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THE NUMBER OF REAL OVALS OF A CYCLIC COVER OF THE SPHERE

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Abstract. A compact Riemann surface X which is a cyclic cover of degree n of the Riemann sphere has a defining equation of the form $y^n = f(x)$ where f is a complex polynomial. If f has real coefficients, then complex conjugation σ leaves X invariant. The fixed point set of σ in X consists of a disjoint union of simple closed curves, called *ovals*. In this paper we determine a procedure to count the exact number of ovals of σ in terms of the multiplicities of the real roots of f.

1. Introduction

Each compact Riemann surface X of genus $g \geq 2$ can be defined by an algebraic equation $F(x, y) = 0$ where $F \in \mathbb{C}[x, y]$ is a polynomial. Some results concerning the relation between Riemann surfaces and defining equations can be found in [11, Section 2]. If F has real coefficients, then complex conjugation σ leaves X invariant. More generally, if X admits an anticonformal involution $\sigma : X \to X$, which we call a symmetry, then X can be defined by an algebraic equation $G(x, y) = 0$ with $G \in \mathbb{R}[x, y]$ and, in this description, σ becomes complex conjugation. The study of symmetries of compact Riemann surfaces and their relation with real algebraic curves is a topic of increasing interest; see, for instance, [1] and the references given therein. The fixed point set of a symmetry σ consists of a disjoint union of closed curves, called *ovals*, homeomorphic to the circle. A result of Harnack states that the number $\|\sigma\|$ of ovals fixed by σ is bounded above by $g + 1$. A natural problem consists of determining the exact number of such ovals. This is a classical problem which has generated a lot of attention in real algebraic geometry and Riemann surface theory.

A fruitful technique for dealing with this problem involves the description of X as the quotient $\mathcal{H}/\Gamma$ of the hyperbolic plane $\mathcal{H}$ under the action of a surface Fuchsian group Γ and the use of combinatorial methods of non-euclidean crystallographic groups; see e.g., [2, Chapter 2]. For instance, this is the method used by Gromadzki in [5] to refine the seminal study of bounds for the sum of the ovals of all symmetries on a compact Riemann surface done previously by Natanzon employing topological methods, [8]. These methods are also used to describe the topological invariants of a symmetry in [9]. Related to the type of surfaces we deal with here, it is worth mentioning the paper [4], where, using again these combinatorial methods, Costa

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and Izquierdo determine the topological features of all possible symmetries of p-cyclic covers of the Riemann sphere for p prime. More recently, the automorphism groups of these types of covers have been studied by Izquierdo and Shaska in [7].

As far as we know, there are few results using algebraic equations to count ovals of a symmetry. This is our approach here: we use the algebraic description of symmetric n-cyclic covers of the sphere, namely, $y^n - f(x) = 0$ where f is a real polynomial, to compute the exact number of ovals fixed by complex conjugation σ. As pointed out above, there may be other symmetries in a cyclic cover of the Riemann sphere, but our results only apply to complex conjugation. In order to deal with another symmetry σ' we first have to find a birational model $w^n - g(z) = 0$ of the surface where σ' becomes complex conjugation. This is done for the Kulkarni surface in [10].

Obviously a point $(x, y) = (r, s)$ is fixed by σ if both r and s are real. If n is odd, then for each real r there is a unique real value of s, so there is one real oval in X which lies above the real line and $x = \infty$. So all through this document, we will assume that n is even. First we factor f into a product of monic linear factors with real roots and a product of monic quadratic factors with conjugate non-real roots; let $g(x)$ denote the latter product times the absolute value of the leading coefficient of f. So assume $f(x) = cg(x)(x - a_1)^{d_1 m_1}(x - a_2)^{d_2 m_2} \cdots (x - a_k)^{d_k m_k}$ where $c = \pm 1$, each a_i is real and $d_i = \gcd(n, d_i m_i)$. For any real r we have $g(r) > 0$, so the sign of $f(x)$ depends on the sign of $c(x - a_1)^{d_1 m_1}(x - a_2)^{d_2 m_2} \cdots (x - a_k)^{d_k m_k}$. For each real r for which $f(r) > 0$, there are two numbers, s^+ and s^-, positive and negative respectively, such that $(x, y) = (r, s^\pm)$ are solutions to $y^n - f(x) = 0$. To determine how many ovals are fixed by σ, we need to see how the branches (r, s^+) and (r, s^-) join at points of X lying over $x = a_1, \ldots, x = a_k$ and $x = \infty$. This is done in Section 3, while the main results are stated and proved in Section 4. Section 2 is dedicated to some preliminary results.

2. Preliminaries

With the above notation, observe that if $d_i m_i > 1$, then $(a_i, 0)$ is a singular point of the defining equation $y^n - f(x) = 0$ of X. So, in order to get an exact picture of X near $(a_i, 0)$ we need to determine a change of coordinates $(x, y) \to (t, w)$ such that the corresponding points are non-singular in the new coordinates. This can be found in [11], but we include it in the following lemma for the reader's convenience.

Lemma 2.1. *Let $y^n - (x - a)^{dm} h(x) = 0$ be a real defining equation for X where $d = \gcd(n, dm)$ and $h(x)$ is a polynomial such that $h(a) \neq 0$. Then there exist d points of X which lie over $x = a$. Let u and v be integers such that $mu + vn/d = 1$. Then the real change of coordinates $(x, y) \to (t, w)$ where*

$$w = y^{n/d}/(x - a)^m, \quad t = y^u (x - a)^v \tag{1}$$

separates each of the points and yields a local parameter at each point. In addition, the d points lying over $(a, 0)$ have the form $(t, w) = (0, \zeta e^{2\pi i k/d})$, where ζ is real and satisfies $\zeta^d = h(a)$.

Proof. Clearly, $w^d - h(x) = 0$ and at any point P of X lying over $x = a$, we have $w^d(P) = h(a) \neq 0$. Therefore the function w has d different possible values when $x = a$, and this means that there must be d different points of X at which w takes on these values. Therefore there are at least d points of X that lie over $x = a$. Let P be any one of them. Then $\mathrm{ord}_P(w) = 0$ and so $(n/d)\mathrm{ord}_P(y) = m\,\mathrm{ord}_P(x - a)$.

Using that n/d and m are relatively prime, we see that $\mathrm{ord}_P(x-a)$ is at least n/d. Since the degree of the zero divisor of $x-a$ is n, this means that there must be exactly d points lying over $x=a$ and that, at each of them, $x-a$ has order n/d and y has order m. Then the function $t = y^u(x-a)^v$ has order $mu+vn/d=1$, so t is a local parameter. On the other hand, $t^m w^v = y$ and $t^{n/d}/w^u = x-a$, as is easy to check. So $\mathbb{C}(w,t) = \mathbb{C}(x,y)$, and therefore X can be defined in terms of w and t. Since $w^d - h(x) = 0$, we obtain a defining equation of the form $w^d - h(a+t^{n/d}/w^u)$. All points that lie over the point $(x,y)=(a,0)$ are of the form $(t,w) = (0, \zeta e^{2\pi i k/d})$, where ζ is real and satisfies $\zeta^d = h(a)$. □

To study points lying over $x=\infty$ we make the real change of coordinates

$$t = 1/x, \quad w = y^{n/d}/x^{\deg(f)/d}, \tag{2}$$

where $d = \gcd(n, \deg(f))$, and study points over $t=0$. A proof similar to the above yields the following; see [11].

Lemma 2.2. *Let $y^n = f(x)$ be a defining equation for X. Then there are d points of X lying over $x=\infty$, where $d = \gcd(n,\deg(f))$.*

As a consequence, n is a divisor of $\deg(f)$ if and only if $x=\infty$ is not a branch point of the projection $(x,y)\mapsto x$. This last can be easily achieved by moving a real non-branch point to ∞. For simplicity, we will assume from now on that ∞ is a not a branch point (although our results can be easily modified to cover the opposite case). In particular, $\deg(f)$ will be even.

If f has no root of odd multiplicity, then f does not change sign and so either $f(x)\geq 0$ for all real x if f is monic or $f(x)\leq 0$ if $-f$ is monic. In the latter case complex conjugation σ fixes no oval, while in the former it fixes either one or two ovals; see Theorem 4.1.

In case f changes sign the next lemma shows that we can assume that f is a monic polynomial.

Lemma 2.3. *Assume n is even, f is real, $-f$ is monic, and f has a real root of odd multiplicity. Assume $y^n = f(x)$ is a defining equation for X with $n|\deg(f)$. Then there is another defining equation of X of the form $w^n = g(t)$ with g monic and real and such that $t=\infty$ is not a branch point.*

Proof. Let us write $f(x) = -\prod_{j=1}^k (x-a_j)^{m_j}$. The hypothesis on f implies that there exists $r\in\mathbb{R}$ such that $f(r)>0$. Let us consider the real change of coordinates given by

$$t := \frac{1}{x-r}, \quad w := \frac{y}{c(x-r)^{\deg(f)/n}}, \quad \text{where } c\in\mathbb{R} \text{ is such that } c^n = f(r).$$

It is straightforward to check that w^n equals the real monic polynomial $g(t) = \prod_{j=1}^k \left(t - \frac{1}{a_j - r}\right)^{m_j}$. Clearly, $\mathbb{C}(x,y) = \mathbb{C}(t,w)$ and so $w^n = g(t)$ is a defining equation for X. Observe that $t=\infty$ is not a branch point in this model since $x=r$ is not a branch point in the (x,y) model. □

3. How the branches join

For the rest of the paper, assume $y^n - f(x) = 0$ is a real defining equation for the Riemann surface X, where f is monic and n is even and divides the degree of f. As said above, for each real r for which $f(r)>0$, there are two numbers, s^+

and s^-, positive and negative respectively, such that $(x, y) = (r, s^\pm)$ are solutions to $y^n - f(x) = 0$. To determine how many ovals are fixed by σ, we need to see how the branches (r, s^+) and (r, s^-) join at points of X lying over $x = a_1, \ldots, x = a_k$ and $x = \infty$. We first deal with finite points.

3.1. **Behavior over $\boldsymbol{x = a}$.** Let a be a real root of f with multiplicity dm where $d = \gcd(n, dm)$. By Lemma 2.1, there are d points of X lying over $x = a$. A consequence of this is that **if $\boldsymbol{d}$ is odd**, then there is only one real valued point lying over a. Hence, when $f(r) > 0$ near a, the two branches of solutions (r, s^+) and (r, s^-) must meet at the real point lying over a. Observe that, since d is odd and n is even, the multiplicity dm is odd so $f(x)$ changes sign at $x = a$. Therefore there are two branches on one side of $x = a$ (and they meet over it), and there are no fixed points in a neighborhood of the other side of $x = a$ (until a new root of odd multiplicity is reached).

We now come to the interesting case, where **$\boldsymbol{d}$ is even**; that is, a has even multiplicity. In this case there are an even number d of points lying over $x = a$. If $f(x) < 0$ near a, then there is no real oval through any of the d points lying over $x = a$. But if $f(x) > 0$ near a, then two of them are real, and, moreover, there are two branches of real points on either side of $x = a$ because the sign of $f(x)$ does not change at $x = a$. In order to see how they join we use the change of coordinates (1). It is important to note that if x and y are real, then the new coordinates t and w given by (1) are real also. Therefore real solutions in the (x, y) plane correspond to real solutions in the (t, w) plane.

If $r \neq a$ but $f(r) > 0$ for a real number r in a small neighborhood of a, then $w^d = y^n/(r - a)^{dm} = f(r)/(r - a)^{dm} > 0$, so there are d solutions for w, and two of them are real. We denote the positive and negative real solutions by Q^+ and Q^- respectively. If a fixed oval goes through Q^+ or Q^-, then in a neighborhood of either point, the local parameter $t = y^u(x - a)^v$ changes sign. We are using the fact that fixed ovals do not have endpoints and that the local parameter t provides a local homeomorphism of the Riemann surface at the points Q^+ and Q^-. On the other hand, the function w does not vanish at Q^+ or Q^- and keeps the same sign in a small neighborhood of each of these points. Defining $\operatorname{sign}(w)$ to be "+" if $w \geq 0$ and "−" otherwise, we see that the fixed oval goes through Q^λ where $\lambda := \operatorname{sign}(x - a)^m \operatorname{sign}\big(y^{n/d}\big) = \operatorname{sign}(w)$. Let us analyze how the two branches of solutions on either side of $x = a$ join at each Q^λ. Clearly, this depends entirely on the parity of m and n/d and the signs of y and $x - a$. Since $mu + vn/d = 1$, we have to consider the following three cases.

- m even, n/d odd. If $x > a$ and $y > 0$, then $\lambda = \operatorname{sign}(x - a)^m \operatorname{sign}(y)^{n/d} =$ " + ", so this branch goes through Q^+. The same happens if $x < a$ and $y > 0$. On the other hand, the branches $(x < a,\ y < 0)$ and $(x > a,\ y < 0)$ go through Q^-. See Figure 1 where we have separated the ovals to show that Q^+ and Q^- both lie over $x = a$.
- m and n/d odd. In this case the branches $(x > a,\ y > 0)$ and $(x < a,\ y < 0)$ meet at Q^+ and the branches $(x > a,\ y < 0)$ and $(x < a,\ y > 0)$ meet at Q^-; see Figure 1.
- m odd, n/d even. Here the two branches of points with $x > a$ join at Q^+, and the two branches of point with $x < a$ join at Q^-; see Figure 2.

In the first two cases, points on a real branch to the right of $x = a$ join with points to the left of $x = a$ and vice-versa. We say that $(a, 0)$ does not mark the end of an oval. So we may ignore the first two types of points when counting the number of ovals.

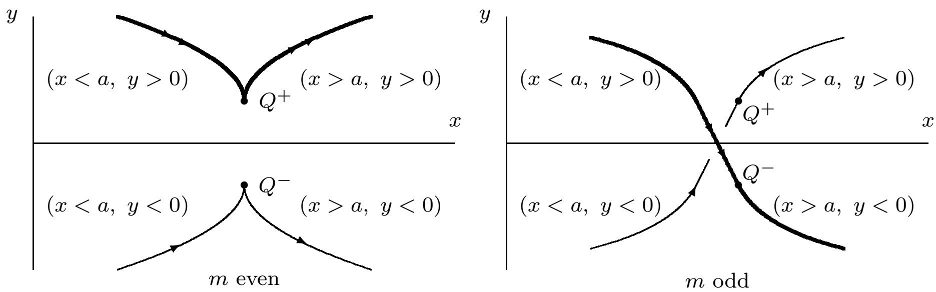

FIGURE 1. d even, n/d odd: the point $(a, 0)$ does not mark the end of an oval.

In the third case, however, the two real branches to the left of $x = a$ go through the same point Q^- and join there. In addition, a new oval starts at Q^+, where the two real branches to the right of $x = a$ join. We will say that Q^- marks the end of an oval and Q^+ marks the beginning of another oval. We will also say that $x = a$ is a *bi-oval root* of f.

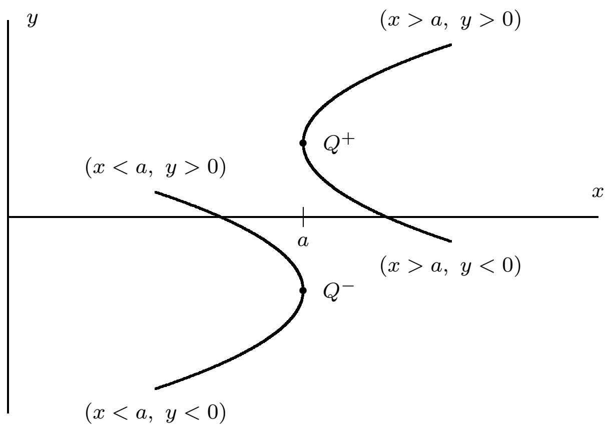

FIGURE 2. d and n/d even: a *bi-oval root* of f.

In summary, the only roots of f that have to be considered when counting ovals are roots with odd multiplicity and bi-oval roots.

3.2. **Behavior over $x = \infty$.** Let $y^n = f(x)$ (n even) be a defining equation for X with f monic and $n | \deg(f)$ (so that ∞ is not a branch point). For large values of $x > 0$ there are two branches of real points, say $(x > 0,\ y > 0)$ and $(x > 0,\ y < 0)$. Analogously, for negative values of x but large in absolute value, there are two branches of real ovals, say $(x < 0,\ y > 0)$ and $(x < 0,\ y < 0)$. Our goal now is to determine how these four branches join at the two real points, say Q^+ and Q^-, which lie over $x = \infty$. To that end we make the real change of variables $t = 1/x$ and see how the corresponding four branches join at points lying over $t = 0$.

Let us write $f(x) = x^{\deg(f)} + \cdots + c_1 x + c_0$. Replacing x by $1/t$ and multiplying both sides of the defining equation $y^n = f(x)$ by $t^{\deg(f)}$ we obtain

$$y^n t^{\deg(f)} = t^{\deg(f)} f\left(\frac{1}{t}\right) = 1 + \cdots + c_1 t^{\deg(f)-1} + c_0 t^{\deg(f)} := g(t).$$

Writing $w = y t^{\deg(f)/n}$ we see that $w^n = g(t)$ is a defining equation of X. In fact, $\mathbb{C}(x, y) = \mathbb{C}(t, w)$, as is easy to see. Observe that x and t have the same sign, but the sign of w depends on the parity of $\deg(f)/n$. In fact:

- If $\deg(f)/n$ is even, then $\operatorname{sign}(w) = \operatorname{sign}(y)$. Therefore the branch $(x < 0,\ y > 0)$ joins with $(x > 0,\ y > 0)$ and the branch $(x < 0,\ y < 0)$ joins with $(x > 0,\ y < 0)$.
- If $\deg(f)/n$ is odd, then $\operatorname{sign}(w) = \operatorname{sign}(y)\operatorname{sign}(t) = \operatorname{sign}(y)\operatorname{sign}(x)$. Therefore, the branch $(x > 0,\ y > 0)$ joins with $(x < 0,\ y < 0)$ and the branch $(x > 0,\ y < 0)$ joins with $(x < 0,\ y > 0)$.

Note that there are always an even number of points lying over ∞ and that an oval never begins or ends at points over ∞. So ∞ does not behave as a bi-oval root of f.

4. Main results

We are now in a position to show the main results of the paper. For simplicity we separate the cases when f does not change sign, Theorem 4.1, and when it does, Theorem 4.2. Recall that a real root of f with multiplicity dm is called bi-oval if both d and n/d are even, where $d = \gcd(n, dm)$. This is equivalent to saying that the highest power of 2 that divides the multiplicity of a is greater than 0 and less than the highest power of 2 that divides n.

Theorem 4.1. *Let $y^n = f(x)$ be a defining equation of an n-cyclic cover of the Riemann sphere with n an even divisor of $\deg(f)$ and f monic with real coefficients. Let us assume that f does not change sign. Let b be the number of bi-oval roots of f. Then the number $\|\sigma\|$ of ovals fixed by complex conjugation σ is*

$$\|\sigma\| = \begin{cases} b & \text{if } b > 0; \\ 1 & \text{if } b = 0 \text{ and } \deg(f)/n \text{ is odd;} \\ 2 & \text{if } b = 0 \text{ and } \deg(f)/n \text{ is even.} \end{cases}$$

Proof. Let us assume first that $b > 0$ and let $a_1 < a_2 < \cdots < a_b$ be the bi-oval roots of f. The two unbounded (in the (x, y) plane) branches $(x < a_1,\ y > 0)$ and $(x < a_1,\ y < 0)$ join at one of the real points lying over a_1 (so that it marks the end of an oval), while a new oval starts at the other real point lying over a_1. This new oval ends at one of the real points lying over a_2, and again a new oval starts over a_2. Continuing with this process we see that there are $b-1$ bounded ovals, the end of the last one being marked by a real point lying over a_b. The two unbounded branches $(x > a_b,\ y > 0)$ and $(x > a_b,\ y < 0)$ join at the other real point over a_b. Our analysis at ∞ in Section 3.2 shows that an oval never begins or ends over ∞. Therefore, the four branches with $x < a_1$ and $x > a_b$ join to form one oval, so the total number of ovals is b, as claimed.

If $b = 0$, then there is no finite branch point over which any pair of unbounded branches may join. So the branches $(x < 0,\ y > 0)$ and $(x > 0,\ y > 0)$ join in the finite (x, y) plane (actually, they join in the upper half plane $\{y > 0\}$), and the branches $(x < 0,\ y < 0)$ and $(x > 0,\ y < 0)$ join in the lower half plane $\{y < 0\}$. We now examine how these branches join over $x = \infty$; see Section 3.2. If $\deg(f)/n$ is even, then $(x < 0,\ y > 0)$ joins with $(x > 0,\ y > 0)$ yielding one oval, and $(x < 0,\ y < 0)$ joins with $(x > 0,\ y < 0)$ yielding another oval. Therefore, if $\deg(f)/n$ is even, there are two fixed ovals. If $\deg(f)/n$ is odd, then $(x < 0,\ y > 0)$ joins with $(x > 0,\ y < 0)$, and $(x < 0,\ y < 0)$ joins with $(x > 0,\ y > 0)$, yielding just one oval. □

We finally consider the case in which f changes sign; that is, f has (an even number of) roots with odd multiplicity. Recall that we may assume that f is monic; see Lemma 2.3.

Theorem 4.2. *Let $y^n = f(x)$ be a defining equation of an n-cyclic cover of the Riemann sphere with n an even divisor of $\deg(f)$ and f monic with real coefficients. Let $a_1 < a_2 < \cdots < a_{2s}$ be the real roots of f with odd multiplicity $(s > 0)$ and let b_j be the number of bi-oval roots of f which lie in the interval (a_{2j}, a_{2j+1}) for $j = 1, \ldots, s-1$. Let also b_0 and b_s be the number of bi-oval roots of f which lie in the intervals $(-\infty, a_1)$ and (a_{2s}, ∞), respectively. Then the number $\|\sigma\|$ of ovals fixed by complex conjugation σ is*

$$\|\sigma\| = s + \sum_{j=0}^{s} b_j.$$

Proof. We just have to consider the intervals $(-\infty, a_1)$, $(a_{2s}, +\infty)$ and (a_{2j}, a_{2j+1}) for $j = 1, \ldots, s-1$, since f is not positive elsewhere. Over each a_i there is a unique real point which marks the beginning of an oval if i is even or the end of an oval if i is odd. It follows easily that over each interval (a_{2j}, a_{2j+1}) there are $b_j + 1$ ovals fixed by σ, over the interval $(-\infty, a_1)$ there are b_0 bounded ovals, and over $(a_{2s}, +\infty)$ there are b_s bounded ovals. Finally, Section 3.2 yields that an oval never begins or ends at ∞, so the four unbounded branches with $x < 0$ or $x > 0$ all join together over $x = \infty$ to yield one unbounded oval. Therefore, the total number of ovals fixed by σ is

$$\|\sigma\| = \sum_{j=1}^{s-1} (b_j + 1) + b_0 + b_s + 1 = \sum_{j=0}^{s} b_j + s.$$

□

Example 4.3. The easiest case $n = 2$ corresponds to real hyperelliptic curves, which have no bi-oval roots. Theorems 4.1 and 4.2 readily apply to yield the well known fact that the number of ovals of complex conjugation depends on (half) the number of real roots of f if any, and on the parity of the genus if there is no real root; see [3, Section 3] and [6, Section 6], for instance.

Other examples where the above results easily apply consist of cyclic covers of the sphere with a low number of real branch points. These include many well known surfaces, such as the Accola-Maclachlan and Kulkarni surfaces. Related to this last surface it is worth mentioning the paper [10], where three defining equations of the Kulkarni surface are given. This surface has three (conjugacy classes of) symmetries with fixed points, and each defining equation exhibits one such symmetry as complex conjugation. Such equations turn out to be of the form $y^n = f(x)$, and so our results apply to each of them to show that the number of ovals fixed by the three conjugacy classes of symmetries are 1, 2 and $(g+1)/4$, where g is the genus of the surface.

Example 4.4. More interesting applications of Theorems 4.1 and 4.2 occur for cyclic covers with a larger number of real branch points. For instance, for each odd prime number p let us consider the surface defined by

$$y^{8p} = x^{4p} \prod_{k=1}^{8p-1} (x-k)^k.$$

Since $8p$ divides the degree of the polynomial $f(x)$ on the right hand side of the defining equation, the hypotheses of Theorem 4.2 are satisfied. Observe that the polynomial $y^{8p} - f(x)$ is irreducible by Eisenstein's criterion because $a = 1$ is a simple root of f. Therefore the polynomial defines a Riemann surface. Let $a_1 = 1 < a_2 = 3 < \cdots < a_k = 2k-1 < \cdots < a_{4p} = 8p-1$ be the $4p$ real roots of f with odd multiplicity. In each interval $(a_{2j}, a_{2j+1}) = (4j-1, 4j+1)$ there is a unique real root, namely $4j$, with even multiplicity $4j$. Since $\gcd(8p, 4j) = 4$ if j is odd and $\gcd(8p, 4j) = 8$ if j is even, we see that $4j$ is a bi-oval root of f if and only if j is odd. Therefore, for each $j = 1, \ldots, 2p-1$, the number b_j of bi-oval roots of f in the interval (a_{2j}, a_{2j+1}) equals 1 if j is odd and 0 if j is even. So $\sum_{j=1}^{2p-1} b_j = p$. On the other hand, the numbers b_0 and b_{2p} of bi-oval roots of f which lie in the intervals $(-\infty, a_1)$ and (a_{4p}, ∞) respectively are $b_0 = 1$ (because $x = 0$ is bi-oval) and $b_{2p} = 0$. Theorem 4.2 then yields that complex conjugation fixes $2p + 1 + p = 3p + 1$ ovals.

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