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CENTERS OF WEIGHT-HOMOGENEOUS POLYNOMIAL VECTOR FIELDS ON THE PLANE

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ABSTRACT. We characterize all centers of planar weight-homogeneous polynomial vector fields. Moreover we classify all centers of planar weight-homogeneous polynomial vector fields of degrees 6 and 7.

1. Introduction and statement of the main results

One of the main problems in the qualitative theory of real planar polynomial differential systems is the center-focus problem. This problem consists of distinguishing when a singular point is either a focus or a center. The notion of center and focus goes back to Poincaré [18]. A singular point p of system (1) is a *center* if there is a neighborhood of p fulfilled by periodic orbits with the unique exception of p. The *period annulus* of a center is the region fulfilled by all the periodic orbits surrounding the center. We say that a center located at the origin is *global* if its period annulus is $\mathbb{R}^2 \setminus \{(0,0)\}$.

The center problem for planar polynomial vector fields has been intensively studied. The center problem for linear type singular points, i.e., singular points with imaginary pure eigenvalues, is the most studied. It started with the study of the quadratic polynomial differential systems with linear type singular points. The works of Dulac [5], Bautin [4], and Żoładek [21] are the principal ones for the quadratic case; see Schlomiuk [20] for an update of these works. But the center-focus problem for polynomial differential systems of degree larger than two remains open. However, for polynomial differential systems of degree larger than two, there are richer partial results on the center problem; see for instance [8, 19, 22, 23].

The inability to go beyond the study of centers for general polynomial differential systems has motivated the study of particular cases as they are the quasi-homogeneous or weight-homogeneous polynomial differential systems.

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Hence we consider the polynomial differential systems of the form

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{1}$$

where $P, Q \in \mathbb{R}[x, y]$ are coprime and the origin is a singularity of system (1). As usual, $\mathbb{R}[x, y]$ denotes the ring of polynomials in the variables x and y with coefficients in $\mathbb{R}$, and the dot denotes derivative with respect to an independent variable t.

We say that system (1) is *weight-homogeneous* if there exist $s = (s_1, s_2) \in \mathbb{N}^2$ and $d \in \mathbb{N}$ such that for arbitrary $\lambda \in \mathbb{R}^+ = \{\lambda \in \mathbb{R} : \lambda > 0\}$ we have

$$P(\lambda^{s_1} x, \lambda^{s_2} y) = \lambda^{s_1 - 1 + d} P(x, y), \quad Q(\lambda^{s_1} x, \lambda^{s_2} y) = \lambda^{s_1 - 1 + d} Q(x, y).$$

We call $s = (s_1, s_2)$ the *weight exponent* of system (1) and d the *weight degree* with respect to the weight exponent s. In the particular case that $s = (1, 1)$, systems (1) are exactly the *homogeneous polynomial differential systems of degree* d. For a weight-homogeneous polynomial differential system (1), a weight vector $w = (\tilde{s}_1, \tilde{s}_2, \tilde{d})$ is *minimal* for system (1) if any other weight vector (s_1, s_2, d) of system (1) satisfies $\tilde{s}_1 \le s_1$, $\tilde{s}_2 \le s_2$ and $\tilde{d} \le d$. Clearly, each weight-homogeneous polynomial differential system has a unique minimal weight vector.

Taking the weighted polar coordinates $x = r^{s_1} \cos\theta$, $y = r^{s_2} \sin\theta$ system (1) becomes

$$\dot{r} = \frac{r^m F(\theta)}{s_1 \cos^2\theta + s_2 \sin^2\theta}, \quad \dot{\theta} = \frac{r^{m-1} G(\theta)}{s_1 \cos^2\theta + s_2 \sin^2\theta}, \tag{2}$$

with

$$F(\theta) = P(\cos\theta, \sin\theta)\cos\theta + Q(\cos\theta, \sin\theta)\sin\theta,$$
$$G(\theta) = s_1\, Q(\cos\theta, \sin\theta)\cos\theta - s_2\, P(\cos\theta, \sin\theta)\sin\theta.$$

We note that $s_1 \cos^2\theta + s_2 \sin^2\theta > 0$ for all $\theta \in \mathbb{R}$ because s_1, $s_2 > 0$. The next well-known result characterizes when the weight-homogeneous polynomial differential system (1) has a center at the origin of coordinates in terms of trigonometric polynomials $F(\theta)$ and $G(\theta)$; see [10] and [13].

Lemma 1. *A quasi-homogeneous system* (1) *has a center (at the origin of coordinates) if and only if* $G(\theta)$ *has no real roots and* $\int_0^{2\pi} \frac{F(\theta)}{G(\theta)}\, d\theta = 0$.

The first main result of this paper is to improve this characterization giving the explicit characterization of all centers of the weight-homogeneous polynomial differential systems.

Following [9] we first introduce a change of variables that transforms system (1) into a system of separable variables. For a proof see [9].

Lemma 2. *The change of variables*

$$x = u^{1/s_2}, \quad y = (uv)^{1/s_1}, \quad \textit{with inverse } u = x^{s_2}, \;\; v = y^{s_1}/x^{s_2}, \tag{3}$$

and the rescaling of time given by $u^{-(d-1)/(s_1 s_2)} v^{-(s_1 - 1 - m)/s_1}$ *with* $m \in \mathbb{N} \cup \{0\}$ *transform a quasi-homogeneous system* (1) *of weight* (s_1, s_2, d) *into a polynomial differential system of the form*

$$\dot{u} = u f(v), \qquad \dot{v} = g(v), \tag{4}$$

where we can choose m *so that* f *and* g *are coprime.*

One immediate consequence of Lemma 2 is that if system (4) has $H(u,v)$ as a first integral, then the quasi-homogeneous system (1) of weight (s_1,s_2,d) has a first integral of the form $H(x^{s_2}, y^{s_1}/x^{s_2})$. On the other hand, if $\tilde{H}(x,y)$ is a first integral of the quasi-homogeneous system (1) of weight (s_1,s_2,d), then $\tilde{H}(u^{1/s_2},(uv)^{1/s_1})$ is a first integral of system (4).

We say that a polynomial $g(v)=\prod_{i=1}^{k}(v-\alpha_i)$ is *square-free* with $\alpha_i \neq \alpha_j$ for $i,j=1,\dots,k$ and $i\neq j$.

The following theorem is our main result. As usual $\mathbb{Q}^-$ denotes the set of negative rational numbers. Let α_i be the roots of the polynomial $g(v)$. Then we define $\gamma_i = f(\alpha_i)/\dot{g}(\alpha_i)$.

Theorem 3. *Consider the quasi-homogeneous system* (1) *of weight* (s_1,s_2,d) *which can be transformed by the change of variables in Lemma* 2 *into system* (4). *If the quasi-homogeneous system* (1) *has a* C^∞ *first integral, then the polynomial* $g(v)$ *is square-free and* $\gamma_i \in \mathbb{Q}^-$ *for* $i=1,\dots,k$. *Reciprocally, if* $g(v)$ *is square-free,* $\deg f < \deg g$ *and* $\gamma_i \in \mathbb{Q}^-$ *for* $i=1,\dots,k$, *then the quasi-homogeneous system* (1) *has a* C^∞ *first integral.*

Theorem 3 is proved in section 3.

Now we recall the following theorem proved in [16], which characterizes when the differential system (1) with an isolated singular point at the origin has a center at this point.

Theorem 4. *Assume that a system has an isolated singular point at the origin. Then it is a center if and only if there exists a first integral of class* C^∞ *with an isolated minimum at the origin.*

We will also use the following result (see [10] for a proof).

Lemma 5. *Consider a weight homogeneous polynomial differential system. If it has a singular point which is a center, then this singular point is at the origin of coordinates.*

Using Theorems 3 and 4 together with Lemma 5 we have the following result.

Theorem 6. *Consider the quasi-homogeneous system* (1) *of weight* (s_1,s_2,d) *which can be transformed by the change of variables in Lemma* 2 *into system* (4). *If system* (1) *has a center, then* $g(v)$ *is square-free and* $\gamma_i \in \mathbb{Q}^-$ *for* $i=1,2,\dots,k$.

It is clear that we only have to prove Theorem 3. To do it, we first recall the following result given in [10].

Lemma 7. *If the polynomial* $g(v)$ *is square-free,* $\deg f < \deg g$ *and* $P(x,y)$ *does not have* x *as a divisor, then* $1+\gamma_1+\gamma_2+\cdots+\gamma_k=0$.

We recall that if we are looking for centers at the origin of system (1) we always have that P does not have x as a divisor, and so Lemma 7 always holds in our case.

To prove Theorem 3 we will first prove the following theorem concerning the existence of C^∞ first integrals for system (4).

Theorem 8. *System* (4) *has a* C^∞ *first integral if and only if the polynomial* $g(v)$ *is square-free and* $\gamma_i \in \mathbb{Q}^-$ *for* $i=1,2,\dots,k$.

The proofs of Theorems 6 and 8 are given in section 3.

Weight-homogeneous polynomial differential systems have also been studied intensively from the point of view of integrability by many authors; see for instance [1, 9, 11, 12]. There are some works related with the study of weight-homogeneous polynomial differential systems and their relation with the center-focus problem. In [3] the authors characterized all cubic weight-homogeneous polynomial differential systems which have a center, and in [14] the authors characterized all weight-homogeneous polynomial differential systems of degrees 2, 3 and 4 which have a center. In [17] the authors studied the center-focus problem for the weight-homogeneous polynomial differential systems with degree 5.

The second main result of this paper is to classify all centers of all weight-homogeneous polynomial differential systems of degrees 6 and 7.

Theorem 9. *Every planar real weight-homogeneous polynomial differential system of degree* 6 *which is not homogeneous can be written as one of the following 20 systems:*

1) $\dot{x} = a_{0,6}y^6 + a_{1,4}xy^4 + a_{2,2}x^2y^2 + a_{3,0}x^3$
$\dot{y} = b_{0,5}y^5 + b_{1,3}xy^3 + b_{2,1}x^2y$
Minimum weight vector of the system: $(2, 1, 5)$

2) $\dot{x} = a_{0,6}y^6 + a_{2,3}x^2y^3 + a_{4,0}x^4$
$\dot{y} = b_{1,4}xy^4 + b_{3,1}x^3y$
Minimum weight vector of the system: $(3, 2, 10)$

3) $\dot{x} = a_{0,6}y^6 + a_{5,0}x^5$
$\dot{y} = b_{4,1}x^4y$
Minimum weight vector of the system: $(6, 5, 25)$

4) $\dot{x} = a_{0,6}y^6$
$\dot{y} = b_{5,0}x^5$
Minimum weight vector of the system: $(7, 6, 30)$

5) $\dot{x} = a_{0,6}y^6 + a_{1,3}xy^3 + a_{2,0}x^2$
$\dot{y} = b_{0,4}y^4 + b_{1,1}xy$
Minimum weight vector of the system: $(3, 1, 4)$

6) $\dot{x} = a_{0,6}y^6$
$\dot{y} = b_{4,0}x^4$
Minimum weight vector of the system: $(7, 5, 24)$

7) $\dot{x} = a_{0,6}y^6$
$\dot{y} = b_{3,0}x^3$
Minimum weight vector of the system: $(7, 4, 18)$

8) $\dot{x} = a_{0,6}y^6$
$\dot{y} = b_{2,0}x^2$
Minimum weight vector of the system: $(7, 3, 12)$

9) $\dot{x} = a_{0,6}y^6$
$\dot{y} = b_{1,0}x$
Minimum weight vector of the system: $(7, 2, 6)$

10) $\dot{x} = a_{1,5}xy^5 + a_{2,3}x^2y^3 + a_{3,1}x^3y$
$\dot{y} = b_{0,6}y^6 + b_{1,4}xy^4 + b_{2,2}x^2y^2 + b_{3,0}x^3$
Minimum weight vector of the system: $(2, 1, 6)$

11) $\dot{x} = a_{1,5}xy^5 + a_{3,2}x^3y^2$
$\dot{y} = b_{0,6}y^6 + b_{2,3}x^2y^3 + b_{4,0}x^4$

Minimum weight vector of the system: $(3, 2, 11)$

12) $\dot{x} = a_{1,5}xy^5 + a_{5,0}x^5$
$\dot{y} = b_{0,6}y^6 + b_{4,1}x^4y$
Minimum weight vector of the system: $(5, 4, 21)$

13) $\dot{x} = a_{1,5}xy^5$
$\dot{y} = b_{0,6}y^6 + b_{5,0}x^5$
Minimum weight vector of the system: $(6, 5, 26)$

14) $\dot{x} = a_{1,5}xy^5 + a_{2,2}x^2y^2$
$\dot{y} = b_{0,6}y^6 + b_{1,3}xy^3 + b_{2,0}x^2$
Minimum weight vector of the system: $(3, 1, 6)$

15) $\dot{x} = a_{1,5}xy^5 + a_{4,0}x^4$
$\dot{y} = b_{0,6}y^6 + b_{3,1}x^3y$
Minimum weight vector of the system: $(5, 3, 16)$

16) $\dot{x} = a_{1,5}xy^5 + a_{3,0}x^3$
$\dot{y} = b_{0,6}y^6 + b_{2,1}x^2y$
Minimum weight vector of the system: $(5, 2, 11)$

17) $\dot{x} = a_{1,5}xy^5 + a_{2,0}x^2$
$\dot{y} = b_{0,6}y^6 + b_{1,1}xy$
Minimum weight vector of the system: $(5, 1, 6)$

18) $\dot{x} = a_{1,5}xy^5$
$\dot{y} = b_{0,6}y^6 + b_{1,0}x$
Minimum weight vector of the system: $(6, 1, 6)$

19) $\dot{x} = a_{2,4}x^2y^4$
$\dot{y} = b_{1,5}xy^5 + b_{5,0}x^5$
Minimum weight vector of the system: $(5, 4, 22)$

20) $\dot{x} = a_{0,6}y^6 + a_{1,0}x$
$\dot{y} = b_{0,1}y$
Minimum weight vector of the system: $(6, 1, 1)$

Theorem 10. *Every planar real weight-homogeneous polynomial differential system of degree* 7 *which is not homogeneous can be written as one of the following* 23 *systems:*

21) $\dot{x} = a_{0,7}y^7 + a_{1,5}xy^5 + a_{2,3}x^2y^3 + a_{3,1}x^3y$
$\dot{y} = b_{0,6}y^6 + b_{1,4}xy^4 + b_{2,2}x^2y^2 + b_{3,0}x^3$
Minimum weight vector of the system: $(2, 1, 6)$

22) $\dot{x} = a_{0,7}y^7 + a_{3,3}x^3y^3$
$\dot{y} = b_{2,4}x^2y^4 + b_{5,0}x^5$
Minimum weight vector of the system: $(4, 3, 18)$

23) $\dot{x} = a_{0,7}y^7 + a_{6,0}x^6$
$\dot{y} = b_{5,1}x^5y$
Minimum weight vector of the system: $(7, 6, 36)$

24) $\dot{x} = a_{0,7}y^7$
$\dot{y} = b_{6,0}x^6$
Minimum weight vector of the system: $(8, 7, 42)$

25) $\dot{x} = a_{0,7}y^7 + a_{5,0}x^5$
$\dot{y} = b_{4,1}x^4y$

Minimum weight vector of the system: $(7,5,29)$

26) $\dot{x} = a_{0,7}y^7 + a_{1,3}xy^3$
$\dot{y} = b_{0,4}y^4 + b_{1,0}x$
Minimum weight vector of the system: $(4,1,4)$

27) $\dot{x} = a_{0,7}y^7 + a_{4,0}x^4$
$\dot{y} = b_{3,1}x^3y$
Minimum weight vector of the system: $(7,4,22)$

28) $\dot{x} = a_{0,7}y^7$
$\dot{y} = b_{4,0}x^4$
Minimum weight vector of the system: $(8,5,28)$

29) $\dot{x} = a_{0,7}y^7 + a_{3,0}x^3$
$\dot{y} = b_{2,1}x^2y$
Minimum weight vector of the system: $(7,3,15)$

30) $\dot{x} = a_{0,7}y^7 + a_{2,0}x^2$
$\dot{y} = b_{1,1}xy$
Minimum weight vector of the system: $(7,2,8)$

31) $\dot{x} = a_{0,7}y^7$
$\dot{y} = b_{2,0}x^2$
Minimum weight vector of the system: $(8,3,14)$

32) $\dot{x} = a_{1,6}xy^6 + a_{2,4}x^2y^4 + a_{3,2}x^3y^2 + a_{4,0}x^4$
$\dot{y} = b_{0,7}y^7 + b_{1,5}xy^5 + b_{2,3}x^2y^3 + b_{3,1}x^3y$
Minimum weight vector of the system: $(2,1,7)$

33) $\dot{x} = a_{1,6}xy^6 + a_{3,3}x^3y^3 + a_{5,0}x^5$
$\dot{y} = b_{0,7}y^7 + b_{2,4}x^2y^4 + b_{4,1}x^4y$
Minimum weight vector of the system: $(3,2,13)$

34) $\dot{x} = a_{1,6}xy^6 + a_{6,0}x^6$
$\dot{y} = b_{0,7}y^7 + b_{5,1}x^5y$
Minimum weight vector of the system: $(6,5,31)$

35) $\dot{x} = a_{1,6}xy^6$
$\dot{y} = b_{0,7}y^7 + b_{6,0}x^6$
Minimum weight vector of the system: $(7,6,37)$

36) $\dot{x} = a_{1,6}xy^6 + a_{2,3}x^2y^3 + a_{3,0}x^3$
$\dot{y} = b_{0,7}y^7 + b_{1,4}xy^4 + b_{2,1}x^2y$
Minimum weight vector of the system: $(3,1,7)$

37) $\dot{x} = a_{1,6}xy^6$
$\dot{y} = b_{0,7}y^7 + b_{5,0}x^5$
Minimum weight vector of the system: $(7,5,31)$

38) $\dot{x} = a_{1,6}xy^6$
$\dot{y} = b_{0,7}y^7 + b_{4,0}x^4$
Minimum weight vector of the system: $(7,4,25)$

39) $\dot{x} = a_{1,6}xy^6$
$\dot{y} = b_{0,7}y^7 + b_{3,0}x^3$
Minimum weight vector of the system: $(7,3,19)$

40) $\dot{x} = a_{1,6}xy^6 + a_{2,0}x^2$
$\dot{y} = b_{0,7}y^7 + b_{1,1}xy$
Minimum weight vector of the system: $(6,1,7)$

41) $\dot{x} = a_{1,6}xy^6$
$\dot{y} = b_{0,7}y^7 + b_{2,0}x^2$
Minimum weight vector of the system: $(7, 2, 13)$

42) $\dot{x} = a_{1,6}xy^6$
$\dot{y} = b_{0,7}y^7 + b_{1,0}x$
Minimum weight vector of the system: $(7, 1, 7)$

43) $\dot{x} = a_{0,7}y^7 + a_{1,0}x$
$\dot{y} = b_{0,1}y$
Minimum weight vector of the system: $(7, 1, 1)$

Theorems 9 and 10 are proved in section 4.

The third and fourth main results present a complete characterization of planar real weight-homogeneous polynomial differential systems with degrees 6 and 7 which are not homogeneous polynomial differential systems having a center at the origin. They are proved in section 5.

Theorem 11. *Real planar weight-homogeneous polynomial differential systems of degree* 6 *which are not homogeneous have no center at the origin.*

Theorem 12. *The unique planar real weight-homogeneous polynomial differential systems of degree* 7 *which are not homogeneous having a center are*

System 21 *with*

$$\frac{a_{3,1} + a_{2,3}r_i + a_{1,5}r_i^2 + a_{0,7}r_i^3}{a_{3,1} - 2b_{2,2} + 2a_{2,3}r_i - 4b_{1,4}r_i + 3a_{1,5}r_i^2 - 6b_{0,6}r_i^2 + 4a_{0,7}r_i^3} \in \mathbb{Q}^+,$$

where the r_i*'s are the four complex simple roots of the polynomial* $2b_{3,0} - a_{3,1}v + 2b_{2,2}v - a_{2,3}v^2 + 2b_{1,4}v^2 - a_{1,5}v^3 + 2b_{0,6}v^3 - a_{0,7}v^4$*, and with an isolated minimum at the origin.*

System 22 *with* $(3a_{3,3} - 4b_{2,4})^2 + 48a_{0,7}b_{5,0} < 0$ *and* $3a_{3,3} + 4b_{2,4} = 0$.

System 26 *with* $a_{1,3}^2 - 8a_{1,3}b_{0,4} + 16b_{0,4}^2 + 16a_{0,7}b_{1,0} < 0$.

We recall that Theorems 11 and 12 solve the center-focus problem for weight-homogeneous polynomial differential systems with degrees 6 and 7 that are not homogeneous. The case of homogeneous polynomials remains open in the sense that the specific families of centers are not known. However their characterization is well-known; see for instance Proposition 3 in [15].

2. Preliminary results

Suppose that the polynomial differential system (1) has its linear part of nilpotent form; that is, its Jacobian matrix is a nilpotent matrix. If the origin is a center it is called a *nilpotent center*. In this case using suitable coordinates, system (1) can be written as

$$\dot{x} = y + P_2(x, y), \qquad \dot{y} = Q_2(x, y), \tag{5}$$

where P_2 and Q_2 are polynomials of degree at least 2. The next theorem proved in [2] solves the monodromy problem for nilpotent singular points.

Theorem 13. *Consider system* (5) *and assume that the origin is an isolated singularity. Define the functions*

$$f(x) = Q_2(x, F(x)) = a\,x^{\alpha} + O(x^{\alpha+1}),$$
$$\phi(x) = \operatorname{div}(y + P_2(x,y), Q_2(x,y))|_{y=F(x)} = b\,x^{\beta} + O(x^{\beta+1}),$$

where $a \neq 0$, $\alpha \geq 2$, $b \neq 0$ *and* $\beta \geq 1$, *or* $\phi(x) \equiv 0$ *and the function* $y = F(x)$ *is the solution of* $y + P_2(x,y) = 0$ *passing through the origin. The origin of system* (5) *is a focus or a center if and only if* a *is negative,* α *is an odd number* ($\alpha = 2n-1$), *and one of the following three conditions holds:* $\beta > n-1$, $\beta = n-1$ *and* $b^2 + 4an$, *or* $\phi(x) \equiv 0$.

3. Proofs of Theorems 6 and 8

We first prove Theorem 8 and later the main Theorem 6.

Proof of Theorem 8. We separate the proof into two cases: the case $\deg f < \deg g$ and $\deg f \geq \deg g$.

Case 1 ($\deg f < \deg g$). Assume that system (4) has a C^∞ first integral in the variables (u,v). We suppose that the C^∞ function $H(u,v)$ is not flat at the origin; otherwise we translate to the origin a point where the first integral $H(u,v)$ is defined and it is not flat. Then in a convenient neighborhood of the origin the function $H(u,v)$ can be written as a power series in u in the form

$$H(u,v) = \sum_{l \geq 0} h_l(v) u^l. \tag{6}$$

We note that by taking the l-th derivative of the C^∞ function $H(u,v)$ with respect to the variable u at $u=0$ we obtain the function $h_l(v)$; consequently this function is C^∞. Then it must satisfy

$$u f(v) \frac{\partial H}{\partial u} + g(v) \frac{\partial H}{\partial v} = 0,$$

that is,

$$0 = \sum_{l \geq 0} l f(v) h_l(v) u^l + \sum_{l \geq 0} g(v) h_l'(v) u^l = \sum_{l \geq 0} (l f(v) h_l(v) + g(v) h_l'(v)) u^l.$$

Hence

$$h_0'(v) = 0, \quad \text{that is,} \quad h_0(v) = \text{constant},$$

and for $l \geq 1$,

$$l f(v) h_l(v) + g(v) h_l'(v) = 0, \quad \text{that is,} \quad \frac{h_l'(v)}{h_l(v)} = -l \frac{f(v)}{g(v)}. \tag{7}$$

We assume now that $g(v)$ is not square-free. Using an affine transformation of the form $v \to v + \alpha$ with $\alpha \in \mathbb{C}$ if it is necessary, we can assume that v is a multiple factor of $g(v)$ with multiplicity $\mu > 1$. Therefore we have that $g(v) = v^\mu r(v)$ with $r(0) \neq 0$. After this affine transformation we know that $f(0) \neq 0$ because f and g are coprime. Now we develop the right-hand side of (7) in simple fractions of v, that is,

$$-l \frac{f(v)}{g(v)} = \frac{c_\mu}{v^\mu} + \frac{c_{\mu-1}}{v^{\mu-1}} + \cdots + \frac{c_1}{v} + \frac{\alpha_1(v)}{r(v)} + \alpha_0(v),$$

where $\alpha_0(v)$ and $\alpha_1(v)$ are polynomials with $\deg \alpha_1(v) < \deg r(v)$ and $c_i \in \mathbb{C}$, for $i = 1, 2, \ldots, \mu$. Equating both expressions, we get that $c_\mu = -l\, f(0)/r(0) \neq 0$. Moreover as $\deg f < \deg g$ we know that $\alpha_0(v) \equiv 0$. Therefore equation (7) becomes

$$\frac{h_l'(v)}{h_l(v)} = \frac{c_\mu}{v^\mu} + \frac{c_{\mu-1}}{v^{\mu-1}} + \cdots + \frac{c_1}{v} + \frac{\alpha_1(v)}{r(v)},$$

with $c_\mu \neq 0$. Now if we integrate this expression we get

$$\begin{aligned} h_l(v) &= C \exp\left[\frac{c_\mu}{1-\mu}\frac{1}{v^{\mu-1}}\right] \\ &\quad \cdot \exp\left[\int \left(\frac{c_{\mu-1}}{v^{\mu-1}} + \ldots + \frac{c_1}{v} + \frac{\alpha_1(v)}{r(v)}\right) dv\right], \end{aligned}$$

where C is a constant of integration. The first exponential factor cannot be simplified with any part of the second exponential factor. Thus, the first integral (6) cannot be a C^∞ first integral having a center at the origin. So, g is square-free. Hence we can write $g(v) = \prod_{j=1}^k (v - \alpha_j)$ and so

$$\frac{f(v)}{g(v)} = \frac{\gamma_1}{v - \alpha_1} + \cdots + \frac{\gamma_k}{v - \alpha_k}.$$

Then

$$\int \frac{f(v)}{g(v)}\, dv = \sum_{j=0}^k \int \frac{\gamma_j}{v - \alpha_j}\, dv = \sum_{j=0}^k \gamma_j \log(v - \alpha_j),$$

and consequently

$$\exp\left(\int \frac{f(v)}{g(v)}\, dv\right) = \prod_{j=0}^k (v - \alpha_j)^{\gamma_j}. \tag{8}$$

Note that in order for this expression to be a C^∞ function we must have $\gamma_j \in \mathbb{Q}^+$ for all $j = 1, \ldots, k$. So $h_l(v)$ is a C^∞ function in v if $\gamma_j \in \mathbb{Q}^+$ for all $j = 1, \ldots, k$.

Conversely, assume that g is square-free and that $\gamma_i \in \mathbb{Q}^-$ for $i = 1, \ldots, k$. We will prove that $H(u, v) = u\varphi(v)$ where

$$\varphi(v) = (v - \alpha_1)^{-\gamma_1}(v - \alpha_2)^{-\gamma_2} \cdots (v - \alpha_k)^{-\gamma_k},$$

with $\gamma_i = f(\alpha_i)/\dot{g}(\alpha_i)$ is a first integral. Indeed, we have

$$uf(v)\frac{\partial H}{\partial u} + g(v)\frac{\partial H}{\partial v} = u(f(v)\varphi(v) + g(v)\dot{\varphi}(v)) = 0.$$

To see that this last expression is identically zero is equivalent to seeing that $\dot{\varphi}(v)/\varphi(v) = -f(v)/g(v)$. Recalling the expression of $\varphi(v)$ we have

$$-\frac{\dot{\varphi}(v)}{\varphi(v)} = \frac{\gamma_1}{v - \alpha_1} + \frac{\gamma_2}{v - \alpha_2} + \cdots + \frac{\gamma_k}{v - \alpha_k}.$$

Taking the common denominator and recalling that $g(v) = c(v - \alpha_1)(v - \alpha_2) \cdots (v - \alpha_k)$ we obtain

$$-\frac{\dot{\varphi}(v)}{\varphi(v)} = \frac{1}{g(v)} \sum_{i=1}^k c\gamma_i \prod_{j=1, j\neq i}^k (v - \alpha_i).$$

Now substituting the values of $\gamma_i = f(\alpha_i)/\dot{g}(\alpha_i)$ and taking into account that

$$\dot{g}(\alpha_i) = c \prod_{j=1, j\neq i}^{k} (\alpha_i - \alpha_j),$$

we obtain

$$-\frac{\dot{\varphi}(v)}{\varphi(v)} = \frac{1}{g(v)} \sum_{i=1}^{k} c f(\alpha_i) \prod_{j=1, j\neq i}^{k} \frac{v-\alpha_i}{\alpha_i - \alpha_j} = \frac{f(v)}{g(v)}. \tag{9}$$

Since $\deg f < \deg g$, the expression in the sum is the Lagrange polynomial which interpolates the k points $(\alpha_i, f(\alpha_i))$ for $i = 1, 2, \ldots, k$. Therefore, this polynomial is $f(v)$, and we conclude that the expression (9) is identically satisfied. Therefore, we obtain that

$$\varphi(v) = C(v-\alpha_1)^{-\gamma_1}(v-\alpha_2)^{-\gamma_2}\cdots(v-\alpha_k)^{-\gamma_k}.$$

Then as $H(u,v) = u\varphi(v)$ and $\gamma_i \in \mathbb{Q}^-$ for $i = 1, \ldots, k$, we have that $H(u,v)$ is a C^∞-function.

Case 2 ($\deg f \geq \deg g$). Again we have that system (4) admits a first integral of the form (6). Then we get that equation (7) holds. Since $\deg f \geq \deg g$ we consider the Euclidean divisions of $f(v)$ and $g(v)$, so we have

$$f(v) = q(v)g(v) + \psi(v),$$

where $\psi(v)$ cannot be zero taking into account that f and g are coprime and $\deg \psi < \deg g$. Hence equation (7) becomes

$$\frac{h'(v)}{h(v)} = -lq(v) + \frac{-l\psi(v)}{g(v)}. \tag{10}$$

Integrating (10) we obtain

$$h(v) = Ce^{-l\tilde{q}(v)} e^{-l\int \frac{\psi(v)}{g(v)}\, dv}, \tag{11}$$

where C is a constant of integration and $\tilde{q}\,'(v) = q(v)$, which is a polynomial. Therefore the first factor is a C^∞ function, and for the second factor we apply the results in Case 1 with ψ replaced by f and get the sufficiency part proved also in this case. For the necessity part we proceed in the same way. Assume that g is square-free and that $f(v) = q(v)g(v) + r(v)$. We will show that

$$H(u,v) = u \exp\Big(-\int q(v)\, dv\Big)(v-\alpha_1)^{-\gamma_1}\cdots(v-\alpha_k)^{-\gamma_k}, \tag{12}$$

with $\gamma_i = r(\alpha_i)/g'(\alpha_i) < 0$ for $i = 1, \ldots, k$, is a C^∞ function. Note that the function defined in (12) is just the function H in (6) with only a term different from zero when $l = 1$, thus satisfying (7) with $l = 1$ and (11) with $\tilde{q}(v) = \int q(v)\, dv$ and with $\exp\Big(\int \frac{r(v)}{g(v)}\, dv\Big)$ as in (8) (with r replaced by f). Now we show that it is indeed a first integral of system (4). We set $\phi(v) = (v-\alpha_1)^{\gamma_1}\cdots(v-\alpha_k)^{\gamma_k}$.

Note that

$$\begin{aligned}0 &= uf(v)\frac{\partial H}{\partial v} + g(v)\frac{\partial H}{\partial v}\\ &= uf(v)\exp(-\int q(v)\,dv)\phi(v) + ug(v)(-q(v)\phi(v) - \phi'(v))\exp(-\int q(v)\,dv)\\ &= u\exp(-\int q(v)\,dv)(f(v)\phi(v) - g(v)q(v)\phi(v) - g(v)\phi'(v))\\ &= u\exp(-\int q(v)\,dv)(r(v)\phi(v) - g(v)\phi'(v)).\end{aligned}$$

To see that this last expression is identically zero it is equivalent to see that

$$\frac{\varphi'(v)}{\varphi(v)} = \frac{r(v)}{g(v)}.$$

Proceeding as we did in Case 1, replacing v by h, we get that this is indeed the case. This completes the proof of the theorem. □

Proof of Theorem 3. We have already pointed out that in order to prove Theorem 6 it is enough to prove Theorem 3 with the assumption that $1+\gamma_1+\gamma_2+\cdots+\gamma_k = 0$. Assume that system (1) has a C^∞ first integral $H = H(x, y)$. By well-known results we have that it is a first integral which is a quasi-homogeneous function of weight exponents (s_1, s_2) and weight-degree d where we can take $d = s_1s_2m$ with $m \in \mathbb{N}$. Let x^iy^j be a monomial with a nonzero coefficient of H. By quasi-homogeneity we have $s_1i+s_2j = m$, which implies that $s_1i+s_2j = s_1s_2\tilde{m}$, that is $s_2j = s_1(s_2\tilde{m}-i)$. As s_1 and s_2 are coprime, we deduce that j is a multiple of s_1, that is, $j = s_1\tilde{j}$ with $\tilde{j} \in \mathbb{N} \cup \{0\}$. Moreover the change of variables described in Lemma 2 implies that $H(u^{1/s_2}, (uv)^{1/s_1})$ is a first integral of system (4), and by homogeneity we have

$$H(u^{1/s_2}, (uv)^{1/s_1}) = H(\alpha^{s_1}, \alpha^{s_2}v^{1/s_1}) = \alpha^{s_1s_2\tilde{m}}H(1, v^{1/s_1}) = u^{\tilde{m}}H(1, v^{1/s_1}).$$

As $H(x, y)$ is a C^∞ function and all the monomials x^iy^j satisfy that j is a multiple of s_1, we have that $h(v) = H(1, v^{1/s_1})$ is also a C^∞ function in v. Therefore we have that $u^{\tilde{m}}h(v)$ is a C^∞ first integral of system (4). In view of Theorem 8 we have that $g(v)$ is square-free and $\gamma_i \in \mathbb{Q}^-$ for $i = 1, 2, \ldots, k$.

Conversely if we assume that $g(v)$ is square-free, $\gamma_i \in \mathbb{Q}^-$ for $i = 1, 2, \ldots, k$ we have that system (4) has a C^∞ first integral. As $\gamma_i \in \mathbb{Q}^-$ there exists $N, n_1, n_2, \ldots, n_k$ such that $\gamma_i = -n_i/N$ for $i = 1, 2, \ldots, k$. Note that by the proof of Theorem 8, we can write the first integral as

$$H(u, v) = u^{-1}e^{\lambda\tilde{g}(v)}(v-\alpha_1)^{-n_1/N}\cdots(v-\alpha_k)^{-n_k/N},$$

where $\lambda = 0$ if $\deg f < \deg g$, and $\lambda = 1$ if $\deg f \ge \deg g$. So

$$\tilde{H}(u, v) = u^Ne^{-N\lambda\tilde{g}(v)}(v-\alpha_1)^{n_1}\cdots(v-\alpha_k)^{n_k}$$

is a C^∞ function of system (4). Recall that $1+\gamma_1+\cdots+\gamma_k = 0$, and so $N - n_1 - \cdots - n_k = 0$. Undoing the change of variables in Lemma 2 we get

$$\begin{aligned}\tilde{H}\Big(x^{s_2}, \tfrac{y^{s_1}}{x^{s_2}}\Big) &= x^{s_2N}e^{-N\lambda\tilde{g}(y^{s_1}/x^{s_2})}\Big(\tfrac{y^{s_1}}{x^{s_2}} - \alpha_1\Big)^{n_1}\cdots\Big(\tfrac{y^{s_1}}{x^{s_2}} - \alpha_k\Big)^{n_k}\\ &= x^{s_2(N-n_1-\cdots-n_k)}e^{-N\lambda\tilde{g}(y^{s_1}/x^{s_2})}(y^{s_1}-\alpha_1x^{s_2})^{n_1}\cdots(y^{s_1}-\alpha_kx^{s_2})^{n_k}.\end{aligned}$$

So in order for $\tilde{H}$ to be a C^∞ function we must have $\lambda = 0$. This concludes the proof. □

4. Proofs of Theorems 9 and 10

We note that from the definition of weight-homogeneous planar polynomial differential systems of weight degree d, the exponents of u and v of any monomial $x^u y^v$ of P and Q are such that they satisfy, respectively, the relations

$$s_1 u + s_2 v = s_1 + d, \quad s_1 u + s_2 v = s_2 + d.$$

We can always assume that $s_1 > s_2$; otherwise we exchange the coordinates x and y. Additionally, we only consider the cases in which P and Q are coprime, since otherwise they can be treated as weight-homogeneous with lower degree.

We also use the following lemma (see [10] for a proof).

Lemma 14. *Given a quasi-homogeneous system of weight* (s_1, s_2, d)*, we can suppose without restriction that* s_1 *and* s_2 *are coprime.*

Using this and Proposition 10 in [6] if a system is weight-homogeneous but not homogeneous of degree n with the weight vector (s_1, s_2, d) and $d > 1$, we find that the system has the minimal vector

$$\tilde{w} = \left(\frac{t+k}{s}, \frac{k}{s}, 1 + \frac{(p-1)t + (n-1)k}{s}\right),$$

with $t \in \{1, 2, \ldots, p\}$, where $p \in \{0, 1, \ldots, n-1\}$ and $k \in \{1, \ldots, n-p-t+1\}$ satisfy

$$s_1 = \frac{(t+k)(d-1)}{(p-1)t + (n-1)k}, \quad s_2 = \frac{k(d-1)}{(p-1)t + (n-1)k},$$

and $s = \gcd(t, k)$. Using this and again [6] we get that the weight-homogeneous but not homogeneous polynomial differential systems of degree n with weight vector (s_1, s_2, d) can be written in the form

$$X_{ptk} = X_n^p + X_{n-t}^{ptk} + \sum_{\mathcal{D}} X_{n-s}^{psk_s},$$

where

$$\mathcal{D} = \{s \in \{1, \ldots, n-p\} \setminus \{t\},\ k_s t = ks,\ k_s \in \{1, \ldots, n-s-p+1\}\},$$

and

$$X_n^p = \left(a_{p,n-p} x^p y^{n-p}, b_{p-1,n-p+1} x^p y^{n-p+1}\right)$$

is the homogeneous part of degree n with coefficients not simultaneously vanishing and

$$X_{n-t}^{ptk} = \left(a_{p+k,n-t-p-k} x^{p+k} y^{n-t-p-k}, b_{p+k-1,n-t-p-k+1} x^{p+k-1} y^{n-t-p-k+1}\right).$$

Moreover, in order that X_{ptk} be weight-homogeneous but not homogeneous of degree n we must have X_n^p not identically zero and at least one of the other elements not identically vanishing.

Using all these previous results, we can prove that with degree 6 we have the 20 families of systems given in Theorem 9 and with degree 7 we have the 20 families of systems given in Theorem 10. In fact in [7] the authors have implemented the algorithm of [10] for computing the weight-homogeneous polynomial differential systems of an arbitrary degree. We have checked the systems of Theorems 9 and 10 using such implementation.

5. Proofs of Theorems 11 and 12

In view of Lemma 5 it is only necessary to look for centers at the origin of coordinates.

Proof of Theorem 11. By Theorem 9 there are 20 families of weight-homogeneous polynomials but not homogeneous with the minimal vector $w = (s_1, s_2, 6)$ with $s_1 > s_2$. We can check that systems $1, 2, 3, 5, 20$ have the invariant line $y = 0$, whereas systems $10, 11, 12, 13, 14, 15, 16, 17, 18, 19$ have the invariant line $x = 0$. So their origin cannot be a center. It remains to study systems 6–9. Note that system 6 has the first integral $H = b_{40}x^5/5 - a_{0,6}y^7/7$, system 7 has the first integral $H = b_{30}x^4/4 - a_{0,6}y^7/7$, for system 8 a first integral is $H = b_{20}x^3/3 - a_{0,6}y^7/7$, and system 9 has the first integral $H = b_{10}x^2/2 - a_{0,6}y^7/7$. Therefore any system 6–9 cannot have a center at the origin because the curves $H = h$ near the origin are not closed. This completes the proof of Theorem 11. □

Proof of Theorem 12. By Theorem 10 there are 23 families of weight-homogeneous polynomials but not homogeneous with the minimal vector $w = (s_1, s_2, 7)$ with $s_1 > s_2$. We can check that systems $23, 25, 27, 29, 30, 32, 33, 34, 43$ have the invariant line $y = 0$, whereas systems $35, 36, 37, 38, 39, 40, 41, 42$ have the invariant line $x = 0$. So their origin cannot be a center. System 24 has the first integral $H = b_{60}x^7/7 - a_{0,7}y^8/8$, system 28 has the first integral $H = b_{40}x^5/5 - a_{0,7}y^8/8$, and for system 31 a first integral is $H = b_{20}x^3/3 - a_{0,7}y^8/8$. So, clearly systems 24, 28 and 31 cannot have a center at the origin.

In short we are left with studying systems 21, 22 and 26.

Lemma 15. *System* 21 *has a center at the origin if and only if*

$$\frac{a_{3,1} + a_{2,3}r_i + a_{1,5}r_i^2 + a_{0,7}r_i^3}{a_{3,1} - 2b_{2,2} + 2a_{2,3}r_i - 4b_{1,4}r_i + 3a_{1,5}r_i^2 - 6b_{0,6}r_i^2 + 4a_{0,7}r_i^3} \in \mathbb{Q}^+,$$

where the r_i's are the four simple roots of the polynomial $2b_{3,0} - a_{3,1}v + 2b_{2,2}v - a_{2,3}v^2 + 2b_{1,4}v^2 - a_{1,5}v^3 + 2b_{0,6}v^3 - a_{0,7}v^4$, and with an isolated minimum at the origin.

Proof. Following [9] we first do the change of variables $u = x$ and $v = y^2/x$ and the rescaling of time which transforms a quasi-homogeneous system 21 into a polynomial system of the form (4) which admits a first integral of the form (6), where the function $h(v)$ satisfies the differential equation (7). We assume now that $V(u, v) = ug(v)$ is not square-free. Recalling the proof of Theorem 8 we obtain that a first integral cannot be a C^∞ first integral having a center at the origin, and the same happens for the transformed first integral in the original variables (x, y) of system 21.

Now we assume that $V(u, v) = ug(v)$ is square-free, i.e., that $g(v)$ is square-free. Using the change of variables (u, v) we arrive at a first integral of the form

$$H(x, y) = (y^2 - r_1x)^{\lambda_1}(y^2 - r_2x)^{\lambda_2}(y^2 - r_3x)^{\lambda_3}(y^2 - r_4x)^{\lambda_4},$$

where r_i are the four complex simple roots of the polynomial $g(v) = 2b_{3,0} - a_{3,1}v + 2b_{2,2}v - a_{2,3}v^2 + 2b_{1,4}v^2 - a_{1,5}v^3 + 2b_{0,6}v^3 - a_{0,7}v^4$ and

$$\lambda_i = \frac{a_{3,1} + a_{2,3}r_i + a_{1,5}r_i^2 + a_{0,7}r_i^3}{a_{3,1} - 2b_{2,2} + 2a_{2,3}r_i - 4b_{1,4}r_i + 3a_{1,5}r_i^2 - 6b_{0,6}r_i^2 + 4a_{0,7}r_i^3},$$

where the denominator is in fact $-g'(r_i)$. Hence the condition for having a C^∞ first integral defined at the origin is $\lambda_i \in \mathbb{Q}^+$. Moreover to have a center at the origin it is also necessary to have an isolated minimum at the origin.

In fact doing the blowup $(X,Y) = (x, y^2)$ and a scaling of time, system 21 becomes a homogeneous system

$$\begin{aligned} \dot{X} &= a_{3,1}X^3 + a_{2,3}X^2Y + a_{1,5}XY^2 + a_{0,7}Y^3, \\ \dot{Y} &= b_{3,0}X^3 + b_{2,2}X^2Y + b_{1,4}XY^2 + b_{0,6}Y^3. \end{aligned} \tag{13}$$

Taking now classical polar coordinates the $G(\theta)$ takes the form

$$\begin{aligned} G(\theta) =& 2b_{3,0}\cos^4 t - (a_{3,1} - 2b_{2,2})\cos^3 t \sin t - (a_{2,3} - 2b_{14})\cos^2 t \sin^2 t \\ & - (a_{1,5} - 2b_{0,6})\cos t \sin^3 t - a_{0,7}\sin^4 t. \end{aligned}$$

Applying Lemma 1 we obtain that $G(\theta)$ must have no real roots, which is the same condition imposed on $g(v)$ above. The second condition of Lemma 1 is equivalent to having an isolated minimum at the origin. □

Lemma 16. *System 22 has a center at the origin if and only if* $(3a_{3,3} - 4b_{2,4})^2 + 48a_{0,7}b_{5,0} < 0$ *and* $3a_{3,3} + 4b_{2,4} = 0$.

Proof. System 22 has the first integral

$$\begin{aligned} H(x,y) = & \left(\left(-3a_{3,3} + 4b_{2,4} + \sqrt{(3a_{3,3} - 4b_{2,4})^2 + 48a_{0,7}b_{5,0}}\right)x^3 - 6a_{0,7}y^4\right)^{\lambda_1} \\ & \cdot \left(\left(3a_{3,3} - 4b_{2,4} + \sqrt{(3a_{3,3} - 4b_{2,4})^2 + 48a_{0,7}b_{5,0}}\right)x^3 + 6a_{0,7}y^4\right)^{\lambda_2}, \end{aligned}$$

where $\lambda_1 = 1 + \frac{3a_{3,3}+4b_{2,4}}{\sqrt{(3a_{3,3}-4b_{2,4})^2+48a_{0,7}b_{5,0}}}$ and $\lambda_2 = 1 - \frac{3a_{3,3}+4b_{2,4}}{\sqrt{(3a_{3,3}-4b_{2,4})^2+48a_{0,7}b_{5,0}}}$, when $(3a_{3,3} - 4b_{2,4})^2 + 48a_{0,7}b_{5,0} \neq 0$. Therefore applying Theorem 4 in order to have a C^∞ first integral we obtain the condition

$$\frac{3a_{3,3} + 4b_{2,4}}{\sqrt{(3a_{3,3} - 4b_{2,4})^2 + 48a_{0,7}b_{5,0}}} \in \mathbb{Q}, \text{ with } \left|\frac{3a_{3,3} + 4b_{2,4}}{\sqrt{(3a_{3,3} - 4b_{2,4})^2 + 48a_{0,7}b_{5,0}}}\right| < 1.$$

Now we apply Lemma 1, take the weighted polar coordinates $x = r^4\cos\theta$, $y = r^3\sin\theta$ and compute $G(\theta)$. The discriminant of $G(\theta)$ in order to have no real roots is $(3a_{3,3} - 4b_{2,4})^2 + 48a_{0,7}b_{5,0} < 0$. Consequently the above condition implies that $3a_{3,3} + 4b_{2,4} = 0$. Under this condition the first integral is polynomial of the form

$$H(x,y) = -4b_{5,0}x^6 + 6a_{3,3}x^3y^4 + 3a_{0,7}y^8$$

and has an isolated minimum at the origin.

In fact we can do the blowup $(X,Y) = (x^3, y^4)$ and a scaling of time, and system 22 becomes a homogeneous system

$$\dot{X} = 3(a_{3,3}X + a_{0,7}Y), \quad \dot{Y} = 4(b_{5,0}X + b_{2,4}Y). \tag{14}$$

Taking now classical polar coordinates the $G(\theta)$ takes the form

$$\tilde{G}(\theta) = -4b_{5,0}\cos^2 t + (3a_{3,3} - 4b_{2,4})\cos t \sin t + 3a_{0,7}\sin^2 t.$$

Applying Lemma 1 we obtain that the condition that $\tilde{G}(\theta)$ has no real roots is the same condition that $G(\theta)$ has no real roots, which is given by $(3a_{3,3} - 4b_{2,4})^2 + 48a_{0,7}b_{5,0} < 0$. The second condition of Lemma 1 gives the condition $3a_{3,3}+4b_{2,4} = 0$ in agreement with the condition found before.

For the case $(3a_{3,3} - 4b_{2,4})^2 + 48a_{0,7}b_{5,0} = 0$ as $a_{0,7} \neq 0$, because otherwise system 22 has the invariant curve $x = 0$, we take $b_{5,0} = -(3a_{3,3} - 4b_{2,4})^2/(48a_{0,7})$, and a first integral is

$$H(x,y) = e^{-\frac{(3a_{3,3}+4b_{2,4})x^3}{3a_{3,3}x^3-4b_{2,4}x^3+6a_{0,7}y^4}}(3a_{3,3}x^3 - 4b_{2,4}x^3 + 6a_{0,7}y^4),$$

which is analytic if and only if $3a_{3,3} + 4b_{2,4} = 0$. However in this case the $G(\theta)$ has real roots and system 22 does not have a center. This last case shows that we can have a system with a C^∞ first integral and without a center at the origin. ☐

Lemma 17. *System* 26 *has a center at the origin if and only if* $a_{1,3}^2 - 8a_{1,3}b_{0,4} + 16b_{0,4}^2 + 16a_{0,7}b_{1,0} < 0$.

Proof. We can consider that $b_{1,0} \neq 0$ because if $b_{1,0} = 0$, then system 26 has the invariant curve $y = 0$ and it cannot have a center at the origin. Hence system 26 has a nilpotent singular point at the origin, and we interchange $x \leftrightarrow y$ to put system 26 into its classical form and we obtain

$$\dot{x} = b_{1,0}y + b_{0,4}x^4, \qquad \dot{y} = a_{0,7}x^7 + a_{1,3}x^3y. \tag{15}$$

Now we do the rescaling $y = Y/b_{1,0}$, and system (15) takes the form

$$\dot{x} = y + b_{0,4}x^4, \qquad \dot{y} = a_{0,7}\, b_{1,0}\, x^7 + a_{1,3}x^3y, \tag{16}$$

replacing Y by y. Now we apply Theorem 13 and obtain that the solution is given by $y = F(x) = -b_{0,4}x^4$ and

$$f(x) = -(a_{1,3}b_{0,4} - a_{0,7}b_{1,0})x^7 + O(x^8),$$
$$\phi(x) = (a_{1,3} + 4b_{0,4})x^3 + O(x^4).$$

Therefore we have $a = -(a_{1,3}b_{0,4} - a_{0,7}b_{1,0})$ with $\alpha = 7$, $b = (a_{1,3} + 4b_{0,4})$ with $\beta = 3$. Here we have $\alpha = 2 \cdot 4 - 1$ and $\beta = n - 1 = 4 - 1 = 3$. Hence the condition that $b^2 - 4an$ is $a_{1,3}^2 - 8a_{1,3}b_{0,4} + 16b_{0,4}^2 + 16a_{0,7}b_{1,0} < 0$ is the condition needed in order that system (16) has a focus or a center.

In this case the origin is a center of (16) and also of (15), because the system above is reversible; i.e. it is invariant by the symmetry $(x,y,t) \to (-x,y,-t)$. ☐

This completes the proof of the theorem. ☐

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