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AN OPERADIC PROOF OF BAEZ-DOLAN STABILIZATION HYPOTHESIS

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Abstract. We prove a stabilization theorem for algebras of n-operads in a monoidal model category $\mathcal{E}$. It implies a version of the Baez-Dolan stabilization hypothesis for Rezk's weak n-categories and some other stabilization results.

1. Introduction

Breen [10] and later Baez and Dolan [1] suggested the following *stabilization hypothesis* in higher category theory.

Hypothesis 1.1. *The category of n-tuply monoidal k-categories is equivalent to the category of $(n+1)$-tuply monoidal k-categories provided $n \geq k+2$.*

Baez and Dolan define n-tuply monoidal k-category as a weak $n+k$-category which has only one cell in each dimension smaller than n. It is known that such a definition, if taken naively, is not completely satisfactory because even if there is a unique cell in lower dimension the action of higher coherence cells associated to it can be nontrivial (see [11] for a discussion).

To get rid of this problem we have to work with weakest possible morphisms of k-categories. Moreover, we want to be able to speak about monoidal structures on k-categories. One technically convenient way to do it is to choose a symmetric monoidal model category $(\mathcal{E}, \otimes, e)$ whose homotopy category is equivalent to the homotopy category of weak k-categories and weak k-functors. For example, for $k=1$ one can take the category of categories with cartesian product and 'folklore' model structure, and for $k=2$ one can consider the category of 2-categories with Gray-product as tensor product and Lack's model structure [18]. For any $k \geq 0$ the category of Θ_k simplicial presheaves $\Theta_k Sp_k$ with Rezk model structure satisfies this requirement [9, 21]. There is a widely accepted understanding that a monoidal k-category is just an E_1-algebra in such a monoidal model category $\mathcal{E}$ [19].

Now, an n-tuply monoidal weak k-category must have n monoidal structures which interact coherently. It is another widely accepted idea that such an interaction of structures is equivalent to an action of an E_n-operad [19]. This is justified by an additivity theorem for E_n-operads ([19, Theorem 5.1.1.2]): the tensor product of an E_n-operad and an E_m-operad is an E_{n+m}-operad. All operads here have to be understood as ∞-operads, and tensor product is a 'derived' Boardman-Vogt tensor product. The statement and proof of this theorem are subtle because of a tricky homotopy behaviour of the Boardman-Vogt tensor product [14]. Lurie's

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Additivity Theorem holds only in a fully homotopised world. As a consequence the stabilisation result ([19, Example 5.1.2.3]) gives an equivalence of $(\infty, 1)$-categories rather than Quillen equivalence of model categories of algebras.[1,2]

In this paper we choose a different approach to coherent interaction of monoidal structures which comes closer to the original Baez-Dolan understanding of n-tuply monoidal k-category as a degenerate $(n+k)$-category[3] and does not require the Additivity Theorem (though we conjecture that the Additivity Theorem can be proved using our techniques). *An n-tuply monoidal k-category for us is an algebra of a cofibrant contractible n-operad in $\mathcal{E}$.*

For $n = 1$ this means that a monoidal k-category is an algebra of a cofibrant nonsymmetric contractible operad, which is well known to be homotopy equivalent to an E_1-algebra structure. This simple observation was extended in [2] to arbitrary dimension. It is shown here that the derived symmetrisation functor on a terminal n-operad is an E_n-operad and, therefore, the homotopy category of algebras of cofibrant contractible n-operads is equivalent to the homotopy category of E_n-algebras. So, homotopically both approaches are equivalent. The difference is that our point of view allows us to avoid the Boardman-Vogt tensor product and ∞-operads. This has some advantage as we can use classical operadic and model theoretic methods and the final result is formulated in terms of Quillen equivalences, which is a stronger statement. Also our techniques allow us to prove stabilisation not only for weakly unital k-categories but also for nonunital algebras. It is known that Additivity Theorem fails in this case even for cofibrant operads [14, Section 3]. Our proof is essentially the same in all cases; we just need to choose an appropriate category of n-operads.

2. Higher operads and symmetrisation

2.1. **n-ordinals and n-operads.** Let $n \geq 0$. Recall [2, Sec. II] that an *n-ordinal* is a finite set T equipped with n binary relations $<_0, \dots, <_{n-1}$ such that

(i) $<_p$ is nonreflexive,
(ii) for every pair a, b of distinct elements of T there exists exactly one p such that

$$a <_p b \quad \text{or} \quad b <_p a,$$

(iii) if $a <_p b$ and $b <_q c$, then $a <_{\min(p,q)} c$.

A morphism of n-ordinals $\sigma : T \to S$ is a map of the underlying sets such that $i <_p j$ in T implies that

(i) $\sigma(i) <_r \sigma(j)$ for some $r \geq p$, or
(ii) $\sigma(i) = \sigma(j)$, or
(iii) $\sigma(j) <_r \sigma(i)$ for $r > p$.

[1] Lurie formulated his argument in the context of $(n, 1)$-categories but observed that it can be extended to a more general context of (n, k)-categories. This has been done by Gepner and Haugseng in [15].

[2] A weaker version of the stabilisation hypothesis was earlier proved by Simpson in [22].

[3] Gepner and Haugseng [15] show that such an interpretation is also possible using their weak enrichment approach.

Let $Ord(n)$ be the skeletal category of n-ordinals and their morphisms. Each n-ordinal can be represented as a pruned planar rooted tree with n levels (*pruned n-tree* for short); cf. [2, Theorem 2.1]. For example, the 2-ordinal

$$0 <_0 2,\ 0 <_0 3,\ 0 <_0 4,\ 1 <_0 2,\ 1 <_0 3,\ 1 <_0 4,\ 0 <_1 1,\ 2 <_1 3,\ 2 <_1 4,\ 3 <_1 4$$

is represented by the following pruned tree:

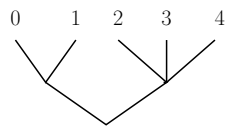


FIGURE 1

The initial n-ordinal $z^n U_0$ has empty underlying set, and its representing pruned n-tree is degenerate: it has no edges but consists only of the root at level 0. The terminal n-ordinal U_n is represented by a linear tree with n levels.

We also would like to consider the limiting case of ∞-ordinals.

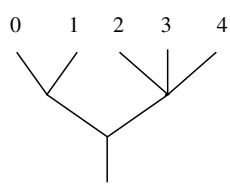
Definition 2.2. Let T be a finite set equipped with a sequence of binary anti-reflexive complimentary relations $<_0, <_{-1} \ldots, <_p, <_{p-1} \ldots$ for all integers $p \leq 0$. The set T is called an ∞-ordinal if these relations satisfy

- $a <_p b$ and $b <_q c$ implies $a <_{min(p,q)} c$.

The definition of morphism between ∞-ordinals coincides with the definition of morphism between n-ordinals for finite n. The category $Ord(\infty)$ denotes *the skeletal category of ∞-ordinals.*

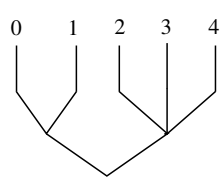
For an n-ordinal R we consider its *vertical suspension* $S(R)$ which is an $(n+1)$-ordinal with the underlying set R, and the order $<_m$ equal the order $<_{m-1}$ on R $<_0$ is empty.

For example, a vertical suspension of the 2-ordinal from Figure 1 is the 3-ordinal



More generally, one can consider a p-suspension S_p where we trivialise the orders $<_p$. So, the vertical suspension is $S = S_0$.

For example, the suspension S_2 of the 2-ordinal from Figure 1 is



Suspension operations give us a family of functors

$$S_p : Ord(n) \to Ord(n+1),\ 0 \le p \le n.$$

We also define an ∞-vertical suspension functor $Ord(n) \to Ord(\infty)$ as follows. For an n-ordinal T its ∞-suspension is an ∞-ordinal $S^\infty T$ whose underlying set is the same as the underlying set of T and $a <_p b$ in $S^\infty T$ if $a <_{n+p-1} b$ in T. It is not hard to see that the sequence

$$Ord(0) \xrightarrow{S} Ord(1) \xrightarrow{S} Ord(2) \longrightarrow \dots \xrightarrow{S} Ord(n) \longrightarrow \dots \xrightarrow{S^\infty} Ord(\infty)$$

exhibits $Ord(\infty)$ as a colimit of $Ord(n)$.

The categories $Ord(n), 0 \le n \le \infty$ are operadic in the sense of Batanin and Markl; cf. [6]. This means that $Ord(n)$ is equipped with cardinality and fiber functors. The cardinality functor

$$|-| : Ord(n) \to \mathrm{FinSet} \tag{1}$$

associates to an n-ordinal T its underlying set. Here FinSet is a skeletal version of the category of finite sets [6]. The fiber functor associates to each morphism of n-ordinals $\sigma : T \to S$ and $i \in |S|$ the preimage $\sigma^{-1}(i)$ with the induced structure of an n-ordinal.

The category FinSet is another example of an operadic category. The fiber functor is given by the preimage as above [6].

Any operadic category $\mathcal{O}$ has an associated category of operads $Op_{\mathcal{O}}(\mathcal{E})$ with values in an arbitrary symmetric monoidal category $(\mathcal{E}, \otimes, e)$ [6]. The category $Op_{\mathrm{FinSet}}(\mathcal{E}) = SOp(\mathcal{E})$ is the category of classical symmetric operads in $\mathcal{E}$.

The category $Op_n(\mathcal{E})$ of n-operads in $\mathcal{E}$ is by definition the category $Op_{Ord(n)}(\mathcal{E})$.

Explicitly an n-operad in $\mathcal{E}$ is a collection $\{A_T\}, T \in Ord(n)$ of objects in $\mathcal{E}$ equipped with the following structure:

- a morphism $\epsilon : e \to A(U_n)$ (unit);

- a morphism $m_\sigma : A(S) \otimes A(T_0) \otimes \dots \otimes A(T_k) \to A(T)$ (multiplication) for each map of n-ordinals $\sigma : T \to S$, where $T_i = \sigma^{-1}(i),\ i \in \{0, \dots, k\} = |S|$.

They must satisfy the following identities:

- for any composite map of n-ordinals

$$T \xrightarrow{\sigma} S \xrightarrow{\omega} R$$

the associativity diagram

$$\begin{tikzcd}
A(R)\otimes A(S_\bullet)\otimes A(T_0^\bullet)\otimes\dots\otimes A(T_i^\bullet)\otimes\dots\otimes A(T_k^\bullet) \arrow[r, "\cong"] \arrow[d] & \begin{array}{c} A(R)\otimes A(S_0)\otimes A(T_0^\bullet)\otimes\dots \\ \otimes A(S_i)\otimes A(T_i^\bullet)\otimes\dots\otimes A(S_k)\otimes A(T_k^\bullet) \end{array} \arrow[d] \\
A(S)\otimes A(T_0^\bullet)\otimes\dots\otimes A(T_i^\bullet)\otimes\dots\otimes A(T_k^\bullet) \arrow[dr] & A(R)\otimes A(T_\bullet) \arrow[dl] \\
& A(T)
\end{tikzcd}$$

commutes, where

$$A(S_\bullet) = A(S_0) \otimes \dots \otimes A(S_k),$$
$$A(T_i^\bullet) = A(T_i^0) \otimes \dots \otimes A(T_i^{m_i})$$

and

$$A(T_\bullet) = A(T_0) \otimes \cdots \otimes A(T_k);$$

- for an identity $\sigma = id : T \to T$ the diagram

$$\begin{tikzcd}
A(T) \otimes e \otimes \cdots \otimes e \arrow[rr] \arrow[dr, "\simeq"'] & & A(T) \otimes A(U_n) \otimes \cdots \otimes A(U_n) \arrow[dl] \\
& A(T) &
\end{tikzcd}$$

commutes;

- for the unique morphism $T \to U_n$ the diagram

$$\begin{tikzcd}
e \otimes A(T) \arrow[rr] \arrow[dr, "\simeq"'] & & A(U_n) \otimes A(T) \arrow[dl] \\
& A(T) &
\end{tikzcd}$$

commutes.

Functors between operadic categories which preserve cardinalities and fibers are called *operadic functors* [6]. The cardinality functor is always an operadic functor. An operadic functor between operadic categories $p : \mathcal{O} \to \mathcal{O}'$ induces a restriction functor $p^* : Op_{\mathcal{O}'}(\mathcal{E}) \to Op_{\mathcal{O}}(\mathcal{E})$ [6]. If $\mathcal{E}$ is a cocomplete symmetric monoidal category, then the restriction functor has a left adjoint $p_! : Op_{\mathcal{O}}(\mathcal{E}) \to Op_{\mathcal{O}'}(\mathcal{E})$.

Any of the suspension functors is an operadic functor. In particular the following diagram commutes:

$$\begin{tikzcd}
Ord(n) \arrow[rr, "S_p"] \arrow[dr, "|-|"'] & & Ord(n+1) \arrow[dl, "|-|"] \\
& \mathrm{FinSet} &
\end{tikzcd} \tag{2}$$

Hence, it induces the following diagram of adjunctions:

$$\begin{tikzcd}
Op_n(\mathcal{E}) \arrow[rr, shift right, "(S_p)_!"'] \arrow[dr, shift left, "sym_n"] & & Op_{n+1}(\mathcal{E}) \arrow[ll, shift right, "S_p^*"'] \arrow[dl, shift left, "sym_{n+1}"'] \\
& SOp(\mathcal{E}) \arrow[ul, shift left, "des_n"] \arrow[ur, shift left, "des_{n+1}"] &
\end{tikzcd} \tag{3}$$

In this diagram the functor des_n is the restriction functor along the cardinality functor, called the *desymmetrisation functor*, and sym_n is its left adjoint, called the *symmetrisation functor* [3].

Example 2.3. Let $Ass_n \in Op_n(\mathcal{E})$ be an n-operad given by $Ass_n(T) = e, T \in Ord(n)(\mathcal{E})$. It is immediate from the definition of the suspension functors that $S_p^*(Ass_{n+1}) = Ass_n$. On the other hand, $sym_1(Ass_1) = Ass$ is the classical symmetric operad for monoids, while for $n \geq 2$, $sym_n(Ass_n) = Com$ is the operad for commutative monoids. This is the classical Eckmann-Hilton argument in disguise; cf. [3].

Now let $\mathcal{E}$ be a closed symmetric monoidal category. An object $X \in \mathcal{E}$ has an associated endomorphism symmetric operad $\mathcal{E}nd_X$:

$$\mathcal{E}nd_X(n) = \underline{\mathcal{E}}(X^{\otimes^n}, X),$$

where $\underline{\mathcal{E}}$ is the internal hom of $\mathcal{E}$.

Definition 2.4. An algebra of a symmetric operad $A \in SOp(\mathcal{E})$ is an object $X \in \mathcal{E}$ equipped with a morphism of operads $A \to \mathcal{E}nd_X$.

An algebra of an n-operad $B \in Op_n(\mathcal{E})$ is an object $X \in \mathcal{E}$ equipped with a morphism of operads $B \to des_n(\mathcal{E}nd_X)$.

Lemma 2.5. *Let $\mathcal{E}$ be a cocomplete closed symmetric monoidal category and let $B \in Op_n(\mathcal{E})$. The following categories are equivalent:*

(1) *the category of algebras of the n-operad B;*
(2) *the category of algebras of the $(n+1)$-operad $(S_p)_!(B)$;*
(3) *the category of algebras of the symmetric operad $sym_n(B)$.*

Proof. If $\mathcal{E}$ is cocomplete the symmetrisation functor sym_n exists, and we use the adjuction $sym_n \dashv des_n$ to transform a B-algebra structure $B \to des_n(\mathcal{E}nd_X)$ to a $sym_n(B)$-algebra structure $sym_n(B) \to \mathcal{E}nd_X$. The proof for the $(S_p)_!(B)$-algebra structure is similar. □

Definition 2.6. A symmetric operad (n-operad) $A \in SOp(\mathcal{E})$ ($A \in Op(\mathcal{E})$) is called constant-free if $A(0)$ ($A(z^nU_0)$) is an initial object in $\mathcal{E}$.

The category $CFSOp(\mathcal{E})$ of a constant-free symmetric operad is equivalent to the category $Op_{\mathrm{FinSet}_0}(\mathcal{E})$ where FinSet_0 is an operadic subcategory of FinSet of nonempty finite sets and surjective maps. Analogously, the category of constant-free n-operads $CFOp_n(\mathcal{E})$ is equivalent to $Op_{Ord_0(n)}(\mathcal{E})$ where $Ord_0(n)$ is the operadic category of nonempty n-ordinals and surjections. This observation allows us to reformulate all previous statements for symmetric operads and n-operads in the context of constant-free operads. So, the commutative triangle of adjunctions (3), as well as the analogue of Lemma 2.5, holds for constant-free operads too.

2.7. **Polynomial monads.** Symmetric and n-operads are examples of algebras of polynomial monads.

Definition 2.8. A finitary polynomial P is a diagram in Set of the form

$$J \xleftarrow{\ s\ } E \xrightarrow{\ p\ } B \xrightarrow{\ t\ } I$$

where $p^{-1}(b)$ is a finite set for any $b \in B$.

Each polynomial P generates a functor called a *polynomial functor* between functor categories

$$\underline{P} : \mathrm{Set}^J \to \mathrm{Set}^I,$$

which is defined as the composite functor

$$\mathrm{Set}^J \xrightarrow{\ s^*\ } \mathrm{Set}^E \xrightarrow{\ p_*\ } \mathrm{Set}^B \xrightarrow{\ t_!\ } \mathrm{Set}^I.$$

So, the functor $\underline{P}$ is given by the formula

$$\underline{P}(X)_i = \coprod_{b \in t^{-1}(i)} \prod_{e \in p^{-1}(b)} X_{s(e)}, \tag{4}$$

which explains the name 'polynomial' that is a sum of products of formal variables.

A cartesian morphism between polynomial functors is their natural transformation such that each naturality square is a pullback. Composition of finitary polynomial functors is again a finitary polynomial functor. Finitary polynomial functors and their cartesian morphisms form a 2-category $\mathbf{Poly}_f$.

Definition 2.9. A finitary polynomial monad is a monad in the 2-category $\mathbf{Poly}_f$.

Remark 2.10. A finitary polynomial functor preserves filtered colimits and pullbacks. A polynomial monad is cartesian; that is, its underlying functor preserves pullbacks, and its unit and multiplication are cartesian natural transformations.

Remark 2.11. One can consider more general polynomial functors of nonfinitary type. Since in this paper we don't need these more general functors, we call finitary polynomial monads simply polynomial monads.

Let $\mathcal{E}$ be a cocomplete symmetric monoidal category and P be a polynomial functor. One can construct a functor $\underline{P}^{\mathcal{E}} : \mathcal{E}^I \to \mathcal{E}^I$ given by the formula similar to (4):

$$\underline{P}^{\mathcal{E}}(X)_i = \coprod_{b \in t^{-1}(i)} \bigotimes_{e \in p^{-1}(b)} X_{s(e)}.$$

If $I = J$ and P was given a structure of a polynomial monad, then $\underline{P}^{\mathcal{E}}$ acquires a structure of a monad on $\mathcal{E}^I$.

Definition 2.12. The category of algebras of a polynomial monad P in a cocomplete symmetric monoidal category $\mathcal{E}$ is the category of algebras of the monad $\underline{P}^{\mathcal{E}}$.

Example 2.13. There is a polynomial monad SO whose category of algebras is isomorphic to the category of symmetric operads. This monad is given by the polynomial

$$\texttt{FinSet} \xleftarrow{s} \texttt{OrderedRootedTrees}^* \xrightarrow{p} \texttt{OrderedRootedTrees} \xrightarrow{t} \texttt{FinSet}$$

in which `FinSet` is the set of isomorphism classes of objects in FinSet and `OrderedRootedTrees` is the set of isomorphism classes of ordered rooted trees. The multiplication in SO is induced by insertion of a tree to a vertex of another tree; cf. [5, Section 9.4].

There is a polynomial monad $O(n)$ whose category of algebras is isomorphic to the category of algebras of n-operads. It is generated by the polynomial

$$\texttt{Ord(n)} \xleftarrow{s} \texttt{nPlanarRootedTrees}^* \xrightarrow{p} \texttt{nPlanarRootedTrees} \xrightarrow{t} \texttt{Ord(n)}$$

where `Ord(n)` is the set of isomorphism classes of n-ordinals and `nPlanarRootedTrees` is the set of isomorphism classes of n-planar trees. The multiplication of the monad is induced by insertion of n-planar trees into vertices of n-planar trees; cf. [5, Proposition 12.15].

The commutative triangle (2) induces in an obvious way a commutative triangle of polynomial monads:

$$\begin{tikzcd} O(n) \arrow[rr] \arrow[dr, "|-|"'] & & O(n+1) \arrow[dl, "|-|"] \\ & SO & \end{tikzcd} \tag{5}$$

and the triangle of adjunctions (3) is also induced by (5).

Example 2.14. There is a polynomial monad $CFSO$ such that its category of algebras is equivalent to the category of constant-free symmetric operads $CFSOp(\mathcal{E})$ [5, Section 9.4]. The corresponding generating polynomial is

$$\mathtt{FinSet}_0 \xleftarrow{s} \mathtt{OrderedRootedTrees}^*_{reg} \xrightarrow{p} \mathtt{OrderedRootedTrees}_{reg} \xrightarrow{t} \mathtt{FinSet}_0$$

where $\mathtt{FinSet}_0$ is the set of isomorphism classes of nonempty finite sets and $\mathtt{OrderedRootedTrees}_{reg}$ is the set of isomorphism classes of *regular* ordered rooted trees. We call a tree regular if for any vertex of the tree the set of incoming edges at this vertex is not empty (so regular trees do not have stumps).

Similarly there is a polynomial monad $CFO(n)$ whose category of algebras is equivalent to the category of constant-free n-operads $CFOp_n(\mathcal{E})$; cf. [5, Proposition 12.19]. It is generated by a polynomial

$$\mathtt{ROrd(n)} \xleftarrow{s} \mathtt{nPlanarRootedTrees}^*_{reg} \xrightarrow{p} \mathtt{nPlanarRootedTrees}_{reg} \xrightarrow{t} \mathtt{ROrd(n)}$$

where $\mathtt{ROrd(n)}$ is the set of isomorphism classes of nonempty n-ordinals and $\mathtt{nPlanarRootedTrees}_{reg}$ is the set of isomorphism classes of regular n-planar trees.

2.15. **Classifiers for maps between polynomial monads.** For any cartesian morphism of cartesian monads $\phi : S \to T$ one can associate a category (in fact a strict categorical T-algebra) $\mathbf{T}^{\mathtt{S}}$ with certain universal property [3, 5, 24]. This category is called *the classifier of internal S-algebras inside categorical T-algebras.*

Classifiers allow us to compute the left adjoint functor between categories of algebras induced by ϕ in terms of a colimit over $\mathbf{T}^{\mathtt{S}}$. In particular, the symmetrisation functor sym_n admits an explicit description as a colimit over the classifier $\mathbf{SO}^{\mathtt{O(n)}}$ of the map of polynomial monads $|-| : O(n) \to SO$. (See [3].)

The homotopy type of the nerve of the classifier $\mathbf{SO}^{\mathtt{O(n)}}$ was computed in [2]:

$$N(\mathbf{SO}^{\mathtt{O(n)}}) = \coprod_k N(\mathbf{SO}^{\mathtt{O(n)}}_k)$$

where $N(\mathbf{SO}^{\mathtt{O(n)}}_k)$ has homotopy type of the configuration space of k points in $\mathbb{R}^n$. It follows that for $n \geq 2$ and $0 \leq i \leq n-2$ the homotopy groups $\pi_i(N(\mathbf{SO}^{\mathtt{O(n)}}_k), a) = 0$. This can be reformulated such that the nerve of the unique map of categories

$$! : \mathbf{SO}^{\mathtt{O(n)}}_k \to 1 \tag{6}$$

is an $(n-2)$-local weak equivalence of simplicial sets [12, Corollary 9.2.15].

For $n = \infty$ we also have a classifier $\mathbf{SO}^{\mathtt{O}(\infty)}$. It is not hard to see that $\mathbf{SO}^{\mathtt{O}(\infty)}$ is the colimit of this sequence of classifiers $\mathbf{SO}^{\mathtt{O(n)}}$ induced by the vertical suspension functor, and so the nerve of $\mathbf{SO}^{\mathtt{O}(\infty)}$ is a contractible simplicial symmetric operad.

3. Stabilization of algebras of n-operads

3.1. **Model categories of symmetric operads, n-operads and their algebras.** Now we assume that our base symmetric monoidal category $(\mathcal{E}, \otimes, e)$ is a cofibrantly generated monoidal model category. Let C be a set and (T, μ, ϵ) be a monad on the category $\mathcal{E}^C$. There is a product model structure on the category $\mathcal{E}^C$, and so one can try to induce a model structure on the category of T-algebras as follows. We define an algebra morphism $f : X \to Y$ to be a weak equivalence (fibration) if $U(f)$ is a weak equivalence (resp. fibration) in $\mathcal{E}^C$, where U is the forgetful functor from T-algebras to $\mathcal{E}^C$. It is more often with this definition that we will get only a semimodel structure [13, 26], not the full model structure on

algebras, but it is sufficient for our purpose. If such a (semi)model structure exists we call it a transferred model structure. An algebra X of T is called *relatively cofibrant* if $U(X)$ is a cofibrant object in $\mathcal{E}^C$.

Proposition 3.2. *If $e \in \mathcal{E}$ is cofibrant, then for any polynomial monad T the category $Alg_T(\mathcal{E})$ admits a transferred semimodel structure in which all cofibrant algebras are relatively cofibrant.*

Proof. The category of algebras of T is isomorphic to the category of algebras of a coloured symmetric operad $O(T)$ [5] whose spaces of operations are of the form $e \otimes O(c_1, \dots, c_m; c) = \bigsqcup_{O(c_1,\dots,c_m;c)} e$ where $O(c_1, \dots, c_m; c)$ is a set with free action of symmetric groups. If e is cofibrant this underlying object of operations is a Σ-cofibrant object and so $O(T)$ is a Σ-cofibrant operad. The statement of the proposition follows now from [26, Theorem 6.3.1]. □

Proposition 3.3. *If e is cofibrant in $\mathcal{E}$, then*

(1) *the categories $Op_n(\mathcal{E}), CFOp_n(\mathcal{E}), SOp(\mathcal{E})$ and $CFSOp(\mathcal{E})$ admit transferred semimodel structures;*
(2) *cofibrant symmetric and n-operads are relatively cofibrant;*
(3) *the triangle* (3) *is a triangle of Quillen adjunctions;*
(4) *the category of algebras of cofibrant symmetric and cofibrant n-operads admits transferred (semi)model structures;*
(5) *for any weak equivalence between cofibrant operads $f : A \to B$ the induced adjunction $f_! \dashv f^*$ between categories of algebras is a Quillen equivalence.*

Proof. Symmetric operads (general or constant-free) as well as n-operads (general or constant-free) are algebras of polynomial monads. So we are in the conditions of Proposition 3.2. The existence of transferred (semi)model structure on algebras of cofibrant symmetric operads is proven in [13, Proposition 4.4.3] and [26]. The existence of transferred model structure on algebras of cofibrant n-operads follows from this and Lemma 2.5. Indeed, since sym_n is a left Quillen functor $sym_n(A)$ is a cofibrant symmetric operad for any cofibrant n-operad A. The last point of the proposition is proven in [13, Proposition 4.4.6]. □

Remark 3.4. This semimodel structure on operads is often a full model structure [5, 26], but not always. For example, the category of symmetric operads (n-operads for $n \geq 2$) in the category of chain complexes of finite characteristic does not admit full model structures [5], but there is a full model structure on the category of constant-free symmetric or n-operads for any compactly generated monoidal model category which satisfies the monoid axiom of Schwede and Shipley [5].

3.5. **Stabilization of algebras.** In this section we prove stabilisation of homotopy categories of algebras of n-operads. The same proof works for constant-free n-operads, so we do not mention them anymore. To simplify notation we fix a $p \geq 0$ and call p-suspension of an n-ordinal simply a suspension and we denote it $S : Ord(n) \to Ord(n+1)$. We also denote by S the map of polynomial monads induced by the suspension. The proof of our main result does not depend on p.

Let $\mathcal{E}$ satisfy all assumptions of Proposition 3.3. Let $G_n \in Op_n(\mathcal{E})$ be a cofibrant replacement for Ass_n. We will denote by $B_n(\mathcal{E})$ the category of G_n-algebras in $\mathcal{E}$. Let also $E_\infty(\mathcal{E})$ be the model category of E_∞-algebras in $\mathcal{E}$, that is, the category of algebras of a cofibrant replacement E of the symmetric operad Com.

Remark 3.6. The category $B_n(\mathcal{E})$ is equivalent to the category of algebras of the symmetric operad $sym_n(G_n)$ which is a cofibrant E_n-operad; cf. [2].

By Lemma 2.5 there is an isomorphism of categories of algebras of an n-operad G_n and an $(n+1)$-operad $S_!(G_n)$. Also observe that $S_!$ is a left Quillen functor and, hence, preserves cofibrations. In particular, the operad $S_!(G_n)$ is cofibrant. There is a map of $(n+1)$-operads $i : S_!(G_n) \to G_{n+1}$. Indeed, since $S^*(Ass_{n+1}) = Ass_n$ by adjunction we have a map $S_!(G_n) \to Ass_{n+1}$. We also have a trivial fibration $G_{n+1} \to Ass_{n+1}$. Since $S_!(G_n)$ is cofibrant there is a lifting $i : S_!(G_n) \to G_{n+1}$. Without loss of generality we can think that i is a cofibration because if it is not we can always factorise it as a cofibration followed by a trivial fibration and so replace G_{n+1} by another cofibrant operad with a trivial fibration to Ass_{n+1}.

The morphism i induces a Quillen adjunction between algebras of $S_!(G_n)$ and algebras of G_{n+1} and so between algebras of G_n and G_{n+1}. Slightly abusing notation we will denote this adjunction $i^* \vdash i_!$.

Recall that a *standard system of simplices* in a monoidal model category $\mathcal{E}$ is a cosimplicial object δ in $\mathcal{E}$ satisfying the following properties [8, Definition A.6]:

(i) δ is cofibrant for the Reedy model structure on $\mathcal{E}^{\Delta}$,
(ii) δ^0 is the unit object I of $\mathcal{E}$ and the simplicial operators $[m] \to [n]$ act via weak equivalences $\delta^m \to \delta^n$ in $\mathcal{E}$, and
(iii) the simplicial realization functor $|-|_\delta = (-)\otimes_\Delta \delta : \mathcal{E}^{\Delta^{op}} \to \mathcal{E}$ is a symmetric monoidal functor whose structural maps

$$|X|_\delta \otimes_V |Y|_\delta \to |X \otimes_V Y|_\delta$$

are weak equivalences for Reedy-cofibrant objects $X, Y \in \mathcal{E}^{\Delta^{op}}$.

Recall also that a model category $\mathcal{E}$ is called *k-truncated* if for all $X, Y \in \mathcal{E}$,

$$\pi_i(\widetilde{\mathcal{E}}(X,Y), a) = 0 \ , \ i > k,$$

for any choice of base point a. Here $\widetilde{\mathcal{E}}(X,Y)$ is a homotopy function complex of $\mathcal{E}$ [16].

Theorem 3.7. *Let $(\mathcal{E}, \otimes, e)$ be a cofibrantly generated monoidal model category whose unit $e \in \mathcal{E}$ is cofibrant. Then*

(a) *for any $2 \le n < \infty$ there is a commutative triangle of Quillen adjunctions:*

$$\begin{tikzcd}
B_n(\mathcal{E}) \arrow[rr, shift right, "i_!"'] \arrow[dr, shift left] & & B_{n+1}(\mathcal{E}) \arrow[ll, shift right, "i^*"'] \arrow[dl, shift right] \\
& E_\infty(\mathcal{E}) \arrow[ul, shift left] \arrow[ur, shift left] &
\end{tikzcd}$$

(b) *If $\mathcal{E}$ has a standard system of simplices, then there is a Quillen equivalence*

$$\begin{tikzcd}
B_\infty(\mathcal{E}) \arrow[rr, shift right] & & E_\infty(\mathcal{E}) \arrow[ll, shift right]
\end{tikzcd}$$

(c) *If, in addition, $\mathcal{E}$ is k-truncated, then the triangle from (a) is a triangle of Quillen equivalences for any $n \ge k+2$.*

Proof. Apply the symmetrisation functor sym_n to the cofibrant replacement $G_n \to Ass_n$. We have a morphism $P_n : sym_n(G_n) \to sym_n(Ass_n) = Com$ and hence a lifting of this morphism to the morphism of operads $sym_n(G_n) \to E$. By (3) we can replace it by a morphism $sym_{n+1}(S_!(G_n)) \to E$. Applying sym_{n+1} to the cofibration $i : S_!(G_n) \to G_{n+1}$ we have a composite

$$sym_{n+1}(S_!(G_n)) \to sym_{n+1}(G_{n+1}) \to sym_{n+1}(1_{n+1}) = Com$$

and since G_n is cofibrant we have a lifting

$$sym_{n+1}(S_!(G_n)) \to E.$$

So, we have a commutative square of operads:

$$\begin{tikzcd}
sym_{n+1}(S_!(G_n)) \arrow[r] \arrow[d, "sym_{n+1}(i)"'] & E \arrow[d] \\
sym_{n+1}(G_{n+1}) \arrow[r, "P_{n+1}"] \arrow[ur, dashed] & Com
\end{tikzcd}$$

and, hence, a lifting $sym_{n+1}(G_{n+1}) \to E$. The upper commutative triangle of operads induces the triangle of Quillen adjunctions. This proves statement (a).

Let us first prove statement (c) of the theorem, so we assume that $\mathcal{E}$ is k-truncated.

By construction the composite of the left vertical morphism and the bottom horizontal morphism is $sym_{n+1}(S_!(P_n))$, and by naturality of isomorphism $sym_n \simeq sym_{n+1}(S_!)$ is isomorphic to P_n. To finish the proof it will be enough to show that P_n and P_{n+1} are weak equivalences of operads, and so $sym_n(G_n)$ and $sym_{n+1}(G_{n+1})$ are both cofibrant replacements of Com in the category of symmetric operads in $\mathcal{E}$. The morphism $sym_{n+1}(i)$ is then a weak equivalence by two out of three properties.

Since G_n is cofibrant the operad $sym_n(G_n)$ is weakly equivalent to the operad $\mathbb{L}sym_n(G_n)$, where $\mathbb{L}sym_n$ is the left derived symmetrisation functor. The underlying object of G_n is cofibrant, and $\mathcal{E}$ has a standard system of simplices so we can apply Theorem 8.2 from [5]. This theorem states that $\mathbb{L}sym_n(G_n)(T)$ is the homotopy colimit in $\mathcal{E}$ of a diagram $\widetilde{G_n} : \mathbf{SO}^{\mathtt{0(n)}} \to \mathcal{E}$.

The functor $\widetilde{G_n}$ representing the n-operad G_n has value on an object $\tau \in \mathbf{SO}^{\mathtt{0(n)}}$ given by a certain tensor product of values of the operad G_n, and, hence, the functor $\widetilde{G_n}$ is equipped with a canonical weak equivalence $\widetilde{G_n}(\tau) \to !^*(e)$, where $!^*(e)$ is the constant functor on $\mathbf{SO}^{\mathtt{0(n)}}$ whose value is the tensor unit e. Since both functors $\widetilde{G_n}$ and $!^*(e)$ are pointwise cofibrant we have a weak equivalence of homotopy colimits. It remains to show that the canonical morphism

$$\mathrm{hocolim}_{\mathbf{SO}^{\mathtt{0(n)}}} !^*(e) \to e$$

is a weak equivalence. For this it is enough to prove that for any fibrant object $S \in \mathcal{E}$ the induced map of simplicial sets

$$\widetilde{\mathcal{E}}(\mathrm{hocolim}_{\mathbf{SO}^{\mathtt{0(n)}}} !^*(e), S) \leftarrow \widetilde{\mathcal{E}}(e, S)$$

is a weak equivalence. Equivalently, we have to prove that for any fibrant k-truncated simplicial set W the map

$$\mathrm{holim}_{\mathbf{SO}^{\mathtt{0}(\mathtt{n})}} !^*(W) \leftarrow W \tag{7}$$

is a weak equivalence. Let $\mathcal{S}(-,-)$ be the internal hom in simplicial sets. We have

$$\begin{aligned}\mathcal{S}(N(\mathbf{SO}^{\mathtt{0}(\mathtt{n})}), W) &\simeq \mathcal{S}(\mathrm{hocolim}_{\mathbf{SO}^{\mathtt{0}(\mathtt{n})}} !^*(1), W)\\ &\simeq \mathrm{holim}_{\mathbf{SO}^{\mathtt{0}(\mathtt{n})}} !^*(\mathcal{S}(1, W)) = \mathrm{holim}_{\mathbf{SO}^{\mathtt{0}(\mathtt{n})}} !^*(W),\end{aligned}$$

and the map (7) is induced by (6), so it is a weak equivalence since $N(!)$ is an $(n-2)$-equivalence and, hence, i-equivalence for each $i \leq n-2$. So, we have proved point (c) of the theorem.

The argument for (b) is identical, but we don't need $\mathcal{E}$ to be truncated because the classifier of ∞-operads inside symmetric operads is contractible. □

Corollary 3.8 (Stabilisation for weak k-groupoids)**.** *The suspension functor induces an equivalence between the homotopy category of n-tuply monoidal weak k-groupoids and $(n+1)$-tuply monoidal weak k-groupoids for $n \geq k+2$.*

Proof. We apply Theorem 3.7 to the category of homotopy k-types Sp_k which is the k-truncation of the model category of simplicial sets $Sp = Set^{\Delta^{op}}$ with its Kan model structure [12]. Weak k-groupoids are fibrant objects in this category. □

Remark 3.9. Corollary 3.8 implies the classical Freudental stabilisation theorem (cf. [1]).

Recall that Rezk's $(m+k, m)$-categories are fibrant objects in the model category $\Theta_m Sp_k, -2 \leq k \leq \infty$, which is a truncation of the model category of Rezk's complete Θ_m-spaces $\Theta_m Sp_\infty$. The category $\Theta_m Sp_\infty$ is itself a certain Bousfield localisation of the category of simplicial presheaves $Sp^{\Theta_m^{op}}$ with its injective model structure. This is a cartesian closed model category which is $(m+k)$-truncated and satisfies all hypotheses of Theorem 3.7 (see [21]).

Definition 3.10. The category of Rezk's n-tuply monoidal $(m+k, m)$-categories is the category of fibrant objects in the (semi)model category $B_n(\Theta_m Sp_k)$.

We immediately have

Corollary 3.11 (Stabilisation for Rezk's $(m+k, m)$-categories)**.** *The suspension functor induces an equivalence between the homotopy category of Rezk's n-tuply monoidal $(m+k, m)$-categories and Rezk's $(n+1)$-tuply monoidal $(m+k, m)$-categories for $n \geq m+k+2$.*

Remark 3.12. If $m = 0$ the category $\Theta_0 Sp_k$ is isomorphic (as a cartesian model category) to the category Sp_k (cf. [21]), and so Corollary 3.11 is a particular case of Corollary 3.8.

If $k = 0$ the fibrant objects of the category $\Theta_m Sp_m$ are weak m-categories, and so we have proved the classical Baez-Dolan Stabilization Hypothesis for Rezk m-categories.

Remark 3.13. The choice of the suspension functor amounts to the choice of a multiplicative structure on an algebra from $B_{n+1}(\mathcal{E})$ which we would like to 'forget'. Theorem 3.7 asserts that up to homotopy this choice in stable dimensions is not important.

Remark 3.14. The argument of Theorem 3.7 works equally well for the swiss cheese type symmetric and n-operads [2]. The stabilisation result amounts then to the stable version of the swiss cheese conjecture of Kontsevich; cf. [17].

Remark 3.15. Another conclusion from the proof of Theorem 3.7 is that some interesting results about equivalence of homotopy categories of algebras can be proved once we have a map of polynomial monads $\phi : S \to T$ such that the classifier $\mathbf{T}^{\mathrm{S}}$ is aspherical with respect to a fixed fundamental localiser $\mathcal{W}$ [12]. We hope to make use of this observation in the future.

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