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INEQUIVALENT TOPOLOGIES ON THE TEICHMÜLLER SPACE OF THE FLUTE SURFACE

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ABSTRACT. The topology defined by the symmetrized Lipschitz metric on the Teichmüller space of an infinite type surface, in contrast to finite type surfaces, need not be the same as the topology defined by the Teichmüller metric. In this paper, we study the equivalence of these topologies on a particular kind of infinite type surface, called the flute surface.

Following a construction by Shiga and using additional hyperbolic geometric estimates, we obtain sufficient conditions in terms of length parameters for these two metrics to be topologically inequivalent. Next, we construct infinite parameter families of quasiconformally distinct flute surfaces with the property that the symmetrized Lipschitz metric is not topologically equivalent to the Teichmüller metric.

1. Introduction

Let S be a hyperbolic Riemann surface. The *Teichmüller space of* S, denoted $T(S)$, is defined to be the equivalence classes of pairs (R, f) where R is a Riemann surface, $f : S \longrightarrow R$ is a quasiconformal map and two pairs (R_1, f_1) and (R_2, f_2) are equivalent if $f_2 \circ f_1^{-1} : R_1 \longrightarrow R_2$ is freely homotopic to a conformal map. The equivalence class of this relation is denoted by $[R, f]$.

Define the *Teichmüller metric* on $T(S)$ to be

$$d_T([R_1, f_1], [R_2, f_2]) = \inf_f \log K(f)$$

where the infimum is taken over all quasiconformal maps $f : R_1 \longrightarrow R_2$ freely homotopic to $f_2 \circ f_1^{-1}$ and $K(f)$ denotes the *maximal dilatation* of f. A standard reference for maximal dilatation and related topics is [7].

Let Σ_S be the set of non-trivial closed geodesics on S. For a closed curve α, let $\ell_R(\alpha)$ denote the hyperbolic length of the closed geodesic on R freely homotopic to α. When there is no chance of confusion, we will drop the subscript and simply write $\ell(\alpha)$.

The *symmetrized Lipschitz metric* on $T(S)$ is defined by

$$d_L([R_1, f_1], [R_2, f_2]) = \log \sup_{\alpha \in \Sigma_S} \max \left\{ \frac{\ell_{R_1}(f_1(\alpha))}{\ell_{R_2}(f_2(\alpha))}, \frac{\ell_{R_2}(f_2(\alpha))}{\ell_{R_1}(f_1(\alpha))} \right\}.$$

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d_L is closely related to Thurston's Lipschitz metrics, defined as

$$d_{P_1}([R_1, f_1], [R_2, f_2]) = \log \sup_{\alpha \in \Sigma_S} \left\{ \frac{\ell_{R_1}(f_1(\alpha))}{\ell_{R_2}(f_2(\alpha))} \right\},$$

and

$$d_{P_2}([R_1, f_1], [R_2, f_2]) = \log \sup_{\alpha \in \Sigma_S} \left\{ \frac{\ell_{R_2}(f_2(\alpha))}{\ell_{R_1}(f_1(\alpha))} \right\}.$$

In fact, d_L is nothing but a symmetrization of d_{P_i}. These (asymmetric) metrics, which are also known as *Thurston's asymmetric metrics*, were defined and studied by Thurston in 1986 [16].

A Riemann surface is said to have *finite topological type* or simply *finite type* if its fundamental group is finitely generated. Otherwise, it is said to be *infinite type.*

Two metrics are said to be *topologically equivalent* if the topologies generated by those two metrics coincide.

The metric d_L and its relationship with d_T has been the topic of a lot of publications over the years. Sorvali defined and studied d_L in 1972 [14] and conjectured about the topological equivalence of this metric, then called *"the length spectrum metric"*, to d_T for finite type Riemann surfaces. In 1975, Sorvali showed that d_L and d_T are metrically equivalent on the Teichmüller space of the torus [15], which implies topological equivalence. In 1986, Li proved that d_L and d_T are topologically equivalent on the Teichmüller space of compact Riemann surfaces [9]. Sorvali's original conjecture was finally proven to be true by Lixin Liu in 1999 [11] and the question was extended to include infinite type surfaces.

In 2003, Shiga [13] proved that d_L and d_T need not be topologically equivalent on the Teichmüller space of infinite type surfaces, in contrast to the finite type case. In the same paper, Shiga also gave a sufficient condition for the topological equivalence of d_L and d_T. He showed that if an infinite type Riemann surface can be decomposed into pairs of pants such that the lengths of all the boundary components except punctures are uniformly bounded from above and below, then d_L is topologically equivalent to d_T. In 2011, Kinjo showed that this condition is not necessary [8] and gave a sufficient condition for the topological inequivalence of d_L and d_T (see Theorem 1.2 below). Our aim in this paper is to obtain sufficient conditions for topological inequivalence in terms of length and twist parameters on a specific kind of infinite type surface, called the "flute surface".

A hyperbolic surface S is a *flute surface* if $S = \mathbb{H}^2/G$ where $G = \langle G_i \rangle_{i=1}^{\infty}$ for some G_i such that $\mathbb{H}^2/G_i$ is a pair of pants for each i and

$$N(G_i) \cap N(G_{i+1}) = A(g_{i+1}),$$

where g_{i+1} is a primitive boundary hyperbolic element both in G_i and G_{i+1}. Here, $N(G)$ denotes the Nielsen convex region of G and $A(g)$ denotes the axis of g. A flute surface $S = \mathbb{H}^2/G$ is complete if the Fuchsian group G representing this surface has an end of the first kind and not complete otherwise. For a more detailed discussion about the completeness of flute surfaces, we refer to [1].

Let $FN(S)$ denote the Fenchel-Nielsen space of S, that is, the space of all hyperbolic structures on S. Note that for finite type surfaces, the Fenchel-Nielsen space and the Teichmüller space are the same, but for infinite type surfaces, the Fenchel-Nielsen space is strictly larger than the Teichmüller space. For a hyperbolic structure $\Gamma \in FN(S)$, we write $T(\Gamma)$ to denote the Teichmüller space of S endowed with the hyperbolic structure Γ.

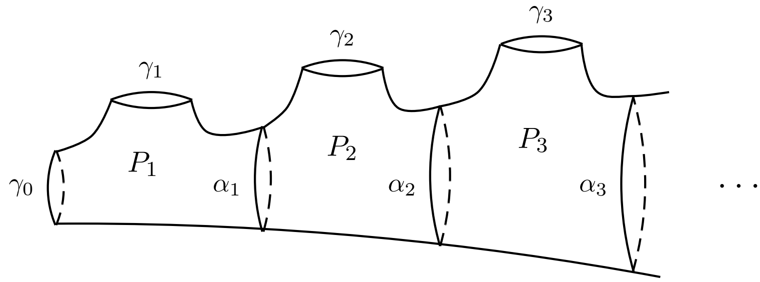

FIGURE 1. A flute surface

A hyperbolic structure on a flute surface is completely determined by the lengths assigned to the boundary components γ_i and the core curves α_j; and a twist parameter for each core curve. If the length assigned to a boundary component is zero, we interpret the boundary component in question as a puncture. Thus, $FN(S)$ can be identified with $\mathbb{R}_+^\infty \times \mathbb{R}^\infty$. The theory of flute surfaces was developed by Basmajian in [2] and [3]; the reader is referred to those papers for further reading.

The flute surface, which is possibly the least complicated among all infinite type surfaces, is a good setting for studying the Teichmüller theory of infinite type surfaces. The main result in this paper is sufficient conditions for d_L and d_T to generate different topologies on flute surfaces.

Let S be a flute surface where α_i and γ_i are as in Figure 1. Define a hyperbolic structure on S to be *strongly flaring* if

(1) $\limsup_{i\to\infty} \ell(\alpha_i) = \infty$,
(2) $\lim_{i\to\infty} \frac{\ell(\gamma_{i+1})+\ell(\alpha_{i+1})}{\ell(\alpha_i)} = \infty$,
(3) $\ell(\gamma_{i+1}) + \ell(\alpha_{i+1}) - \ell(\alpha_i)$ is eventually strictly increasing;

where ℓ is the hyperbolic length with respect to this hyperbolic structure.

Theorem 1.1. *Let $\Gamma \in FN(S)$ be a strongly flaring hyperbolic structure on the flute surface S. Then, d_L and d_T generate different topologies on $T(\Gamma)$.*

Theorem 1.1 is similar in spirit to Theorem A of Choi-Rafi's 2007 paper [5], where they construct a sequence in the Teichmüller space of an arbitrary hyperbolic surface to show that d_L and d_T are not metrically equivalent. In the proof of Theorem 1.1, we will construct a sequence to show the topological inequivalence of d_L and d_T in the Teichmüller space of a strongly flaring flute surface.

This theorem is also related to Theorem 1.5 of Kinjo's 2011 paper [8], which is stated below:

Theorem 1.2 (Kinjo, 2011). *Let R_0 be a Riemann surface. Suppose that there exists a sequence $\{\alpha_n\}_{n=1}^\infty \subseteq \Sigma'_{R_0}$ such that for an arbitrary sequence $\{\beta_n\}_{n=1}^\infty \subseteq \Sigma'_{R_0}$ with $\alpha_n \cap \beta_n \neq \emptyset$ $(n = 1, 2, \ldots)$,*

$$\frac{\ell_{R_0}(\beta_n)}{\#(\alpha_n \cap \beta_n)\ell_{R_0}(\alpha_n)} \to \infty \qquad (n \to \infty).$$

Then, d_L does not define the same topology as that of d_T on $T(R_0)$.

Here, Σ'_{R_o} denotes the set of non-trivial simple closed geodesics in R_0. For the case of a flute surface, the strongly flaring condition ensures that one can always find a sequence $\{\alpha_n\}_{n=1}^{\infty}$ satisfying the requirements of this theorem.

Next, we will use Theorem 1.1 to construct infinite parameter families of quasiconformally distinct Riemann surfaces with the property that d_L is not topologically equivalent to d_T on the Teichmüller space of any of the surfaces belonging to the family.

Theorem 1.3. *Let $\Gamma \in FN(S)$ be a hyperbolic structure on the flute surface S, let $V \subseteq \mathbb{R}[x]$ be the set of polynomials with non-negative coefficients and constant term zero, and write $\mathcal{O} \in V$ for the zero polynomial. Suppose that one of the following holds:*

(1) *Γ is strongly flaring, and the boundary lengths $\ell(\gamma_i)$ are strictly increasing and unbounded.*
(2) *Γ is strongly flaring, the boundary lengths are bounded and eventually non-decreasing, and the cuff lengths eventually satisfy $\ell(\alpha_{i+1}) \geq i \cdot \ell(\alpha_i)$.*
(3) *The cuff lengths tend to zero.*

Then, in each case V parametrizes a family of hyperbolic structures on S by a continuous embedding $\iota : V \to FN(S)$ with $\iota(\mathcal{O}) = \Gamma$ such that

- *d_L is not topologically equivalent to d_T on $T(\iota(p))$ for any $p \in V$; and*
- *$\iota(p)$ and $\iota(q)$ are not quasiconformally equivalent unless $p = q$.*

Remark. In case (1), the whole family $\iota(V)$ of hyperbolic structures is obtained by modifying the boundary lengths $c_i = \ell(\gamma_i)$ from Γ by setting $c'_i = c_i e^{p(c_i)}$ for each $p \in V$ while leaving the cuff lengths and twists unchanged. On the other hand, we put $a'_i = a_i e^{p(a_i)}$ in case (2) and $a'_i = a_i e^{-p(1/a_i)}$ in case (3), maintaining the boundary lengths and only modifying cuff lengths in both cases.

Theorem 1.3 illustrates the fact that the Fenchel-Nielsen space of an infinite type surface is strictly greater than its Teichmüller space. Since $\iota(p)$ and $\iota(q)$ are not quasiconformally equivalent for distinct p and q, $T(\iota(p))$ and $T(\iota(q))$ are disjoint. This means that for each $p \in V$, we have a distinct Teichmüller space $T(\iota(p))$ sitting inside $FN(S)$. A similar result was obtained by Basmajian in his 1997 paper [3].

Item (3) of Theorem 1.3 uses the following result by Liu, Sun and Wei [10]:

Theorem 1.4 (Liu et al., 2008)**.** *Let X be a Riemann surface of infinite topological type such that there exists a sequence of simple closed curves $\{\alpha_n\}$, $n = 1, 2, \ldots$; $\alpha_n \in \Sigma_0(X)$ with $\lim_{n\to\infty} \ell_X(\alpha_n) = 0$. Here, $\Sigma_0(X)$ is the collection of homotopy classes of all simple closed curves on X. Then, in the Teichmüller space $T(X)$, d_T is not topologically equivalent to d_L.*

When the cuff lengths in a flute surface tend to zero, one can simply take them to be the α_n's in Theorem 1.4 and guarantee that d_L is topologically distinct from d_T on $T(\Gamma)$. Moreover, the way ι is defined for this case, the cuff lengths on S with the hyperbolic structure $\iota(p)$ will tend to zero for any $p \in V$ and therefore d_L will be topologically distinct from d_T on $T(\iota(p))$ as well.

The theorems in this paper appeared in the author's Ph.D. Thesis [6] completed at the Graduate Center, City University of New York.

2. Examples

In this section, we provide examples of hyperbolic structures on flute surfaces with length parameters of various growth rates that satisfy the conditions of cases (1) and (2) of Theorem 1.3.

- *A hyperbolic structure with linearly growing boundary*:
  Take $\ell(\alpha_i) = \log i$ and $\ell(\gamma_i) = i$ for $i = 1, 2, 3, \ldots$ and let the twist parameters be arbitrary. This hyperbolic structure is strongly flaring. Moreover, $\ell(\gamma_i) = i$ is strictly increasing and goes to infinity. So, case (1) of Theorem 1.3 applies and we obtain a family that has boundary components with lengths $ie^{p(i)}$ for $p \in V$.
- *A hyperbolic structure with polynomially growing boundary:*
  Fix an integer $n > 0$. Take $\ell(\alpha_i) = i^n$ and $\ell(\gamma_i) = i^{n+1}$ for $i = 1, 2, 3, \ldots$. Let the twist parameters be arbitrary. The hyperbolic structure we obtain is strongly flaring and conditions of case (1) of Theorem 1.3 is met. The family that we get has boundary components with lengths $i^n e^{p(i^n)}$.
- *A hyperbolic structure with exponentially growing boundary:*
  Take $\ell(\alpha_i) = i$ and $\ell(\gamma_i) = e^i$ for $i = 1, 2, 3, \ldots$; and let the twist parameters be arbitrary. It can be easily shown that this hyperbolic structure is strongly flaring. Since, moreover, $\ell(\gamma_i) = e^i$ is strictly increasing to infinity, we obtain a family by case (1) of Theorem 1.3. The boundary components of this family have length $e^i e^{p(e^i)} = e^{i+p(e^i)}$.
- *A hyperbolic structure with factorially growing core curves:*
  Take $\ell(\alpha_i) = \sqrt{i!}$ and take the boundary curves to have arbitrary bounded and non-decreasing lengths, for instance, take $\ell(\gamma_i) = 0$ for all i. Once more, the hyperbolic structure described here is strongly flaring.
  Moreover, we observe that $\ell(\alpha_i) = \sqrt{i!}$ is strictly increasing and
  $$\sqrt{(i+1)!} = \sqrt{i+1}\sqrt{i!} \leq i \cdot \sqrt{i!}$$
  is eventually satisfied. Therefore, applying case (2) of Theorem 1.3, we obtain a family where all hyperbolic structures have the same boundary data and the core curves are of length $\sqrt{i!}e^{p(\sqrt{i!})}$.

3. Proof of Theorem 1.1

As $\ell(\gamma_{i+1}) + \ell(\alpha_{i+1}) - \ell(\alpha_i)$ is eventually strictly increasing, there exists $K \in \mathbb{N}$ such that
$$\ell(\gamma_{j+1}) + \ell(\alpha_{j+1}) - \ell(\alpha_j) < \ell(\gamma_{j+2}) + \ell(\alpha_{j+2}) - \ell(\alpha_{j+1})$$
for every $j > K$.

Let $f_0 = \mathrm{Id}$ and f_i be the Dehn twist around α_i for $i \geq 1$. Look at the sequence $\{[S, f_i]\} \subseteq T(S)$. We claim that $d_L([S, f_0], [S, f_i]) \to 0$ as $i \to \infty$.

We note the following lemma which is based on an observation from [4]:

Lemma 3.1. *Let P be a pair of pants equipped with a hyperbolic structure where the boundary components α, β and γ are geodesics. Suppose δ is a geodesic arc whose endpoints are on α, but not homotopic to a subarc of α, as in Figure 2. Then,*
$$\ell(\delta) \geq \frac{1}{2}\Big(\ell(\beta) + \ell(\gamma) - \ell(\alpha)\Big).$$

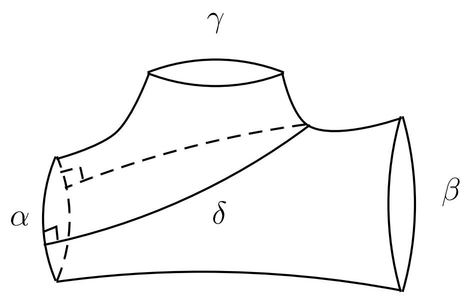

FIGURE 2. The surface P

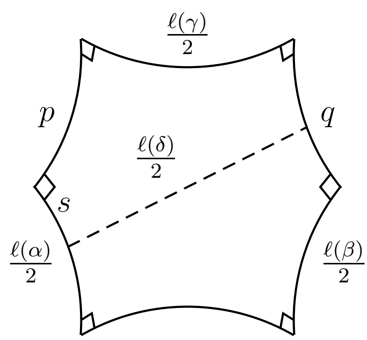

FIGURE 3. Decomposition of P into hyperbolic hexagons

Proof. First, assume that none of the boundary components are punctures. We can decompose the pair of pants into two isometric hexagons as in Figure 3.

Observe that the arc of length $\ell(\gamma)/2$ is the common perpendicular between the sides labeled p and q; therefore it is shorter than any other curve joining p to q. Since the arc labeled s followed by the arc of length $\ell(\delta)/2$ is a curve which joins side p to side q, we have

$$\ell(s) + \frac{\ell(\delta)}{2} \geq \frac{\ell(\gamma)}{2}.$$

Applying the same argument to the opposite side, we get

$$\frac{\ell(\alpha)}{2} - \ell(s) + \frac{\ell(\delta)}{2} \geq \frac{\ell(\beta)}{2}.$$

Combining these two inequalities, we obtain

$$\ell(\delta) \geq \frac{\ell(\gamma)}{2} + \frac{\ell(\beta)}{2} - \frac{\ell(\alpha)}{2}.$$

If one of the boundary components is a puncture, then we decompose the pair of pants into two ideal pentagons as in Figure 4.

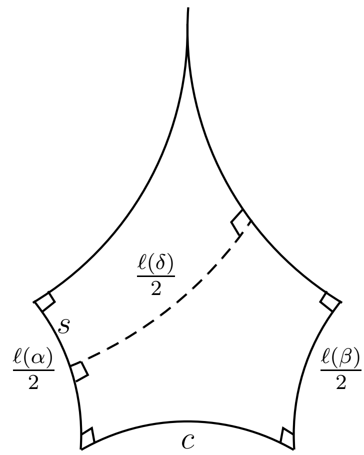

FIGURE 4. Decomposition of P into ideal pentagons

Clearly,

$$\ell(s) + \frac{\ell(\delta)}{2} \geq 0.$$

On the other hand, the side of length $\ell(\beta)/2$ is the common perpendicular between the sides labeled c and one of the sides which go out to infinity; thus

$$\frac{\ell(\alpha)}{2} - \ell(s) + \frac{\ell(\delta)}{2} \geq \frac{\ell(\beta)}{2}.$$

Combining these two inequalities gives us

$$\ell(\delta) \geq \frac{\ell(\beta)}{2} - \frac{\ell(\alpha)}{2}.$$

Observe that the same result is obtained if we take $\ell(\gamma) = 0$ in the previous case.

If two of the boundary components are punctures, then the inequality holds trivially since

$$\ell(\delta) \geq -\frac{\ell(\alpha)}{2}.$$

Note that we did not consider the case where the boundary component α is a puncture, since in this case δ has infinite length. $\square$

Now, look at P_i, i.e., the pair of pants bounded by α_i, α_{i+1} and γ_{i+1} in S. By Lemma 3.1, the length of any arc contained in P_i with endpoints on α_i must be greater than $\frac{1}{2}\big(\ell(\gamma_{i+1}) + \ell(\alpha_{i+1}) - \ell(\alpha_i)\big)$.

Let β be a closed geodesic in S which is distinct from each of the α_i's and assume that $\beta \cap \alpha_i \neq \emptyset$ for some $i > K$. The geometric intersection number $\#(\beta \cap \alpha_i)$ must be an even number, say $2k$. Then, we get $2k$ arcs with endpoints on α_i; k of which are contained in $P_i \cup P_{i+1} \cup \cdots$. Let β_0 be any one of these arcs.

If β_0 is contained in P_i, then, as noted earlier,

$$\ell(\beta_0) \geq \frac{1}{2}\big(\ell(\gamma_{i+1}) + \ell(\alpha_{i+1}) - \ell(\alpha_i)\big).$$

Otherwise, β_0 has a subarc with endpoints on α_j and is contained in P_j for some $j > i$. Since $\ell(\gamma_{i+1}) + \ell(\alpha_{i+1}) - \ell(\alpha_i)$ is strictly increasing, we have

$$\begin{aligned}\ell(\beta_o) &\geq \frac{1}{2}\Big(\ell(\gamma_{j+1}) + \ell(\alpha_{j+1}) - \ell(\alpha_j)\Big)\\ &\geq \frac{1}{2}\Big(\ell(\gamma_{i+1}) + \ell(\alpha_{i+1}) - \ell(\alpha_i)\Big).\end{aligned}$$

Hence, we have

$$\ell(\beta) \geq k \cdot \frac{1}{2}\Big(\ell(\gamma_{i+1}) + \ell(\alpha_{i+1}) - \ell(\alpha_i)\Big).$$

Let us investigate the ratio

$$\frac{\ell(f_i(\beta))}{\ell(\beta)}$$

for a closed geodesic β in S.

If $\beta \cap \alpha_i = \emptyset$, then this ratio is 1. Assume that $\#(\beta \cap \alpha_i) = 2k > 0$. Then,

$$\begin{aligned}\frac{\ell(f_i(\beta))}{\ell(\beta)} &\leq \frac{\ell(\beta) + 2k\ell(\alpha_i)}{\ell(\beta)}\\ &\leq 1 + \frac{4k\ell(\alpha_i)}{k(\ell(\gamma_{i+1}) + \ell(\alpha_{i+1}) - \ell(\alpha_i))}.\end{aligned}$$

We conclude that

$$\frac{\ell(f_i(\beta))}{\ell(\beta)} \leq 1 + \frac{4}{\frac{\ell(\gamma_{i+1})+\ell(\alpha_{i+1})}{\ell(\alpha_i)} - 1} \tag{3.1}$$

for an arbitrary closed geodesic β in S.

We can apply the same arguments to f_i^{-1} to obtain

$$\frac{\ell(f_i^{-1}(\beta))}{\ell(\beta)} \leq 1 + \frac{4}{\frac{\ell(\gamma_{i+1})+\ell(\alpha_{i+1})}{\ell(\alpha_i)} - 1}$$

for an arbitrary β. Applying this estimate to the geodesic in the homotopy class of $f_i(\beta)$, we get

$$\begin{aligned}\frac{\ell(f_i^{-1}(f_i(\beta)))}{\ell(f_i(\beta))} &= \frac{\ell(\beta)}{\ell(f_i(\beta))}\\ &\leq 1 + \frac{4}{\frac{\ell(\gamma_{i+1})+\ell(\alpha_{i+1})}{\ell(\alpha_i)} - 1}.\end{aligned} \tag{3.2}$$

The inequality (3.1) together with (3.2) implies

$$\max\left\{\frac{\ell(f_i(\beta))}{\ell(\beta)}, \frac{\ell(\beta)}{\ell(f_i(\beta))}\right\} \leq 1 + \frac{4}{\frac{\ell(\gamma_{i+1})+\ell(\alpha_{i+1})}{\ell(\alpha_i)} - 1}.$$

Since this is true for an arbitrary β, it is true for the supremum:

$$\sup_{\beta\in\Sigma_S} \max\left\{\frac{\ell(f_i(\beta))}{\ell(\beta)}, \frac{\ell(\beta)}{\ell(f_i(\beta))}\right\} \leq 1 + \frac{4}{\frac{\ell(\gamma_{i+1})+\ell(\alpha_{i+1})}{\ell(\alpha_i)} - 1}.$$

Then,

$$\lim_{i\to\infty} \sup_{\beta\in\Sigma_S} \max\left\{\frac{\ell(f_i(\beta))}{\ell(\beta)}, \frac{\ell(\beta)}{\ell(f_i(\beta))}\right\} = 1;$$

hence $\lim_{i\to\infty} d_L([S, f_0], [S, f_i]) = 0$.

Now, since $\limsup_{i\to\infty} \ell(\alpha_i) = \infty$, $\{\ell(\alpha_i)\}$ has a subsequence $\{\ell(\alpha_{i_k})\}$ such that $\lim_{k\to\infty} \ell(\alpha_{i_k}) = \infty$. To see that $\lim_{k\to\infty} d_T([S, f_0], [S, f_{i_k}]) = \infty$, we note the following theorem of Matsuzaki from [12]:

Theorem 3.2 (Matsuzaki, 2003). *Let α be a simple closed geodesic on a Riemann surface R_0 and let $f : R_0 \to R_0$ be the Dehn twist along α. Then, the maximal dilatation $K(f)$ of an extremal quasiconformal automorphism of f satisfies*

$$K(f) \geq \left\{\left(\frac{\ell_{R_0}(\alpha)}{\pi}\right)^2 + 1\right\}^{1/2}.$$

Matsuzaki proves this theorem for an arbitrary infinite type Riemann surface, so we can apply this theorem for each α_{i_k} to get that $K(f_{i_k}) \to \infty$ as $k \to \infty$. This shows that $\lim_{k\to\infty} d_T([S, f_0], [S, f_{i_k}]) \to \infty$ as desired.

On the other hand, observe that $\{d_L([S, f_0], [S, f_{i_k}])\}$ is a subsequence of $\{d_L([S, f_o], [S, f_i])\}$; hence $d_L([S, f_0], [S, f_{i_k}]) \to 0$ as well. This establishes that d_T is not topologically equivalent to d_L on $T(S)$.

Note that the same result can be obtained by taking α_n's in Theorem 1.2 to be α_i for $i > K$.

Remark. Using (3.1) and (3.2) separately, one can easily show that for the same sequence, $d_{P_1}([S, f_0], [S, f_{i_k}]) \to 0$ and $d_{P_2}([S, f_0], [S, f_{i_k}]) \to 0$ while

$$d_T([S, f_0], [S, f_{i_k}]) \to \infty$$

is satisfied.

4. Proof of Theorem 1.3

Case (1). Let $a_i = \ell(\alpha_i)$ and $c_i = \ell(\gamma_i)$. Define ι to take $p \in V$ to the hyperbolic structure on S where the boundary components have length $c_i e^{p(c_i)}$ and where the lengths of the core curves and twist parameters are the same as Γ.

Now, the boundary components of $\iota(\mathcal{O})$ has length $c_i e^{\mathcal{O}(c_i)} = c_i$ and the lengths of the core curves and twist parameters are also the same as those of Γ; therefore $\iota(\mathcal{O}) = \Gamma$.

Next, we show that d_L is not topologically equivalent to d_T on $T(\iota(p))$ for any $p \in V$. Let $p \in V$ be arbitrary.

By Theorem 1.1, it suffices to prove that $\frac{c_{i+1}e^{p(c_{i+1})}+a_{i+1}}{a_i} \to \infty$ and that $c_{i+1}e^{p(c_{i+1})} + a_{i+1} - a_i$ is eventually strictly increasing.

To prove the former, recall that Γ is strongly flaring; therefore, we must have $\frac{a_{i+1}}{a_i} \to \infty$ or $\frac{c_{i+1}}{a_i} \to \infty$. In the first case, there is nothing to prove; so assume the second. Since $c_i \to \infty$ and the coefficients of p are non-negative, we get

$$\frac{c_{i+1}e^{p(c_{i+1})}}{a_i} \geq \frac{c_{i+1}}{a_i} \to \infty.$$

Now, we show that $c_{i+1}e^{p(c_{i+1})}+a_{i+1}-a_i$ is eventually strictly increasing for any $p \in V$. Since Γ is strongly flaring, $c_{i+1}+a_{i+1}-a_i$ is eventually strictly increasing; therefore there exists a natural number $K>0$ such that for all $i>K$,

$$c_{i+2}+a_{i+2}-a_{i+1} > c_{i+1}+a_{i+1}-a_i;$$

or equivalently,

$$c_{i+2}-c_{i+1} > 2a_{i+1}-a_{i+2}-a_i. \tag{4.1}$$

Let $f(x)=xe^{p(x)}-x$. Since p is a polynomial with non-negative coefficients, we have

$$\begin{aligned} f'(x) &= e^{p(x)}(1+xp'(x))-1 \\ &\geq (1+xp'(x))-1 \\ &= xp'(x) > 0 \end{aligned}$$

for $x>0$. Then, f is strictly increasing and as $\{c_i\}$ is also strictly increasing,

$$f(c_{i+2}) > f(c_{i+1})$$

for all i, i.e.,

$$c_{i+2}e^{p(c_{i+2})}-c_{i+2} > c_{i+1}e^{p(c_{i+1})}-c_{i+1}$$

for all i. Combining this with (4.1), we get

$$c_{i+2}e^{p(c_{i+2})}-c_{i+1}e^{p(c_{i+1})} > 2a_{i+1}-a_{i+2}-a_i$$

for all i.

Now, applying Theorem 1.1 to the hyperbolic structure $\iota(p)$, we conclude that d_L is not topologically equivalent to d_T on $T(\iota(p))$ for any $p \in V$.

Next, we show that $\iota(p)$ and $\iota(q)$ are not quasiconformally equivalent unless $p=q$. For this, we need the following lemma about the topology of the flute surfaces:

Lemma 4.1. *Let S be the flute surface from Figure* 1*, possibly without any hyperbolic structure. Let $f: S \longrightarrow S$ be a homeomorphism. If $f(\alpha_i)$ is not homotopic to α_i, then $f(\alpha_i)$ is not contained in the finite type component of $S\backslash\alpha_i$.*

Proof. Since f is a homeomorphism, topological invariants are preserved under f. In particular, f must map boundary components to boundary components. It follows that a surface with n boundary components must be mapped to a surface with a total of n boundary components under f.

Let S_i be the finite type component of $S\backslash\alpha_i$. Towards a contradiction, assume that $f(\alpha_i) \subseteq S_i$. Observe that S_i is a surface with $i+2$ boundary components, namely α_i and $\gamma_0,\ldots,\gamma_i$; therefore $f(S_i)$ has to be a surface with $i+2$ boundary components.

Since α_i is not a boundary component of S, f cannot map α_i to γ_j for any j. Since $f(\alpha_i)$ is assumed to be distinct from α_i, $f(\alpha_i)$ has to be a simple closed curve in the interior of S_i.

Since α_i separates S into two surfaces, one of finite topological type and one of infinite topological type; $f(\alpha_i)$ must do the same, however, no simple closed curve in the interior of S_i can separate S into surfaces with the same total number of boundary components and punctures as the components of $S\backslash\alpha_i$. A contradiction. □

Now, assume that $f : \iota(p) \longrightarrow \iota(q)$ is a K-quasiconformal mapping. The following result by Wolpert [17] is well known:

Lemma 4.2 (Wolpert's lemma)**.** *Let $f : R_1 \longrightarrow R_2$ be quasiconformal and let β be a closed geodesic in R_1. Then,*

$$\ell_{R_2}(f(\beta)) \leq K(f) \cdot \ell_{R_1}(\beta).$$

By Lemma 4.1, $f(\alpha_i)$ cannot be contained in the finite type component of $S \backslash \alpha_i$ for any i; therefore one and only one of the following is true:

(1) For infinitely many i, $f(\alpha_i)$ is transverse to α_{j_i} for some $j_i > i$.
(2) There exists k such that $f(\alpha_i)$ is homotopic to α_i for $i \geq k$.

Applying Lemma 4.2 to f, one obtains

$$\ell(f(\alpha_i)) \leq K \cdot \alpha_i. \tag{4.2}$$

Now, let j be the greatest index such that $f(\alpha_i)$ intersects α_j. Decomposing P_j (the pair of pants bounded by α_j, α_{j+1} and γ_{j+1}) into two hyperbolic hexagons as in the proof of Theorem 1.1, we get

$$\ell(f(\alpha_i)) \geq c_{j+1} e^{q(c_{j+1})} + a_{j+1} - a_j.$$

Since $c_{j+1} e^{q(c_{j+1})} + a_{j+1} - a_j$ is strictly increasing, it follows that

$$\ell(f(\alpha_i)) \geq c_{i+1} e^{q(c_{i+1})} + a_{i+1} - a_i. \tag{4.3}$$

Combining inequalities (4.2) and (4.3), we get

$$K \cdot \alpha_i \geq c_{i+1} e^{q(c_{i+1})} + a_{i+1} - a_i,$$

which leads to

$$K \geq \frac{c_{i+1} e^{q(c_{i+1})} + a_{i+1}}{a_i} - 1 \geq \frac{c_{i+1} + a_{i+1}}{a_i} - 1 \to \infty,$$

by the strongly flaring condition. This contradicts the quasiconformality of f.

In the second case, look at P_i for large i. Since $f(\alpha_i) = \alpha_i$ for $i > k$ and since pairs of pants are mapped to pairs of pants under homeomorphisms, we must have $f(\gamma_{i+1}) = \gamma_{i+1}$. Note that since $c_k \to \infty$, either $p(c_k) - q(c_k) \to \infty$ or $q(c_k) - p(c_k) \to \infty$ as $k \to \infty$.

If $p(c_k) - q(c_k) \to \infty$, then

$$K \geq \frac{c_k e^{p(c_k)}}{c_k e^{q(c_k)}} \to \infty,$$

by Lemma 4.2 (Wolpert's lemma), which contradicts the fact that $K < \infty$. Otherwise, the same estimation applied to $f^{-1} : \iota(q) \longrightarrow \iota(p)$ yields

$$K \geq \frac{c_k e^{q(c_k)}}{c_k e^{p(c_k)}} \to \infty,$$

which again by Lemma 4.2 leads to a contradiction.

Therefore, $\iota(p)$ and $\iota(q)$ cannot be quasiconformally equivalent unless $p = q$.

Case (2). Let $\ell(\alpha_i) = a_i$. Define ι to take $p \in V$ to the hyperbolic structure on S where the lengths of the core curves are $a_i e^{p(a_i)}$ and where the lengths of the boundary components and twist parameters are the same as Γ.

It is clear that $\iota(\mathcal{O}) = \Gamma$.

To prove that d_L is not topologically equivalent to d_T on $T(\iota(p))$, we will again use Theorem 1.1. Items (1) and (2) of the strongly flaring condition for the hyperbolic structure $\iota(p)$ follow from the assumption that $\ell(\alpha_{i+1}) \geq i \cdot \ell(\alpha_i)$. To verify item (3), we need to show that

$$\ell(\gamma_{i+1}) + a_{i+1}e^{p(a_{i+1})} - a_i e^{p(a_i)}$$

is eventually strictly increasing for any $p \in V$. As $\ell(\gamma_i)$ was chosen to be eventually non-decreasing, it is enough to show that $a_{i+1}e^{p(a_{i+1})} - a_i e^{p(a_i)}$ is eventually strictly increasing, i.e., that for large i,

$$a_{i+2}e^{p(a_{i+2})} - a_{i+1}e^{p(a_{i+1})} > a_{i+1}e^{p(a_{i+1})} - a_i e^{p(a_i)}. \tag{4.4}$$

Dividing each side by a_i and rearranging terms, (4.4) is seen to be equivalent to

$$\frac{a_{i+2}}{a_i}e^{p(a_{i+2})} - 2\frac{a_{i+1}}{a_i}e^{p(a_{i+1})} + e^{p(a_i)} > 0.$$

Since $a_{i+2} \geq (i+1) \cdot a_{i+1}$, we have

$$\begin{aligned} &\frac{a_{i+2}}{a_i}e^{p(a_{i+2})} - 2\frac{a_{i+1}}{a_i}e^{p(a_{i+1})} + e^{p(a_i)} \\ &\qquad\qquad \geq \frac{(i+1)a_{i+1}}{a_i}e^{p(a_{i+2})} - 2\frac{a_{i+1}}{a_i}e^{p(a_{i+1})} + e^{p(a_i)}. \end{aligned}$$

The right-hand side of this inequality can be rewritten as

$$\frac{a_{i+1}}{a_i}\Big((i+1)e^{p(a_{i+2})} - 2e^{p(a_{i+1})}\Big) + e^{p(a_i)}.$$

We note that it suffices to show that $(i+1)e^{p(a_{i+2})} - 2e^{p(a_{i+1})}$ is positive, which is equivalent to

$$\frac{e^{p(a_{i+2})}}{e^{p(a_{i+1})}} > \frac{2}{i+1} \tag{4.5}$$

for large i.

Consider the function $f(x) = e^{p(x)}$. We have

$$f'(x) = e^{p(x)}p'(x).$$

Since the coefficients of p are non-negative, $p'(x)$ is non-negative. Note that $p'(x) = \mathcal{O}$ implies that $p = \mathcal{O}$, which has already been dealt with. Then, $p'(x)$ is strictly positive and therefore $f'(x)$ is strictly positive; which means f is strictly increasing on $(0, \infty)$. Now, since a_i was chosen to be strictly increasing, we have $a_{i+2} > a_{i+1}$. As f is also strictly increasing, we conclude $f(a_{i+2}) > f(a_{i+1})$; i.e.

$$e^{p(a_{i+2})} > e^{p(a_{i+1})};$$

which is the same as saying

$$\frac{e^{p(a_{i+2})}}{e^{p(a_{i+1})}} > 1.$$

Since $\frac{2}{i+1} \leq 1$, the inequality (4.5) is satisfied for large i. This concludes the proof of item (1).

Finally, we prove that $\iota(p)$ and $\iota(q)$ are quasiconformally distinct for $p \neq q$. As in the proof of case (1), assume towards a contradiction that $f : \iota(p) \longrightarrow \iota(q)$ is a K-quasiconformal mapping with $p \neq q$.

First of all, assume that there exists infinitely many i such that there exists j_i with $f(\alpha_i) \cap \alpha_{j_i} \neq \emptyset$. Then,

$$\ell(f(\alpha_i)) \geq \ell(\gamma_{j_i+1}) + a_{j_i+1} - a_{j_i}.$$

By Lemma 4.1, j_i must be greater than or equal to i. Since Γ is strongly flaring, $\ell(\gamma_{j_i+1}) + a_{j_i+1} - a_{j_i}$ is strictly increasing; therefore we get

$$\ell(f(\alpha_i)) \geq \ell(\gamma_{j_i+1}) + a_{j_i+1} - a_{j_i} \geq \ell(\gamma_{i+1}) + a_{i+1} - a_i$$

for infinitely many i. On the other hand, by Lemma 4.2 (Wolpert's lemma), we have

$$\ell(f(\alpha_i)) \leq K \cdot a_i.$$

Combining these inequalities and dividing each side by a_i, one obtains

$$\frac{\ell(\gamma_{i+1}) + a_{i+1}}{a_i} - 1 \leq K.$$

Since Γ is strongly flaring, the left-hand side has a subsequence which goes out to infinity; a contradiction with the assumption that $K < \infty$.

It follows that for only finitely many i, the intersection $f(\alpha_i) \cap \alpha_{j_i} \neq \emptyset$ for $j_i > i$; which means that there exists an $N \in \mathbb{N}$ such that for all $i \geq N$, $f(\alpha_i) = \alpha_i$ setwise. Then,

$$a_i e^{q(a_i)} = \ell(f(\alpha_i)) \leq K \cdot a_i e^{p(a_i)},$$

which can be rewritten as

$$\frac{e^{p(a_i)}}{e^{q(a_i)}} \leq K.$$

Carrying out the same computation for f^{-1} gives us

$$\frac{e^{q(a_i)}}{e^{p(a_i)}} \leq K;$$

which is a contradiction since either one of $p(a_i) - q(a_i)$ or $q(a_i) - p(a_i)$ goes to infinity while $K < \infty$.

Case (3). As before, let $\ell(\alpha_i) = a_i$. Define ι to take $p \in V$ to the hyperbolic structure on S where the core curves have length $a_i e^{-p\left(\frac{1}{a_i}\right)}$ and where the lengths of the boundary components and the twist parameters are the same as those of Γ.

Once more, items (2) and (3) follow immediately from the definitions.

To prove item (1), one just need to observe that $a_i e^{-p\left(\frac{1}{a_i}\right)} \to 0$ whenever $a_i \to 0$ and then apply Theorem 1.4.

Let us show that $\iota(p)$ and $\iota(q)$ are not quasiconformally equivalent unless $p = q$. Towards a contradiction, assume $f : \iota(p) \longrightarrow \iota(q)$ is K-quasiconformal. By Lemma 4.2 (Wolpert's lemma), we have

$$\ell(f(\alpha_i)) \leq K \cdot a_i \tag{4.6}$$

for all i. Note that there exists $N > 0$ such that for all $i > N$, $f(\alpha_i)$ is homotopic to α_i, because otherwise, by Lemma 4.1, there would be infinitely many i such that $f(\alpha_i) \cap \alpha_{j_i} \neq \emptyset$ for some $j_i > i$ and since $\ell(\alpha_i) \to 0$ as $i \to \infty$, $\ell(f(\alpha_i)) \to \infty$ by the Collar lemma; which would imply

$$K \geq \frac{\ell(f(\alpha_i))}{\alpha_i} \to \infty$$

as $i \to \infty$, a contradiction with the fact that $K < \infty$.

Now, for any $i > N$, $f(\alpha_i)$ is homotopic to α_i; therefore we have $\ell(f(\alpha_i)) = a_i e^{-q\left(\frac{1}{a_i}\right)}$ and $\ell(\alpha_i) = a_i e^{-p\left(\frac{1}{a_i}\right)}$. If we substitute these in (4.6), we obtain

$$a_i e^{-q\left(\frac{1}{a_i}\right)} \leq K \cdot a_i e^{-p\left(\frac{1}{a_i}\right)}.$$

Dividing each side by $e^{-p\left(\frac{1}{a_i}\right)}$, we obtain

$$e^{p\left(\frac{1}{a_i}\right)-q\left(\frac{1}{a_i}\right)} \leq K$$

for all $i > N$. Carrying out the same computation for f^{-1} in the place of f, one also obtains

$$e^{q\left(\frac{1}{a_i}\right)-p\left(\frac{1}{a_i}\right)} \leq K.$$

This is a contradiction, since either $p\left(\frac{1}{a_i}\right) - q\left(\frac{1}{a_i}\right) \to \infty$ or $q\left(\frac{1}{a_i}\right) - p\left(\frac{1}{a_i}\right) \to \infty$.

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