \mathrm{IBr}(K)$. From [10, Theorem 2.13], the set

$$\{\Phi_\mu \mid \mu \in \mathrm{IBr}(K)\}$$

is a basis of the space of complex functions of K vanishing off the set K° of all p-regular elements of K. Therefore Q stabilizes the characteristic functions of the p-regular conjugacy classes of K and thus Q stabilizes all the p-regular conjugacy classes of K.

(3) If $x \in K$ is p-regular, then $x^K \cap \mathbf{C}_K(Q) \neq \emptyset$. Let $\mathcal{C}$ be a p-regular class of K. By (2), Q acts on $\mathcal{C}$, K acts transitively on $\mathcal{C}$ and Q acts coprimely on K. By Corollary 1 of Theorem 4 in [3], we obtain that $\mathcal{C} \cap \mathbf{C}_K(Q) \neq \emptyset$. This proves ($i$).

(4) $y^G \cap H \neq \emptyset$ for every p-regular element $y \in G$. Let y be a p-regular element of G. If $y \in K$, then $y^K \cap \mathbf{C}_K(Q) \neq \emptyset$ so $y^G \cap H \neq \emptyset$ and we are done. So, we can assume that $y \notin K$.

Suppose that the order of y is $q^a \cdot m$, where $a, m \geq 1$ and $q \nmid m$. Since $\gcd(q^a, m) = 1$, there exist integers u, v such that $1 = uq^a + vm$. Let $y_1 = y^{vm}$ and $y_2 = y^{uq^a}$.

Then $y = y_1y_2 = y_2y_1$, where y_1 is a q-element and y_2 is a q'-element. By Sylow theorem, $y_1^t \in Q$ for some $t \in G$. Since y_2^t is a q'-element and $K \trianglelefteq G$ is a Hall q'-subgroup of G, we deduce that $y_2^t \in K$. Thus $y^t = y_1^t y_2^t = y_2^t y_1^t$, where $y_1^t \in Q$ and $y_2^t \in K$. Replacing y by y^t, we can assume $y = y_1y_2 = y_2y_1$ with $y_1 \in Q$, $y_2 \in K$. Since y is p-regular, we see that $y_2 \in K$ is also p-regular and thus $y_2^k \in \mathbf{C}_K(Q)$ for some $k \in K$ by (3) above. Hence $Q^{k^{-1}} \leq \mathbf{C}_G(y_2)$. By the Sylow theorem, there exists $l \in \mathbf{C}_G(y_2)$ such that $Q^{k^{-1}} = Q^l$ so that $Q^{lk} = Q$ or equivalently $lk \in H$.

Since $y = y_1y_2$ and $y_1 \in Q \leq H$, we deduce that $y_1^{lk} \in H$. Moreover, as $l \in \mathbf{C}_G(y_2)$, we have $y_2^{lk} = y_2^k \in \mathbf{C}_K(Q) \leq H$. Therefore,

$$y^{lk} = y_1^{lk} y_2^{lk} \in H.$$

So, we have shown that $y^G \cap H \neq \emptyset$ for all p-regular elements $y \in G$. This completes the proof of (ii). □

We are now ready to prove Theorem A which we restate here.

Theorem 2.7. *Let p and q be distinct primes and let G be a finite p-solvable group. Suppose that $q \nmid \varphi(1)$ for all $\varphi \in \mathrm{IBr}(G)$. Then $x^G \cap \mathbf{N}_G(Q) \neq \emptyset$ for all p-regular elements $x \in G$, where Q is a Sylow q-subgroup of G.*

Proof. Let $Q \in \mathrm{Syl}_q(G)$ and $H = \mathbf{N}_G(Q)$. Suppose that $q \nmid \varphi(1)$ for all $\varphi \in \mathrm{IBr}(G)$. We proceed by induction on $|G|$.

(1) If $N \trianglelefteq G$ is non-trivial, then $y^G \cap HN \neq \emptyset$ for all p-regular elements $y \in G$. Since G/N satisfies the hypothesis of the theorem with $|G/N| < |G|$, by induction we deduce that $\mathbf{N}_{G/N}(QN/N) = \mathbf{N}_G(Q)N/N = HN/N$ meets all the p-regular classes of G/N. It follows that $y^G \cap HN \neq \emptyset$ for every p-regular element $y \in G$.

(2) $\mathbf{O}_p(G) = 1 = \mathbf{O}_q(G)$ and $H_G = 1$. If $H_G \trianglelefteq G$ is non-trivial, then the conclusion of the theorem holds by applying (1). So, we can assume $H_G = 1$ and so $\mathbf{O}_q(G) = 1$.

Suppose that $\mathbf{O}_p(G)$ is non-trivial. By (1) again, we see that $y^G \cap H\mathbf{O}_p(G)$ is non-empty for every p-regular element $y \in G$. Replacing y by its conjugate if necessary, we can assume $y \in H\mathbf{O}_p(G)$. Since y is an element of p'-order and H contains a Hall p'-subgroup T of $H\mathbf{O}_p(G)$, some $H\mathbf{O}_p(G)$-conjugate of y lies in $T \subseteq H$ and thus $y^G \cap H$ is non-empty.

(3) Let $L = \mathbf{O}^{q'}(G)$. Then $L = QK$ where $K = \mathbf{O}_{q'}(L)$ is solvable. Since $L \trianglelefteq G$ and $\mathbf{O}_p(G) = 1 = \mathbf{O}_q(G)$ by (2), we deduce that $\mathbf{O}_p(L) = \mathbf{O}_q(L) = 1$ so that $\mathbf{O}_{p,q}(L) = 1$. Therefore, by Lemma 2.2 L is solvable and has q-length at most 1. Since $L = \mathbf{O}^{q'}(L)$, we must have that $L = Q\mathbf{O}_{q'}(L)$ as wanted.

(4) $x^K \cap \mathbf{C}_K(Q) \neq \emptyset$ for every p-regular element $x \in K$. This follows from (3) and Lemma 2.6(i).

(5) $y^G \cap H \neq \emptyset$ for every p-regular element $y \in G$. Since $L = QK$ is solvable by (3) and $\mathbf{O}_q(L) = 1$ by (2), we see that K is non-trivial and solvable. Notice that K is a solvable normal subgroup of G. Hence K contains a minimal normal subgroup of G, say N. Then N is an elementary abelian r-group for some prime r different from both p and q.

From (1), we have that $y^G \cap HN \neq \emptyset$ for all p-regular elements $y \in G$. Hence it suffices to show that $y^G \cap H \neq \emptyset$ for all p-regular elements $y \in HN$. Now fix a p-regular element $y \in HN$. Then $y = hn$ for some $h \in H$ and $n \in N$. If $n = 1$, then $y = h \in H$ and we are done. So we assume that n is non-trivial. By (4), $n^k \in \mathbf{C}_K(Q)$ for some $k \in K$. Hence $Q^{k^{-1}} \leq \mathbf{C}_{QK}(n)$, and thus by the Sylow theorem, $Q^{k^{-1}} = Q^l$ for some $l \in \mathbf{C}_{QK}(n)$ so $Q^{lk} = Q$ and hence $lk \in H$. Since $n^{lk} = n^k$ and $h^{lk} \in H$, we obtain that

$$y^{lk} = (hn)^{lk} = h^{lk}n^{lk} = h^{lk}n^k \in H.$$

Therefore, we have shown that $y^G \cap H$ is not empty. □

3. Examples

In the example below, we show that the solvable assumption on G in Theorem A is necessary.

Let p be a prime. We refer the readers to [6] for some basic information on modular representation theory of $\mathrm{SL}_2(p^f)$ in defining characteristic p, where $f \geq 1$.

It follows from Remark 4.5 [6] that every p-modular irreducible representation of $\mathrm{SL}_2(p)$ has degree $k+1$ for some k with $0 \leq k \leq p-1$ and for odd p, it is unfaithful if and only if $k+1$ is odd. The p-modular irreducible representations of $\mathrm{SL}_2(p^f)$ are then obtained by using Steinberg's tensor product theorem (see [6, Theorem 2.2]). It follows that every p-Brauer character degree of $\mathrm{SL}_2(p^f)$ is a product of at most f Brauer character degrees of $\mathrm{SL}_2(p)$.

Now, if $p = 2$, then all irreducible p-Brauer characters of $\mathrm{SL}_2(2^f)$ have 2-power degrees since every irreducible p-Brauer characters of $\mathrm{SL}_2(2)$ has degree 1 or 2.

Denote by $\mathrm{cd}_p(G)$ the degrees of irreducible p-Brauer characters of G. Assume $p > 2$. Then $\mathrm{cd}_p(\mathrm{SL}_2(p))$ consists of all integers from 1 to p; and $\mathrm{cd}_p(\mathrm{PSL}_2(p))$ consists of all odd integers from 1 to p. Finally, $\mathrm{cd}_p(\mathrm{SL}_2(p^2))$ consists of all integers of the form ab where $a, b \in \{1, 2, \cdots, p\}$.

With these results on p-Brauer character degrees of $\mathrm{SL}_2(p^f)$ and $\mathrm{PSL}_2(p)$, we have the following.

Examples. Let p and q be distinct primes and let $f \geq 1$ be an integer.

(i) Assume $f \geq 4$, $p = 2$, and q is a prime divisor of $2^f + 1$. Let $G = \mathrm{SL}_2(2^f)$ and $Q \in \mathrm{Syl}_q(G)$. Then $\mathbf{N}_G(Q) \cong \mathrm{D}_{2(2^f+1)}$ and all irreducible 2-Brauer characters of G have 2-power degree. In particular, $q \nmid \varphi(1)$ for all 2-Brauer characters $\varphi \in \mathrm{IBr}(G)$, but $\mathbf{N}_G(Q)$ contains no element of order $2^f - 1$.

(ii) Let $p \geq 5$ be a prime and let q be a prime divisor of p^2+1 such that $q > p$. Let $G = \mathrm{SL}_2(p^2)$. Then q divides the degree of no irreducible p-Brauer character of G. However, $\mathbf{N}_G(Q) \cong \mathrm{D}_{p^2+1} \cdot 2$ contains no p-regular element of order $p^2 - 1$.

(iii) Let p be a prime of the form $2^f \pm 1 \geq 17$ and let $G = \mathrm{PSL}_2(p)$. Then all irreducible p-Brauer characters of G have odd degree and that the Sylow 2-subgroup Q of G is maximal in G. Then $2 \nmid \varphi(1)$ for all $\varphi \in \mathrm{IBr}(G)$ but $\mathbf{N}_G(Q) = Q$ contains no odd p-regular element of G.

Next, we have examples of p-solvable groups G where $q \nmid \varphi(1)$ for all $\varphi \in \mathrm{IBr}(G)$ and G has a non-abelian Sylow q-subgroup. The easiest example is to take $p = 3$, $q = 2$, and $G = \mathrm{S}_4$. For another example, let V be the additive group of the field $\mathbf{F}$ of order 3^3, let C be the subgroup of the multiplicative group of $\mathbf{F}$ having order 13, and let A be the Galois group of F over $\mathbb{Z}_3$. Now, C acts on V and A acts on VC, and we take G to be VCA. With $p = 13$ and $q = 3$, we see that the irreducible 13-Brauer characters have degrees 1 and 13 and G has a non-abelian Sylow 3-subgroup.

Finally, we present examples of groups that satisfy the conclusion of Theorem A and the conditions of Manz and Wolf, yet have irreducible p-Brauer characters whose degrees are divisible by q. We begin by noting that all $\{p,q\}$-groups trivially satisfy the conclusion of Theorem A since the p-regular elements will have q-power order and hence necessarily be conjugate to elements of the given Sylow q-subgroup. Also, by Burnside's theorem, we know that any $\{p,q\}$-group is necessarily solvable. Thus, it suffices to find a $\{p,q\}$-group G where in the q-series for G, the q-factors are abelian, the q-length of $G/\mathbf{O}_{p,q}(G)$ is at most 1, and the Sylow q-subgroups are metabelian, and there exists a p-Brauer character whose degree is divisible by q. A specific example when $p = 3$ and $q = 2$ can be found by taking the semidirect product of S_3 acting on two copies of the Klein 4-group where the action of the S_3 on each Klein 4-group is the action found in S_4. Obviously, a Sylow 2-subgroup is metabelian, the 2-factors in 2-series for G will be abelian, and G will have 2-length 1. Finally, it is not difficult to see that there exist irreducible 3-Brauer characters for G that have degree 6. We note that there is nothing particular about 3 and 2 that are needed for an example. We claim that for any two distinct primes p and q, the iterated wreath product of $\mathbb{Z}_q$ by $\mathbb{Z}_p$ and then $\mathbb{Z}_q$ again will yield an example, but we leave the details to the reader.

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