

## MEAN PROXIMALITY AND MEAN LI-YORKE CHAOS

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**ABSTRACT.** We prove that if a topological dynamical system is mean sensitive and contains a mean proximal pair consisting of a transitive point and a periodic point, then it is mean Li-Yorke chaotic (DC2 chaotic). On the other hand we show that a system is mean proximal if and only if it is uniquely ergodic and the unique measure is supported on one point.

### 1. INTRODUCTION

In the study of topological dynamical systems different versions of chaos, which represent complexity in various ways, have been defined and studied. Some properties that are considered forms of chaos are positive entropy, topological mixing, Devaney chaos, and Li-Yorke chaos. The relationship among them has been one of the main interests of this topic. Solving open questions Huang and Ye [15] proved that Devaney chaos implies Li-Yorke chaos; and Blanchard, Glasner, Kolyada and Maass [3] showed that positive entropy also implies Li-Yorke chaos. It is also known that topological weak mixing implies Li-Yorke chaos [16]. For all the other implications among these notions there are counterexamples.

Many of the classical notions in topological dynamics have an analogous version in the mean sense (e.g. mean sensitivity, mean equicontinuity, and mean distality [6, 12, 19, 22]). In a similar way mean forms of Li-Yorke chaos, also known as distributional chaos (DC) have been defined and studied [5, 23]. A nice generalization proved by Downarowicz [5] says that positive topological entropy implies mean Li-Yorke chaos, which strengthens the result of [3]. It was also mentioned in [5] that mean Li-Yorke chaos is equivalent to the so-called chaos DC2 which was first studied for interval systems in [23]. In our present paper, we would like to use the former terminology (i.e. calling it mean Li-Yorke). Mean Li-Yorke chaos does not imply positive entropy (see, e.g., [9, 11] for details). For related topics we also recommend [4, 20].

Besides positive topological entropy we do not know of any other condition that implies mean Li-Yorke chaos. Oprocha showed that Devaney chaos does not imply mean Li-Yorke chaos [21]. Furthermore there are topologically mixing systems with dense periodic points and no mean Li-Yorke pairs [3].

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Motivated by the ideas and results above, we ask if there is some strong form of Devaney chaos (“mean Devaney chaos” one could say), that is stronger than mean Li-Yorke chaos. We show that with mean sensitivity and a stronger relationship between transitivity and periodicity we can obtain mean Li-Yorke chaos. As a consequence of this result we show that it is easy to construct subshifts with dense mean Li-Yorke subsets.

We also characterize mean proximal systems, which are a subclass of the classic proximal systems. It is known that a system is proximal if and only if its unique minimal subset is a fixpoint [2]. Among other characterizations we show that a system is mean proximal if and only if its unique invariant measure is a delta measure on a fixpoint  $x_0$ . As a corollary we also obtain that mean proximal systems contain no mean Li-Yorke pairs.

We recall some necessary definitions in the following. By a *topological dynamical system* (TDS, for short), we mean a pair  $(X, T)$ , where  $X$  is a compact metric space with the metric  $d$ , and  $T : X \rightarrow X$  is continuous.

Let  $(X, T)$  be a TDS. A point  $x \in X$  is called a *transitive point* if its orbit is dense in  $X$ , i.e.,  $\overline{\{T^n x : n \geq 0\}} = X$ ; and called a *periodic point* of period  $n \in \mathbb{N}$  if  $T^n x = x$  but  $T^i x \neq x$  for  $1 \leq i \leq n - 1$ .

A pair  $(x, y) \in X \times X$  is said to be *proximal* if

$$\liminf_{n \rightarrow \infty} d(T^n x, T^n y) = 0.$$

Given  $\eta > 0$ , a pair  $(x, y) \in X \times X$  is called a *Li-Yorke pair* if  $(x, y)$  is proximal and

$$\limsup_{n \rightarrow \infty} d(T^n x, T^n y) > \eta.$$

A subset  $S \subset X$  is called a *Li-Yorke set* (or a scrambled set) if every pair  $(x, y) \in S \times S$  of distinct points is a Li-Yorke pair. We say that  $(X, T)$  is *Li-Yorke chaotic* if  $X$  contains an uncountable Li-Yorke subset. It was shown in [8] that Li-Yorke sets have measure zero for every invariant measure.

We say that  $(X, T)$  is *sensitive*, if there exists  $\delta > 0$  such that for every  $x \in X$  and every  $\epsilon > 0$ , there is  $y \in B(x, \epsilon)$  satisfying

$$\limsup_{n \rightarrow \infty} d(T^n x, T^n y) > \delta.$$

Originally a TDS was defined to be *Devaney chaotic* if it was transitive, sensitive, and had dense periodic points (it was shown later that the sensitivity hypothesis can be removed). As we noted Devaney chaos implies Li-Yorke chaos [15].

Now we define the equivalent “mean” forms.

A pair  $(x, y) \in X \times X$  is said to be *mean proximal* if

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) = 0.$$

A TDS  $(X, T)$  is *mean proximal* if every pair  $(x, y) \in X \times X$  is mean proximal. Note that when studying invertible TDS this property is known as the so-called “forward mean proximal” [7, 22].

Given  $\eta > 0$ , a pair  $(x, y) \in X \times X$  is called a *mean Li-Yorke pair* (with modulus  $\eta$ ) if  $(x, y)$  is mean proximal and

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) (\geq \eta) > 0.$$

A subset  $S \subset X$  is called a *mean Li-Yorke set* (with modulus  $\eta$ ) if every pair  $(x, y) \in S \times S$  of distinct points is a mean Li-Yorke pair (with modulus  $\eta$ ). We say that  $(X, T)$  is *mean Li-Yorke chaotic* if  $X$  contains an uncountable mean Li-Yorke subset.

We say that  $(X, T)$  is *mean sensitive* if there exists  $\delta > 0$  such that for every  $x \in X$  and every  $\epsilon > 0$ , there is  $y \in B(x, \epsilon)$  satisfying

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) > \delta.$$

Next we turn to introducing a new version of chaos. Our aim is to add something to the version of Devaney chaos such that the chaos in the new sense implies mean Li-Yorke chaos. It is worth mentioning that there are Devaney chaotic examples (even with positive entropy) that are not mean sensitive [13]. So the first extra hypothesis we add is mean sensitivity. Now, by observing that if a TDS is Devaney chaotic, then we can find a periodic point and a transitive point such that they are proximal, we consider the stronger condition: “there exists a mean proximal pair which consists of a periodic point and a transitive point”. Indeed, this condition together with the mean sensitivity is enough, and we do not need the dense periodic points.

**Theorem 1.1.** *If a TDS  $(X, T)$  is mean sensitive and there is a forward mean proximal pair of  $X \times X$  consisting of a transitive point and a periodic point, then  $(X, T)$  is mean Li-Yorke chaotic; more precisely, there exist a positive number  $\eta$  and a subset  $K \subset X$  which is a union of countably many Cantor sets such that  $K$  is a mean Li-Yorke set with modulus  $\eta$ .*

As an application of this result we show that with this condition it is easy to construct mean Li-Yorke chaotic systems. See Example 2.1.

Another aim of this paper is to characterize mean proximal systems using mean asymptotic pairs. For a TDS  $(X, T)$ , we say that a pair  $(x, y) \in X \times X$  is *asymptotic* if

$$\lim_{n \rightarrow \infty} d(T^n x, T^n y) = 0,$$

and we say that  $(x, y) \in X \times X$  is *mean asymptotic* if

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) = 0.$$

A TDS is mean asymptotic if every pair  $(x, y) \in X \times X$  is mean asymptotic.

It is known that proximal systems (i.e., systems  $(X, T)$  satisfying that every pair  $(x, y) \in X \times X$  is proximal) may have no asymptotic pairs [18, Theorem 6.1] (and hence it may happen that every pair is a Li-Yorke pair). However for mean proximal systems it will be completely different; see Corollary 1.1.

Clearly, mean proximal pairs are proximal (hence mean proximal systems are proximal), but not vice versa; while asymptotic pairs are mean asymptotic, but not

vice versa. Thus, a priori, we have the following implications (both for pairs and for systems):

$$\text{asymptotic} \Rightarrow \text{mean asymptotic} \Rightarrow \text{mean proximal} \Rightarrow \text{proximal}.$$

The next theorem reverses the central implication for systems.

We denote by  $M(X, T)$  the set of all  $T$ -invariant Borel probability measures on  $X$ . Given a set  $Y$ , we denote the diagonal in the product space as  $\Delta_Y := \{(y, y) : y \in Y\}$ .

**Theorem 1.2.** *Suppose that  $(X, T)$  is a TDS. Then the following are equivalent:*

- (1)  $(X, T)$  is mean proximal.
- (2)  $(X, T)$  is mean asymptotic.
- (3) Every measure  $\lambda \in M(X \times X, T \times T)$  satisfies  $\lambda(\Delta_X) = 1$ .
- (4)  $(X, T)$  is uniquely ergodic and the unique measure of  $M(X, T)$  is a delta measure  $\delta_{x_0}$  on a fixed point  $x_0$  (the unique fixed point).
- (5)  $(X \times X, T \times T)$  is mean proximal.

As we mentioned before proximal systems may have no asymptotic pairs; thus on one hand the situation for mean proximal systems is very different. Nonetheless, the implications 1)  $\Leftrightarrow$  4) provide a measure theoretic analogy of a characterization of proximal systems: a system is proximal if and only if its unique minimal subset is a fixpoint  $x_0$  [2] (note that in this case invariant measures other than  $\delta_{x_0}$  are possible). Mean proximality is a stronger condition: the fixpoint supports a unique invariant measure.

In particular, we also have the following corollary.

**Corollary 1.1.** *If  $(X, T)$  is a mean proximal system, then  $(X, T)$  has no mean Li-Yorke pairs.*

This paper is organized as follows. In Section 2, we prove Theorem 1.1 . In Section 3, using measure-theoretic tools, we focus on mean proximal pairs; we provide a proof of Theorem 1.2.

## 2. PROOF OF THEOREM 1.1

We first state the following useful result due to Mycielski. For details see [1, Theorem 5.10].

**Lemma 2.1** (Mycielski’s lemma). *Let  $Y$  be a perfect compact metric space and  $C$  be a symmetric dense  $G_\delta$  subset of  $Y \times Y$ . Then there exists a dense subset  $K \subset Y$  which is a union of countably many Cantor sets such that  $K \times K \subset C \cup \Delta_Y$ .*

*Proof.* Denote by  $d$  the metric on  $X$ . Since  $(X, T)$  is mean sensitive, there exists  $\delta > 0$  such that for every  $x \in X$  and every  $\epsilon > 0$ , there is  $y \in B(x, \epsilon)$  with

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} d(T^k x, T^k y) > \delta.$$

Let  $\eta = \delta/2 > 0$ , and

$$D_\eta = \{(x, y) \in X \times X : \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} d(T^k x, T^k y) \geq \eta\}.$$

By noting that  $D_\eta$  can also be written as in the following form:

$$D_\eta = \bigcap_{m=1}^\infty \bigcap_{l=1}^\infty \bigcup_{n \geq l} \{(x, y) \in X \times X : \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) > \eta - \frac{1}{m}\},$$

we know that  $D_\eta$  is a  $G_\delta$  subset of  $X \times X$ . If  $D_\eta$  is not dense in  $X \times X$ , then there exist  $\epsilon > 0$  and  $x, z \in X$  such that for every  $y \in B(x, \epsilon)$ , we have

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} d(T^k y, T^k z) < \eta.$$

It follows that for every  $y \in B(x, \epsilon)$ , we have

$$\begin{aligned} & \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} d(T^k x, T^k y) \\ & \leq \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} (d(T^k x, T^k z) + d(T^k y, T^k z)) \\ & \leq \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} d(T^k x, T^k z) + \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} d(T^k y, T^k z) \\ & < \eta + \eta = \delta. \end{aligned}$$

This is a contradiction with the fact that  $(X, T)$  is mean sensitive. So  $D_\eta$  is dense in  $X \times X$ . Thus,

(2.1)  $D_\eta$  is a dense  $G_\delta$  subset of  $X \times X$ .

Let

$$MP = \{(x, y) \in X \times X : \liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} d(T^k x, T^k y) = 0\}$$

be the set of all mean proximal pairs of  $X \times X$ . Then

(2.2)  $MP$  is a  $G_\delta$  subset of  $X \times X$

since it is easy to check that

$$MP = \bigcap_{m=1}^\infty \bigcap_{l=1}^\infty \bigcup_{n \geq l} \{(x, y) \in X \times X : \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) < \frac{1}{m}\}.$$

Take

$$MLY_\eta = MP \cap D_\eta.$$

Clearly,  $MLY_\eta \subset X \times X$  is the set of all mean Li-Yorke pairs with modulus  $\eta$  in  $X \times X$ .

By hypothesis there exist  $p \in X$  a periodic point of period  $t$  and  $q \in X$  a transitive point such that the pair  $(p, q) \in X \times X$  is mean proximal. Let

$$X_j = \overline{\{T^{nt+j} q : n \geq 0\}}$$

for  $0 \leq j \leq t - 1$ .

Since  $(p, q) \in MP$ , there exists an increasing sequence of positive integers  $\{N_i\}_{i=1}^\infty$  with  $N_i \rightarrow \infty$  such that

$$\lim_{i \rightarrow \infty} \frac{1}{N_i} \sum_{k=0}^{N_i-1} d(T^k p, T^k q) = 0.$$

Hence for any fixed  $n \in \mathbb{N}$  and  $0 \leq j \leq t - 1$  we have

$$\lim_{i \rightarrow \infty} \frac{1}{N_i} \sum_{k=0}^{N_i-1} d(T^{k+nt+j} p, T^{k+nt+j} q) = 0$$

which, together with the fact that  $T^t p = p$ , implies that

$$\lim_{i \rightarrow \infty} \frac{1}{N_i} \sum_{k=0}^{N_i-1} d(T^{k+nt+j} q, T^{k+j} p) = 0.$$

Thus, for any  $n_1, n_2 \geq 0$  and  $0 \leq j \leq t - 1$ , we have

$$\begin{aligned} & \lim_{i \rightarrow \infty} \frac{1}{N_i} \sum_{k=0}^{N_i-1} d(T^{k+n_1t+j} q, T^{k+n_2t+j} q) \\ & \leq \lim_{i \rightarrow \infty} \frac{1}{N_i} \sum_{k=0}^{N_i-1} d(T^{k+n_1t+j} q, T^{k+j} p) + \lim_{i \rightarrow \infty} \frac{1}{N_i} \sum_{k=0}^{N_i-1} d(T^{k+n_2t+j} q, T^{k+j} p) \\ & = 0 \end{aligned}$$

which implies that  $(T^{n_1t+j} q, T^{n_2t+j} q) \in MP$ . Thus,

(2.3)  $MP$  is dense in  $X_j \times X_j$  for each  $0 \leq j \leq t - 1$ .

Since it is clear that

$$X = \bigcup_{j=0}^{t-1} X_j,$$

there exists some  $j$  such that  $X_j$  has non-empty interior, which is denoted by  $A_j$ . Put

$$A = \overline{A_j}.$$

Since  $A_j \times A_j$  is open (for both  $X \times X$  and  $X_j \times X_j$ ), by (2.1) and (2.3), we know that  $D_\eta$  and  $MP$  are dense in  $A_j \times A_j$ , and hence are dense in  $A \times A$ . Thus, by (2.1) and (2.2), we have that

(2.4)  $MP \cap D_\eta \cap (A \times A)$  is a symmetric dense  $G_\delta$  subset of  $A \times A$ .

From the definition of the mean sensitivity of  $(X, T)$ , we know that  $X$  is perfect. Then by noting that  $A_j$  is open, we have that  $A_j$  is perfect, and hence  $A$  is perfect. Thus,

(2.5)  $A$  is a perfect compact metric space.

Now applying Mycielski's lemma (Lemma 2.1) with (2.5) and (2.4), and by the definition of  $MLY_\eta$ , we obtain a dense subset  $K$  of  $A$  such that  $K$  is a union of countably many Cantor sets and satisfies

$$K \times K \subset MLY_\eta \cup \Delta_X.$$

This implies that  $(X, T)$  is mean Li-Yorke chaotic and  $K \subset X$  is a mean Li-Yorke set with modulus  $\eta$ . This completes the proof. □

*Remark 2.1.* According to the proof of Theorem 1.1, we know that if in addition, either  $(X, T)$  is *totally transitive*, i.e.,  $T^n$  is transitive for each  $n \geq 1$ , or the periodic point  $p$  is a fixed point, then  $(X, T)$  is *densely mean Li-Yorke chaotic*; that is, it admits a dense uncountable mean Li-Yorke subset (which is  $K$  in the proof).

An important class of topological dynamical systems are subshifts. Let  $\mathcal{A}$  be a finite set. For  $x \in \mathcal{A}^{\mathbb{Z}_+}$  and  $i \in \mathbb{Z}_+$  we use  $x_i$  to denote the  $i$ th coordinate of  $x$  and  $\sigma : X \rightarrow X$  to denote the *shift map* ( $x_{i+j} = (\sigma^i x)_j$  for all  $x \in \mathcal{A}^{\mathbb{Z}_+}$  and  $j \in \mathbb{Z}_+$ ). Using the product topology of discrete spaces, we have that  $\mathcal{A}^{\mathbb{Z}_+}$  is a compact metrizable space. The metric of this space is equivalent to the metric

$$d(x, y) = \begin{cases} 1/\inf \{m + 1 : x_m \neq y_m\} & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

A subset  $X \subset \mathcal{A}^{\mathbb{Z}_+}$  is a *subshift (or shift space)* if it is closed and  $\sigma$ -invariant; in this case  $(X, \sigma)$  is a TDS.

As it was noted in [19] and [12] mean sensitivity and (with similar arguments) mean proximality can be expressed in terms of densities. In particular, if  $(X, \sigma)$  is a subshift,  $x, y \in X$ , and

$$\liminf_{n \rightarrow \infty} \frac{|\{i \leq n \mid x_i \neq y_i\}|}{n} = 0,$$

then  $x$  and  $y$  are mean proximal.

A finite string of symbols is a *word*. Given a word  $w$  the set  $Pow(w)$  is the set that contains all the possible subwords of  $w$ . For example, if  $w = 011$ ,  $Pow(w) = \{0, 1, 01, 11, 011\}$ .

**Example 2.1.** There exists a mean sensitive subshift  $X \subset \{0, 1\}^{\mathbb{Z}_+}$  with a transitive point  $x \in X$  such that  $x$  and  $0^\infty$  are mean proximal (and thus this system contains a dense uncountable mean Li-Yorke subset).

*Proof.* We will construct a sequence of words inductively with

$$w_1 = 0.$$

Let  $A_k := Pow(w_{k-1}) := \{v_i\}_{i=1}^{|Pow(w_{k-1})|}$ .

We define

$$w_k := w_{k-1} 0^{n_k} v_1 0^k v_1 1^k v_2 0^k v_2 1^k \dots v_{|Pow(w_{k-1})|} 0^k v_{|Pow(w_{k-1})|} 1^k,$$

where  $n_k \geq |w_k|(1 - 1/k)$ .

Let  $x = \lim_{k \rightarrow \infty} w_k$  and let  $X$  be the shift orbit closure of  $x$ .

The construction implies that for every finite word  $v$  appearing in  $x$  we have that  $v0^\infty \in X$  and  $v1^\infty \in X$ . This means that  $(X, \sigma)$  is mean sensitive.

Using that  $n_k \geq |w_k|(1 - 1/k)$  and that every  $w_k$  contains  $0^{n_k}$  we obtain that the density of 0s in  $x$  is one. This implies that  $x$  and  $0^\infty$  are mean proximal.  $\square$

### 3. ON MEAN PROXIMAL SETS

In this section we characterize mean proximal systems.

We remark here that as an application of (the corollary of) Theorem 1.2 we obtain that the whole space  $X$  cannot become a mean Li-Yorke set for any non-trivial TDS  $(X, T)$ . However, we also note that if we only want to observe that for any  $\eta > 0$ ,  $X$  cannot be a mean Li-Yorke set with modulus  $\eta$ , it will be much

easier. In fact, if this holds, then, on the one hand, for every pair  $(x, y) \in X \times X$  of distinct points, we have

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) \geq \eta;$$

on the other hand, according to [17, Theorem 2.1], we can find a pair  $(x, y) \in X \times X$  of distinct points such that  $d(T^i x, T^i y) < \eta/2$  for all  $i \in \mathbb{N}$ . This implies that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) \leq \eta/2 < \eta,$$

a contradiction.

Before we give the proof of Theorem 1.2, we provide another result which only concerns the condition of mean proximality. We remind the reader that  $M(X, T)$  represents the set of all  $T$ -invariant Borel probability measures on  $X$ .

**Theorem 3.1.** *Let  $(X, T)$  be a TDS and  $A$  be a Borel subset of  $X$  such that every pair  $(x, y) \in A \times A$  is mean proximal. Then for every non-atomic measure  $\mu \in M(X, T)$ , we have  $\mu(A) = 0$ .*

*Proof.* Suppose  $\mu \in M(X, T)$  is non-atomic and  $\mu(A) > 0$ . Let

$$\mu \times \mu = \int_{\Omega} \lambda_{\omega} d\xi(\omega)$$

be the ergodic decomposition of  $\mu \times \mu$  with respect to  $T \times T$ . Since  $\mu$  is non-atomic, which implies that  $\mu \times \mu(\Delta_X) = 0$ , we have  $\mu \times \mu(A \times A \setminus \Delta_X) > 0$ . Hence there exists an ergodic measure  $\lambda_{\omega} \in M(X \times X, T \times T)$  with  $\lambda_{\omega}(A \times A \setminus \Delta_X) > 0$ . By Birkhoff's Pointwise Ergodic Theorem, we can find a generic point  $(x_0, y_0) \in A \times A \setminus \Delta_X$  of  $\lambda_{\omega}$  for  $T \times T$ , i.e., there exists a pair  $(x_0, y_0) \in A \times A \setminus \Delta_X$  such that

$$\frac{1}{N} \sum_{i=0}^{N-1} \delta_{(T \times T)^i(x_0, y_0)} \rightarrow \lambda_{\omega}$$

under the weak\*-topology as  $N \rightarrow \infty$ . Here the probability measure  $\delta_{(T \times T)^i(x_0, y_0)}$  denotes the point mass on  $(T \times T)^i(x_0, y_0) = (T^i x_0, T^i y_0)$  of  $X \times X$ .

On the one hand, since the pair  $(x_0, y_0) \in A \times A \setminus \Delta_X$  is mean proximal, we have

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{i=0}^{N-1} d(T^i x_0, T^i y_0) = 0.$$

Now since  $\lambda_{\omega}(X \times X \setminus \Delta_X) \geq \lambda_{\omega}(A \times A \setminus \Delta_X) > 0$ , there exists  $\epsilon > 0$  satisfying  $\lambda_{\omega}(D_{\epsilon}) > 0$ , where  $D_{\epsilon} = \{(x, y) \in X \times X : d(x, y) > \epsilon\}$ . By noting that  $D_{\epsilon}$  is open, it then follows that

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{i=0}^{N-1} d(T^i x_0, T^i y_0) \geq \epsilon \cdot \liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{i=0}^{N-1} \delta_{(T^i x_0, T^i y_0)}(D_{\epsilon}) \geq \epsilon \lambda_{\omega}(D_{\epsilon}) > 0,$$

a contradiction. This proves the result. □

*Proof.* Proof of Theorem 1.2 (2) $\implies$ (1). Obvious.

(3) $\implies$ (2). Suppose that (2) does not hold; then we can find a pair  $(x_1, y_1) \in X \times X$  such that  $(x_1, y_1)$  is not mean asymptotic. This implies that the orbit of  $(x_1, y_1)$  (in  $X \times X$ ) spends outside some neighborhood  $U$  of the diagonal  $\Delta_X$

time which has positive lower density. Thus, the pair  $(x_1, y_1)$  semi-generates (i.e., generates along a subsequence of averages) a measure  $\nu \in M(X \times X, T \times T)$  with  $\nu(X \times X \setminus U) > 0$ . This means that the measure  $\nu$  is not supported by  $\Delta_X$ .

(1) $\implies$ (3). Suppose that (3) does not hold; then there is some

$$\lambda \in M(X \times X, T \times T)$$

satisfying  $\lambda(\Delta_X) < 1$  which implies

$$\lambda(X \times X \setminus \Delta_X) > 0.$$

From now on, we begin to use the similar argument as in the proof of Theorem 3.1 to complete our proof. We do this as follows.

Since  $\lambda(X \times X \setminus \Delta_X) > 0$  and  $\lambda \in M(X \times X, T \times T)$ , by the Ergodic Decomposition Theorem, there exists an ergodic measure  $\lambda_\omega \in M(X \times X, T \times T)$  with

$$\lambda_\omega(X \times X \setminus \Delta_X) > 0.$$

It then follows from the Birkhoff Pointwise Ergodic Theorem that there exists a pair  $(x_2, y_2) \in X \times X \setminus \Delta_X$  such that

$$\frac{1}{N} \sum_{k=0}^{N-1} \delta_{(T \times T)^k(x_2, y_2)} \rightarrow \lambda_\omega$$

as  $N \rightarrow \infty$  under the weak\*-topology.

Since  $\lambda_\omega(X \times X \setminus \Delta_X) > 0$ , we can take some  $\epsilon > 0$  and put

$$D_\epsilon = \{(x, y) \in X \times X : d(x, y) > \epsilon\}$$

such that  $\lambda_\omega(D_\epsilon) > 0$ .

Then, by noting that  $D_\epsilon$  is open, we have

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} d(T^k x_2, T^k y_2) \geq \epsilon \cdot \liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} \delta_{(T^k x_2, T^k y_2)}(D_\epsilon) \geq \epsilon \lambda_\omega(D_\epsilon) > 0,$$

which is a contradiction with the assumption that every pair  $(x, y)$  of  $X \times X$  is mean proximal.

(3) $\implies$ (4). Suppose that there exists  $\mu \in M(X, T)$  such that the support contains at least two different points,  $x_0$  and  $y_0$ . This implies that the support of  $\lambda = \mu \times \mu$  contains the point  $(x_0, y_0)$ . We conclude that  $\lambda(\Delta_X) < 1$ . This implies that if (3) holds, then every invariant measure is supported on a fixed point. Nonetheless we already know (3) implies  $(X, T)$  is proximal and hence it only contains one fixed point [2]; thus  $(X, T)$  is uniquely ergodic.

(4) $\implies$ (3). Let  $x_0$  be the only fixed point,  $\delta_{x_0}$  the unique invariant measure of  $(X, T)$ , and  $\lambda \in M(X \times X, T \times T)$ . This implies that  $\lambda(x_0 \times X) = \delta_{x_0}(x_0) = 1$  and analogously  $\lambda(X \times x_0) = 1$ ; thus  $\lambda(x_0 \times x_0) = 1$ .

(4) $\implies$ (5). In the previous step we showed that  $\lambda(x_0 \times x_0) = 1$ . To conclude simply apply (4) $\implies$ (1) for the system  $(X \times X, T \times T)$ .

(5) $\implies$ (3). Using (1) $\implies$ (4) for the system  $(X \times X, T \times T)$  we conclude it has a unique fixed point and a unique invariant (delta) measure. The fixed point must lie on the diagonal, so for the unique invariant measure  $\lambda \in M(X \times X, T \times T)$  we have that  $\lambda(\Delta_X) = 1$ . □

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