

INTERSECTION PATTERNS OF FINITE SETS AND OF CONVEX SETS

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ABSTRACT. The main result is a common generalization of results on lower bounds for the chromatic number of r -uniform hypergraphs and some of the major theorems in Tverberg-type theory, which is concerned with the intersection pattern of faces in a simplicial complex when continuously mapped to Euclidean space. As an application we get a simple proof of a generalization of a result of Kriz for certain parameters. This specializes to a short and simple proof of Kneser’s conjecture. Moreover, combining this result with recent work of Mabillard and Wagner we show that the existence of certain equivariant maps yields lower bounds for chromatic numbers. We obtain an essentially elementary proof of the result of Schrijver on the chromatic number of stable Kneser graphs. In fact, we show that every neighborly even-dimensional polytope yields a small induced subgraph of the Kneser graph of the same chromatic number. We furthermore use this geometric viewpoint to give tight lower bounds for the chromatic number of certain small subhypergraphs of Kneser hypergraphs.

1. INTRODUCTION

Kneser conjectured [18] that for any partition of the n -element subsets of $\{1, 2, \dots, 2n + k\}$ into $k + 1$ classes there exists one class that contains two disjoint sets. This can be easily reformulated into a question about the chromatic number of a graph: the vertices of the *Kneser graph* $\text{KG}(n, 2n + k)$ correspond to the n -element subsets of $\{1, \dots, 2n + k\}$ with an edge between two vertices if the corresponding sets are disjoint. Kneser’s conjecture then states that $\chi(\text{KG}(n, 2n + k)) \geq k + 2$. A simple greedy coloring shows $\chi(\text{KG}(n, 2n + k)) \leq k + 2$. This conjecture was proved by Lovász [23] using the Borsuk–Ulam theorem in one of the earliest applications of algebraic topology to a combinatorial problem.

More generally, one could ask for sufficient conditions on a finite system G of finite sets such that in any partition of G into k classes there is one class with r pairwise disjoint sets. This can be reformulated into a question about lower bounds for the chromatic number of r -uniform hypergraphs. A rather general lower bound is due to Kriz [19, 20]. While his proof is topological – using equivariant cohomology – the condition on G that Kriz derives is purely combinatorial. The approach in Section 4 yields an elementary proof of Kriz’s result (for certain parameters) and extends it by also taking the topology of G into account, more precisely the topology of the simplicial complex of all sets with no subset in G .

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Tverberg-type theory is a geometric analog of the intersection patterns of finite sets, where now instead of finite sets and their intersections one is given a finite set of points in Euclidean space and is interested in which restrictions prohibit convex hulls of r pairwise disjoint subsets to have a common point of intersection. Usually this is formulated as the existence of a point of r -fold incidence among pairwise disjoint faces of a simplicial complex K when K is affinely mapped to \mathbb{R}^d . Continuous analogs of this theory have turned out to be of major interest. We will summarize the main results in Tverberg-type theory in Section 2.

An idea already present in papers of Sarkaria [35, 36] is to relate Tverberg-type theory and colorings of hypergraphs to one another. For a simplicial complex L and a subcomplex $K \subseteq L$ denote by $\text{KG}^r(K, L)$ the r -uniform hypergraph with vertices corresponding to the inclusion-minimal faces of L that are not contained in K and a hyperedge spanned by r vertices if the corresponding faces of L are pairwise disjoint. We use the constraint method of Blagojević, Ziegler, and the author [9] to give a simple and elementary proof of the following result relating the combinatorics of missing faces of a simplicial complex to Tverberg-type intersection results; see Theorem 4.5. (See Section 4 for the definition of chromatic number $\chi(H)$ of a hypergraph H .)

Theorem. *Let $d, k \geq 0$ and $r \geq 2$ be integers, $K \subseteq L$ simplicial complexes such that for every continuous map $F: L \rightarrow \mathbb{R}^{d+k}$ there are r pairwise disjoint faces $\sigma_1, \dots, \sigma_r$ of L such that $F(\sigma_1) \cap \dots \cap F(\sigma_r) \neq \emptyset$. Suppose $\chi(\text{KG}^r(K, L)) \leq k$. Then for every continuous map $f: K \rightarrow \mathbb{R}^d$ there are r pairwise disjoint faces $\sigma_1, \dots, \sigma_r$ of K such that $f(\sigma_1) \cap \dots \cap f(\sigma_r) \neq \emptyset$.*

We do not distinguish between an abstract simplicial complex and its geometric realization and denote both by the same symbol. Given a simplicial complex L such that for every continuous map $F: L \rightarrow \mathbb{R}^{d+k}$ there are r pairwise disjoint faces $\sigma_1, \dots, \sigma_r$ of L such that $F(\sigma_1) \cap \dots \cap F(\sigma_r) \neq \emptyset$, this theorem can be used in two directions: By establishing the upper bound $\chi(\text{KG}^r(K, L)) \leq k$ one can deduce an r -fold intersection result for continuous maps $f: K \rightarrow \mathbb{R}^d$, whereas by exhibiting a continuous map $f: K \rightarrow \mathbb{R}^d$ without such an r -fold intersection one can deduce the lower bound $\chi(\text{KG}^r(K, L)) \geq k + 1$. Thus this relates intersection patterns of continuous images of faces in a simplicial complex to intersection patterns of finite sets in the guise of chromatic numbers of intersection hypergraphs.

The theorem has the following results as simple corollaries:

- the generalized van Kampen–Flores theorem of Sarkaria [36] and Volovikov [40],
- the colored Tverberg theorems of type A of Živaljević and Vrećica [44] as well as the generalization of type B due to Vrećica and Živaljević [41] (see Corollary 4.6 for a common generalization of these colored Tverberg theorems and the generalized van Kampen–Flores theorem),
- Kneser’s conjecture proven by Lovász [23], see Theorem 3.2, and Dol’nikov’s theorem [12],
- more generally, Kriz’s lower bound for the chromatic number of r -uniform intersection hypergraphs [19] for certain parameters; see Corollary 4.7,
- Schrijver’s theorem on stable Kneser graphs [37]; see Section 5.

The Tverberg-type result follows from combining the topological Tverberg theorem, see Theorem 2.2, with greedy colorings of hypergraphs. The lower bounds

for the chromatic number follow from a codimension count for a general position affine map or from understanding the geometry of points in cyclic position.

Special cases of this result are already implicit in Sarkaria's papers [35, 36] and more explicit in Matoušek's book [28, Theorem 6.7.3], where the special case that r is a prime and L is the simplex on the vertex set of K is proven. However, the proof presented here is significantly simpler and does not need to appeal to \mathbb{Z}/r -indices or other methods from algebraic topology. It is a combination of Sarkaria's coloring ideas with the constraint method of [9].

While the main focus of this paper is to significantly simplify proofs of known results and relate intersection patterns of convex sets and of finite sets to one another, we can also use the main theorem to further extend the known results. We derive the following new results:

- The consequences above follow from combining the main result and the topological Tverberg theorem. We get proper extension for r a prime by using the optimal colored Tverberg theorem of Blagojević, Matschke, and Ziegler [11] instead; see Corollary 4.13.
- Missing faces of neighborly even-dimensional polytopes induce subgraphs of the Kneser graph $\text{KG}(n, 2n + k)$ that still have chromatic number $k + 2$, where Schrijver's theorem is the special case of cyclic polytopes. We thus obtain many subgraphs of $\text{KG}(n, 2n + k)$ on few vertices and with chromatic number $k + 2$; see Corollary 5.4.
- More generally, we show that if K triangulates S^{d-1} on n vertices, then the graph with vertex set the missing faces of K and an edge for each pair of disjoint faces has chromatic number $n - d$; see Theorem 5.3.
- We show that the chromatic number of Kneser hypergraphs does not decrease if one restricts to k -element sets that are $(\frac{r(k-3)}{2(k-1)} + 1)$ -stable on average; see Theorem 6.5.
- We remark that lower bounds for chromatic numbers of Kneser hypergraphs can be established by exhibiting equivariant maps — and not as usual by showing the nonexistence of an equivariant map, which often is more difficult. This follows from combining the main theorem with recent work of Mabillard and Wagner [25]; see Corollary 4.15.

2. TVERBERG-TYPE THEOREMS

Here we collect some of the main results from Tverberg-type theory and refer to Matoušek's book [28] for details. We denote the N -dimensional simplex by Δ_N . A classical lemma of Radon [34] states that any $d + 2$ points in \mathbb{R}^d can be split into two sets with intersecting convex hulls. Equivalently, this can be phrased as: any affine map $f: \Delta_{d+1} \rightarrow \mathbb{R}^d$ identifies points from two disjoint faces of Δ_{d+1} . The following theorem states that this remains true if one replaces affine by continuous.

Theorem 2.1 (Topological Radon theorem, Bajmóczy and Bárány [6]). *For any continuous map $f: \Delta_{d+1} \rightarrow \mathbb{R}^d$ there are two disjoint faces σ_1 and σ_2 of Δ_{d+1} such that $f(\sigma_1) \cap f(\sigma_2) \neq \emptyset$.*

This theorem follows from the Borsuk–Ulam theorem stating that any continuous map $S^d \rightarrow \mathbb{R}^d$ identifies two antipodal points. We will use the topological Radon theorem to establish lower bounds on the chromatic number of Kneser graphs.

To obtain lower bounds for the chromatic number of r -uniform hypergraphs, we need a generalization of the topological Radon theorem to multiple intersections. In the case of an affine map such a theorem is due to Tverberg [39]. He showed that for $N = (r - 1)(d + 1)$ any affine map $f: \Delta_N \rightarrow \mathbb{R}^d$ identifies points from r pairwise disjoint faces. Extending this result to continuous maps turned out to be a major problem. This was accomplished for r a prime by Bárány, Shlosman, and Szűcz [8] and more generally for r a power of a prime by Özaydin [31].

Theorem 2.2 (Topological Tverberg theorem, Bárány, Shlosman, and Szűcz [8], Özaydin [31]). *Let $r \geq 2$ be a prime power, $d \geq 0$ an integer, and $N = (r - 1)(d + 1)$. Then for any continuous map $f: \Delta_N \rightarrow \mathbb{R}^d$ there are r pairwise disjoint faces $\sigma_1, \dots, \sigma_r$ of Δ_N such that $f(\sigma_1) \cap \dots \cap f(\sigma_r) \neq \emptyset$.*

Özaydin already showed that the obstruction used to prove the topological Tverberg theorem for r a prime power vanishes for all other r . Counterexamples to the topological Tverberg conjecture, that is, to the statement that Theorem 2.2 holds for all r , came about when Mabillard and Wagner [24] proved that for simplicial complexes of dimension at most $d - 3$ (and other technical conditions) the vanishing of the r -fold van Kampen obstruction implies the existence of a continuous map that does not identify points from r pairwise disjoint faces. This together with [9] – the reduction needed here was also sketched earlier by Gromov [17, p. 445f.] – yields counterexamples to the topological Tverberg conjecture for any r that is not a prime power as pointed out in [10, 14]. The smallest counterexample is due to Avvakumov, Mabillard, Skopenkov, and Wagner [5]. By $N(r, d)$ we denote the smallest integer N such that the statement of Theorem 2.2 holds for parameters N, r , and d . For lower and upper bounds on $N(r, d)$ for r not a prime power see [10].

Three important variations of Theorem 2.2 are the following. Here we denote by $K^{(k)}$ the k -skeleton of the simplicial complex K , and $K * L$ denotes the join of the complexes K and L . See [28] for more details.

Theorem 2.3 (Generalized van Kampen–Flores theorem, Sarkaria [36], Volovikov [40]). *Let $r \geq 2$ be a prime power, $d \geq 0$ an integer, $N = (r - 1) \cdot (d + 2)$, and k an integer with $r(k + 2) > N + 1$. Then for any continuous map $f: \Delta_N^{(k)} \rightarrow \mathbb{R}^d$ there are r pairwise disjoint faces $\sigma_1, \dots, \sigma_r$ of $\Delta_N^{(k)}$ such that $f(\sigma_1) \cap \dots \cap f(\sigma_r) \neq \emptyset$.*

Theorem 2.4 (Colored Tverberg theorem, Vrećica and Živaljević [41]). *Let $r \geq 2$ be a prime power and let $d \geq 0$ and $m \geq 1$ be integers. Let C_1, \dots, C_m be disjoint sets of cardinality at most $2r - 1$ such that $|C_1 \cup \dots \cup C_m| \geq (r - 1)(d + 1 + m) + 1$. Then for any continuous map $f: C_1 * \dots * C_m \rightarrow \mathbb{R}^d$ there are r pairwise disjoint faces $\sigma_1, \dots, \sigma_r$ of $C_1 * \dots * C_m$ such that $f(\sigma_1) \cap \dots \cap f(\sigma_r) \neq \emptyset$.*

For short and elementary proofs see [9]. We will obtain a common generalization of Theorem 2.3 and Theorem 2.4; see Corollary 4.6.

Theorem 2.5 (Optimal colored Tverberg theorem, Blagojević, Matschke, and Ziegler [11]). *Let $r \geq 2$ be a prime, $d \geq 0$ an integer, and $N = (r - 1)(d + 1)$. Let C_1, \dots, C_m be disjoint sets of cardinality at most $r - 1$ such that $|C_1 \cup \dots \cup C_m| = N + 1$. Then for any continuous map $f: C_1 * \dots * C_m \rightarrow \mathbb{R}^d$ there are r pairwise disjoint faces $\sigma_1, \dots, \sigma_r$ of $C_1 * \dots * C_m$ such that $f(\sigma_1) \cap \dots \cap f(\sigma_r) \neq \emptyset$.*

3. A PROOF OF KNESER’S CONJECTURE

We recall that $\text{KG}(k, n)$ denotes the graph with vertex set the k -element subsets of $\{1, \dots, n\}$ and an edge between two vertices if the corresponding sets are disjoint. We denote the chromatic number of a graph G by $\chi(G)$. Kneser’s conjecture that $\chi(\text{KG}(n, 2n+k)) = k+2$ was proved by Lovász [23]. Bárány gave a different proof [7] that also used the Borsuk–Ulam theorem in an essential way and the geometry of finite point sets on the sphere. Subsequent proofs are due to Greene [16] and Matoušek [27]. The latter proof is the first purely combinatorial proof of Kneser’s conjecture.

Here we first present a short proof of Kneser’s conjecture, which we will extend to the hypergraph setting in Section 4. The proof builds on the topological Radon theorem, which is a corollary to the Borsuk–Ulam theorem.

Lemma 3.1. *Let $c: V(\text{KG}(n, N + 1)) \rightarrow \{1, \dots, k\}$ be a proper k -coloring. Then there is a continuous map $C: \Delta_N \rightarrow \mathbb{R}^k$ with the property that for disjoint faces σ_1 and σ_2 of Δ_N we have that $C(x_1) = C(x_2)$ for $x_1 \in \sigma_1$ and $x_2 \in \sigma_2$ implies $x_1, x_2 \in \Delta_N^{(n-2)}$.*

Proof. Let G be the graph of subsets of $\{1, \dots, N + 1\}$ that have cardinality at least n and an edge between two vertices if the corresponding subsets are disjoint. Then $\text{KG}(n, N + 1)$ naturally is a subgraph of G of the same chromatic number. The coloring c can be extended to a proper k -coloring c' of G : for any subset $\sigma \subseteq \{1, \dots, N + 1\}$ of cardinality at least n choose some $\tau \subseteq \sigma$ of cardinality n and define $c'(\sigma) = c(\tau)$. For example, we could choose τ such that $c(\tau)$ is minimal among all n -subsets of σ . The map c' maps disjoint sets to distinct values since c does.

We can now define the map C as an affine map on the barycentric subdivision Δ'_N of Δ_N . The vertices of Δ'_N correspond to the faces of Δ_N . Given some ℓ -face σ of Δ_N define $C(\sigma) = e_{c'(\sigma)}$, where e_1, \dots, e_k denotes the standard basis of \mathbb{R}^k , for $\ell \geq n - 1$ and $C(\sigma) = 0$ otherwise. Then extend C affinely onto the faces of Δ'_N .

Let σ be a face of Δ_N and $x \in \sigma$ some point. Then $C(x) \in \text{conv}(\{e_{c'(\tau)} \mid \tau \subseteq \sigma\} \cup \{0\})$. Let σ_1 and σ_2 be disjoint faces of Δ_N and $x_1 \in \sigma_1, x_2 \in \sigma_2$ with $C(x_1) = C(x_2)$. Since for any subfaces $\tau_1 \subseteq \sigma_1$ and $\tau_2 \subseteq \sigma_2$ of dimension $\geq n - 1$ we have that $c'(\tau_1) \neq c'(\tau_2)$, we conclude that $C(x_1) = 0 = C(x_2)$ and thus $x_1, x_2 \in \Delta_N^{(n-2)}$. □

The map C constructed in Lemma 3.1 can be used as a constraint function in the sense of [9]. This immediately yields a proof of Kneser’s conjecture:

Theorem 3.2 (Kneser’s conjecture, Lovász [23]). $\chi(\text{KG}(n, 2n + k)) \geq k + 2$.

Proof. Let $N = 2n+k-1$ and let $f: \Delta_N \rightarrow \mathbb{R}^{2n-3}$ be a general position affine map. Suppose there was a proper $(k + 1)$ -coloring $c: V(\text{KG}(n, 2n+k)) \rightarrow \{1, \dots, k + 1\}$. Then by Lemma 3.1 this induces a continuous map $C: \Delta_N \rightarrow \mathbb{R}^{k+1}$ such that if x_1 and x_2 are in disjoint faces of Δ_N with $C(x_1) = C(x_2)$, then $x_1, x_2 \in \Delta_N^{(n-2)}$.

Consider the continuous map $F: \Delta_N \rightarrow \mathbb{R}^{2n+k-2}, x \mapsto (f(x), C(x))$. By the topological Radon theorem there are points x_1 and x_2 in disjoint faces of Δ_N with $F(x_1) = F(x_2)$. Then $C(x_1) = C(x_2)$ implies that $x_1, x_2 \in \Delta_N^{(n-2)}$. Since also $f(x_1) = f(x_2)$ we get the contradiction that two $(n - 2)$ -faces intersect in \mathbb{R}^{2n-3} for the general position map f . □

A slightly more careful analysis shows that this reasoning actually implies Dol'nikov's generalization [12] of Theorem 3.2. We do not carry this out here since the generalization to hypergraphs presented in the next section will be even more general. See also Section 5 for further generalizations.

4. GENERALIZATIONS TO HYPERGRAPH COLORINGS

The proof of Theorem 3.2 easily generalizes to the hypergraph setting, where now instead of the topological Radon theorem we use the topological Tverberg theorem. A *hypergraph* on vertex set V is a set E of subsets of V . A *hyperedge* is any element of E . We will restrict our attention to hypergraphs where the hyperedges have cardinality at least two. A hypergraph is *r-uniform* if all hyperedges have cardinality r . A *k-coloring* of a hypergraph on vertex set V is a function $c: V \rightarrow \{1, \dots, k\}$ such that any hyperedge contains vertices v and w with $c(v) \neq c(w)$. The least k such that the hypergraph H has a k -coloring is its *chromatic number* $\chi(H)$. A *partial hypergraph* of a hypergraph H is obtained by removing hyperedges from H . A *subhypergraph* is obtained by removing vertices.

Let L be a simplicial complex, $K \subseteq L$ a subcomplex, and $r \geq 2$ an integer. Denote by $\text{KG}^r(K, L)$ the associated *generalized Kneser hypergraph*, that is, the r -uniform hypergraph with vertices the inclusion-minimal faces in L that are not contained in K and a hyperedge of r vertices precisely if the r corresponding faces are pairwise disjoint. The minimal nonfaces of K are sometimes also called *missing faces*.

If K is a skeleton of the simplex, say $K = \Delta_N^{(n-2)}$, then $\text{KG}(K, \Delta_N)$ is the usual Kneser graph $\text{KG}(n, N+1)$ of n -subsets of $\{1, \dots, N+1\}$. The r -uniform hypergraph $\text{KG}^r(K, \Delta_N)$ is called *Kneser hypergraph* and denoted by $\text{KG}^r(n, N+1)$.

Let G be a system of nonempty subsets of $\{1, \dots, N+1\}$. Then Kriz [19] defines the *r-uniform intersection hypergraph* $[G, r]$ to have vertex set G and a hyperedge $\{M_1, \dots, M_r\}$ for $M_i \in G$ precisely if the M_i are pairwise disjoint. Our first goal will be to show that there is no loss of generality in considering the hypergraphs $\text{KG}^r(K, \Delta_N)$ compared to $[G, r]$. Not every hypergraph $[G, r]$ is isomorphic to some $\text{KG}^r(K, \Delta_N)$, but it dismantles to such a hypergraph, that is, deleting a set from G that is not inclusion-minimal in G has the effect of deleting a vertex v of $[G, r]$ that is dominated by another vertex w in the sense that replacing v by w in any hyperedge yields another hyperedge.

Lemma 4.1. *Given a finite system of nonempty sets G , let G' consist of those sets in G that are inclusion-minimal in G . Then for any integer $r \geq 2$ we have $\chi([G, r]) = \chi([G', r])$.*

Proof. Since $[G', r]$ is a subhypergraph of $[G, r]$ we have that $\chi([G, r]) \geq \chi([G', r])$. Let c be a proper vertex coloring of $[G', r]$. Given two sets $\sigma' \subset \sigma$ with $\sigma' \in G'$ and $\sigma \in G$ and corresponding vertices v' of σ' and v of σ , let $c(v) = c(v')$. This results in a proper vertex coloring of $[G, r]$. This is due to the fact that for any hyperedge $\{v, w_2, \dots, w_r\}$ of $[G, r]$, the set $\{v', w_2, \dots, w_r\}$ is a hyperedge of $[G', r]$ since $\sigma' \subset \sigma$. \square

From now on we will assume that G does not contain two distinct sets $\sigma, \tau \in G$ with $\sigma \subset \tau$ since deleting supersets does not affect the chromatic number of $[G, r]$.

Lemma 4.2. *Let G' be a system of nonempty subsets of $\{1, \dots, N + 1\}$ such that for any two distinct sets $\sigma, \tau \in G'$ neither is a subset of the other. Let K be the simplicial complex on vertex set $\{1, \dots, N + 1\}$ that contains all sets $\tau \subseteq \{1, \dots, N + 1\}$ as faces such that no subset of τ is in G' . Then $\text{KG}^r(K, \Delta_N) = [G', r]$.*

Proof. Let $\sigma \in G'$. Then all proper subsets of σ are not contained in G' . Thus all proper subsets of σ are faces of K , while σ itself is not. This implies that σ is a vertex of $\text{KG}^r(K, \Delta_N)$. Conversely, if $\sigma \subseteq \{1, \dots, N + 1\}$ is a minimal nonface of K , then no proper subset of σ is in G' . Since σ itself is not a face of K it must be in G' . This shows that $\text{KG}^r(K, \Delta_N)$ and $[G', r]$ have the same vertices. They coincide as hypergraphs since vertices are connected by hyperedges by the same condition of pairwise disjointness of the corresponding sets for both hypergraphs. \square

Kriz defines the r -width $\omega(G, r)$ of $[G, r]$ as the minimal integer k such that there exist r subsets M_i of $\{1, \dots, N + 1\}$ and such that no subset of any M_i is in G and $|\bigcup_i M_i| = N + 1 - k$. Kriz's main result then is:

Theorem 4.3 (Kriz [19, Theorem 2.4]; see also [20]). $\chi([G, r]) \geq \frac{\omega(G, r)}{r - 1}$.

This contains as a special case the result of Alon, Frankl, and Lovász [4] on the chromatic number of Kneser hypergraphs, conjectured by Erdős [13]. A simpler proof of Theorem 4.3 using the \mathbb{Z}/p -index is due to Matoušek [26]. Combinatorial proofs of this result and several of its variants and extensions are due to Ziegler [42]. See also the erratum [43] and the clarifications by Lange and Ziegler [22].

That no subset of M_i is contained in G is equivalent to M_i spanning a face of the simplicial complex K constructed above. By definition of $\omega([G, r])$ we then have that $\sum_{i=1}^r \dim \sigma_i \leq N + 1 - \omega([G, r]) - r$ for any r pairwise disjoint faces $\sigma_1, \dots, \sigma_r$ of K with K as in Lemma 4.2. We will denote $\omega([G, r])$ also by $\omega(K, r)$. The inequality of Theorem 4.3 is purely combinatorial. We will extend the proof in Section 3 to obtain a lower bound for the chromatic number of intersection hypergraphs that takes the topology of the simplicial complex K into account. The following lemma is the r -fold analog of Lemma 3.1.

Lemma 4.4. *For simplicial complexes $K \subseteq L$ let $c: V(\text{KG}^r(K, L)) \rightarrow \{1, \dots, k\}$ be a proper vertex coloring. Then there is a continuous map $C: L \rightarrow \mathbb{R}^k$ such that $C(x_1) = \dots = C(x_r)$ for x_1, \dots, x_r in r pairwise disjoint faces of L implies that $x_i \in K$ for all $i = 1, \dots, r$.*

Proof. Denote by H the r -uniform hypergraph with vertex set the faces of L that are not contained in K and a hyperedge for any set of r pairwise disjoint faces. The hypergraph $\text{KG}^r(K, L)$ is naturally a subhypergraph of H and the coloring c can be extended to a proper k -coloring c' of H : for any face σ of L that is not contained in K let

$$c'(\sigma) = \min\{c(\tau) \mid \tau \subseteq \sigma, \tau \text{ is a minimal nonface of } K\}.$$

The vertices of the barycentric subdivision L' of L correspond to the faces of L . Thus we can think of c' as a map $c': V(L') \setminus V(K') \rightarrow \{1, \dots, k\}$. Define the affine map $C: L' \rightarrow \mathbb{R}^k$ by setting $C(x) = 0$ for $x \in K$ and $C(x) = e_{c'(x)}$ for any vertex x of L' that is not contained in K , where (e_1, \dots, e_k) denotes the standard basis of \mathbb{R}^k .

As before for $x \in \sigma$ we have that $C(x) \in \text{conv}(\{e_{c'(\tau)} \mid \tau \subseteq \sigma\} \cup \{0\})$. Let $A_1, \dots, A_r \subseteq \{1, \dots, k\}$. Then $\bigcap_{i=1}^r \text{conv}(\{e_t \mid t \in A_i\} \cup \{0\}) \neq \{0\}$ if and only if $\bigcap_{i=1}^r A_i \neq \emptyset$. Thus since c' is a proper coloring we have that for r pairwise disjoint faces $\sigma_1, \dots, \sigma_r$ of L the intersection $\bigcap_{i=1}^r C(\sigma_i)$ is equal to $\{0\}$. This concludes the proof since $C(x) = 0$ implies that $x \in K$. \square

We are now ready to prove the main result. Analogous to the case $r = 2$ presented in Section 3 it follows from using the map C of Lemma 4.4 as a constraint function in the sense of [9].

Theorem 4.5. *Let $d, k \geq 0$ and $r \geq 2$ be integers, $K \subseteq L$ simplicial complexes such that for every continuous map $F: L \rightarrow \mathbb{R}^{d+k}$ there are r pairwise disjoint faces $\sigma_1, \dots, \sigma_r$ of L such that $F(\sigma_1) \cap \dots \cap F(\sigma_r) \neq \emptyset$. Suppose $\chi(\text{KG}^r(K, L)) \leq k$. Then for every continuous map $f: K \rightarrow \mathbb{R}^d$ there are r pairwise disjoint faces $\sigma_1, \dots, \sigma_r$ of K such that $f(\sigma_1) \cap \dots \cap f(\sigma_r) \neq \emptyset$.*

Proof. Extend f to a continuous map $f': L \rightarrow \mathbb{R}^d$. Let $C: L \rightarrow \mathbb{R}^k$ be the continuous map given by any proper k -coloring of the vertices of L and Lemma 4.4. Consider the continuous map $F: L \rightarrow \mathbb{R}^{d+k}$, $x \mapsto (f'(x), C(x))$. Then there are x_1, \dots, x_r in r pairwise disjoint faces of L with $F(x_1) = \dots = F(x_r)$. The equality $C(x_1) = \dots = C(x_r)$ guarantees that $x_1, \dots, x_r \in K$ by Lemma 4.4. \square

In light of the topological Tverberg theorem, Theorem 2.2, the following corollary is immediate.

Corollary 4.6. *Let $d, k \geq 0$ be integers, $r \geq 2$ be a prime power, $N \geq (r - 1) \cdot (d + k + 1)$, and $K \subseteq \Delta_N$ with $\chi(\text{KG}^r(K, \Delta_N)) \leq k$. Then for every continuous map $f: K \rightarrow \mathbb{R}^d$ there are r pairwise disjoint faces $\sigma_1, \dots, \sigma_r$ of K such that $f(\sigma_1) \cap \dots \cap f(\sigma_r) \neq \emptyset$.*

This corollary remains true for any integer $r \geq 2$ if the inequality $N \geq (r - 1) \cdot (d + k + 1)$ is replaced by $N \geq N(r, d + k)$, that is, if the topological Tverberg conjecture holds for parameters $r, d + k$, and N . Corollary 4.6 is a common generalization of the generalized van Kampen–Flores theorem, Theorem 2.3, and the colored Tverberg theorem, Theorem 2.4: If $K = \Delta_N^{(k)}$, then for $r(k + 2) > N + 1$ we have that $\text{KG}^r(K, \Delta_N)$ does not contain any hyperedges by the pigeonhole principle and thus admits a 1-coloring. This proves Theorem 2.3. If $K = C_1 * \dots * C_m$ is the join of m discrete sets of cardinality at most $2r - 1$, then the missing faces of K — which correspond to vertices of $\text{KG}^r(K, \Delta_N)$ — are precisely those edges with both endpoints in one C_j . Color such an edge with j . Then at most $r - 1$ pairwise disjoint edges can receive color j by the pigeonhole principle. Thus this defines an m -coloring of $\text{KG}^r(K, \Delta_N)$, and this proves Theorem 2.4.

Whenever we can exhibit a continuous map $f: K \rightarrow \mathbb{R}^d$ that does not identify points from r pairwise disjoint faces, we get a lower bound for the chromatic number of $\text{KG}^r(K, \Delta_N)$. We explicitly formulate this contrapositive as a corollary:

Corollary 4.7. *Let $d \geq 0$ be an integer, $r \geq 2$ be a prime power, and $N \geq (r - 1)(d + 1)$. Let $K \subseteq \Delta_N$ such that there exists a continuous map $f: K \rightarrow \mathbb{R}^d$ with the property that for any r pairwise disjoint faces $\sigma_1, \dots, \sigma_r$ of K we have that $f(\sigma_1) \cap \dots \cap f(\sigma_r) = \emptyset$. Then $\chi(\text{KG}^r(K, \Delta_N)) \geq \lfloor \frac{N}{r-1} \rfloor - d$.*

Proof. Let $k \geq 0$ and $0 \leq m < r - 1$ be integers such that $N = (r - 1)(d + k + 1) + m$. By Corollary 4.6 we have that $\chi(\text{KG}^r(K, \Delta_N)) \geq k + 1$. Now $k + 1 = \frac{N - m}{r - 1} - d > \frac{N - (r - 1)}{r - 1} - d \geq \lfloor \frac{N}{r - 1} \rfloor - d - 1$. Thus $k + 1 \geq \lfloor \frac{N}{r - 1} \rfloor - d$. \square

Corollary 4.7 is more general than Kriz’s bound $\chi(\text{KG}^r(K, \Delta_N)) \geq \frac{\omega(K, r)}{r - 1}$ if $r - 1$ divides N . By definition of $\omega(K, r)$ we have that $r \dim K \leq N + 1 - \omega(K, r) - r$ and thus $\frac{\omega(K, r)}{r - 1} \leq \frac{N + 1 - r \dim K - r}{r - 1}$. Choose the integer $d \geq 0$ such that $(r - 1)d = r \dim K + m$ for some $0 < m \leq r - 1$. Then we have for any (strong) general position affine map $f: K \rightarrow \mathbb{R}^d$ that $f(\sigma_1) \cap \dots \cap f(\sigma_r) = \emptyset$ for pairwise disjoint faces σ_i by a codimension count. (See Perles and Sigron [33] for the notion of strong general position.)

We now get that $\omega(K, r) \leq N + 1 - r \dim K - r = N - (r - 1)d + m - (r - 1) \leq N - (r - 1)d$. Hence, if N is divisible by $r - 1$ we have that $\frac{\omega(K, r)}{r - 1} \leq \lfloor \frac{N}{r - 1} \rfloor - d$.

Example 4.8. This naive general position bound for $\chi(\text{KG}^r(K, \Delta_N))$ can for certain parameters be worse by one compared to Kriz’s bound: let $r = 3$, $d = 1$, $N = 5$, and $K = \Delta_N^{(0)}$. Then $\omega(K, r) = N + 1 - r \dim K - r = 3$ and thus $\chi(\text{KG}^r(K, \Delta_N)) \geq \omega(K, r)/(r - 1) = \frac{3}{2}$. While the bound in Corollary 4.7 works out to be $\chi(\text{KG}^r(K, \Delta_N)) \geq \lfloor \frac{N}{r - 1} \rfloor - d = \lfloor \frac{5}{2} \rfloor - 1 = 1$.

We recover (or almost recover) Kriz’s bound for $\chi(\text{KG}^r(K, \Delta_N))$ if the images of r pairwise disjoint faces of K under a general position affine map $K \rightarrow \mathbb{R}^d$ do not intersect for codimension reasons. The bound of Corollary 4.7 improves whenever we can produce a continuous map $f: K \rightarrow \mathbb{R}^d$ such that the images of any r pairwise disjoint faces do not intersect, while a codimension count is not sufficient to ascertain that. For example, the bound of Corollary 4.7 is better than the bound of Theorem 4.3 by an arbitrarily large margin even for $r = 2$, by producing simplicial complexes of dimension d that embed into \mathbb{R}^d .

Example 4.9. Let K be the simplicial complex whose facets are $\{1, 2, \dots, t\}$, $\{2, 3, \dots, t + 1\}, \dots, \{N + 1, 1, 2, \dots, t - 1\}$. Then for $N \geq 2t + 1$ odd K is homeomorphic to $B^{t - 2} \times S^1$ and thus embeds into $\mathbb{R}^{t - 1}$. This follows from extending Kühnel’s arguments [21]. (For example, one can realize K in the Schlegel diagram of a cyclic polytope; see Section 5 for more general results.) Hence Corollary 4.7 gives the bound $\chi(\text{KG}^2(K, \Delta_N)) \geq N - t + 1$, while Theorem 4.3 only guarantees that $\chi(\text{KG}^2(K, \Delta_N)) \geq N - 2t + 1$. A greedy coloring shows that the bound $\chi(\text{KG}^2(K, \Delta_N)) \geq N - t + 1$ is tight.

Remark 4.10. In fact, a greedy coloring shows that the bound of Corollary 4.7 is tight whenever $r - 1$ divides N and $\dim K \geq (r - 1)d$: label the vertices of K by $1, 2, \dots, N + 1$ in such a way that $\{N - (r - 1)d + 1, N - (r - 1)d + 2, \dots, N + 1\}$ determines a face. Given a missing face σ of K let $k \in \sigma$ be the vertex with the least label. Now color σ by $\lceil \frac{k}{r - 1} \rceil$. This is a proper hypergraph coloring which uses at most $\frac{N}{r - 1} - d$ colors.

Theorem 4.3 holds for any integer $r \geq 2$ while Corollary 4.7 only holds for prime powers. In fact, Kriz’s proof of Theorem 4.3 works for primes, and the general case is established by an induction over the number of prime divisors [20]. Such an induction must fail for the more general setting of Corollary 4.7 since it turns out to be wrong for r not a prime power.

Example 4.11. Let $r \geq 6$ be an integer that is not a power of a prime, $N = (r - 1)(rk + 2)$, and let $K = \Delta_N^{((r-1)k)}$. Then for $k \geq 2$ there is a continuous map $f: K \rightarrow \mathbb{R}^{rk}$ such that for any r pairwise disjoint faces $\sigma_1, \dots, \sigma_r$ of K we have that $f(\sigma_1) \cap \dots \cap f(\sigma_r) = \emptyset$; see [25]. The minimal nonfaces of K have $(r - 1)k + 2$ elements and thus r pairwise disjoint minimal nonfaces involve $r((r - 1)k + 2) = (r - 1)rk + 2r = (r - 1)(rk + 2) + 2 > N + 1$ vertices. Thus $\text{KG}^r(K, \Delta_N)$ does not contain any hyperedges, which implies that $\chi(\text{KG}^r(K, \Delta_N)) = 1$, while if Corollary 4.7 was true for r we would obtain the lower bound $\chi(\text{KG}^r(K, \Delta_N)) \geq \lfloor \frac{N}{r-1} \rfloor - rk = 2$.

Conjecture 4.12. *Corollary 4.7 remains true for every integer $r \geq 2$ if the map f is required to be affine.*

We can get a proper extension of Corollary 4.6 for r a prime by applying the optimal colored Tverberg theorem of Blagojević, Matschke, and Ziegler [11], Theorem 2.5, to Theorem 4.5:

Corollary 4.13. *Let $d, k \geq 0$ be integers and $r \geq 2$ be a prime. Let C_1, \dots, C_m be disjoint sets of at most $r - 1$ points with $|\bigcup_{i=1}^m C_i| > (r - 1)(d + k + 1)$, $L = C_1 * \dots * C_m$, and $K \subseteq L$ with $\chi(\text{KG}^r(K, L)) \leq k$. Then for every continuous map $f: K \rightarrow \mathbb{R}^d$ there are r pairwise disjoint faces $\sigma_1, \dots, \sigma_r$ of K such that $f(\sigma_1) \cap \dots \cap f(\sigma_r) \neq \emptyset$.*

Lastly, as we obtain lower bounds for the chromatic number of $\text{KG}^r(K, \Delta_N)$ by exhibiting a continuous map $K \rightarrow \mathbb{R}^d$ without r -fold intersections among its pairwise disjoint faces, and the existence of such a map is equivalent to the existence of a certain \mathfrak{S}_r -equivariant map in the r -metastable range due to a recent result of Mabillard and Wagner [25], we can obtain a lower bound in these cases by establishing the existence of equivariant maps. Notice that this is different from the usual approach of proving the nonexistence of a certain equivariant map.

To formulate the result of Mabillard and Wagner we first need some notation. For a simplicial complex K denote by

$$K_\Delta^r = \{(x_1, \dots, x_r) \in \sigma_1 \times \dots \times \sigma_r \mid \sigma_i \text{ face of } K, \sigma_i \cap \sigma_j = \emptyset \forall i \neq j\}$$

the r -fold deleted product of K , which is a polytopal cell complex in a natural way with faces that are products of simplices. Denote by W_r the vector space $\{(x_1, \dots, x_r) \in \mathbb{R}^r \mid \sum x_i = 0\}$ with the action by the symmetric group \mathfrak{S}_r that permutes coordinates. Given a normed vector space V , denote the unit sphere in V by $S(V)$.

Theorem 4.14 (Mabillard and Wagner [25]). *Let $d \geq 1$ and $r \geq 2$ be integers, and let K be a finite simplicial complex of dimension $\dim K \leq \frac{rd-3}{r+1}$. Suppose there is an \mathfrak{S}_r -equivariant map $K_\Delta^r \rightarrow S(W_r^{\oplus d})$. Then there is a continuous map $f: K \rightarrow \mathbb{R}^d$ such that $f(\sigma_1) \cap \dots \cap f(\sigma_r) \neq \emptyset$ for all r pairwise disjoint faces $\sigma_1, \dots, \sigma_r$ of K .*

Combining Theorem 4.14 and Theorem 4.5 yields:

Corollary 4.15. *Let $d, k \geq 0$ and $r \geq 2$ be integers, $N \geq N(r, d + k)$, and $K \subseteq \Delta_N$ with $\dim K \leq \frac{rd-3}{r+1}$ such that an \mathfrak{S}_r -equivariant map $K_\Delta^r \rightarrow S(W_r^{\oplus d})$ exists. Then $\chi(\text{KG}^r(K, \Delta_N)) \geq k + 1$.*

In Section 6 we point out that a potential approach to the notoriously open Conjecture 6.1 is given by Corollary 4.15 and exhibiting a certain equivariant map.

5. VERTEX-CRITICAL SUBGRAPHS OF KNESER GRAPHS
VIA NEIGHBORLY POLYTOPES

Schrijver found vertex-critical subgraphs of Kneser graphs $KG(n, 2n + k)$, that is, for each choice of parameters $n \geq 1$ and $k \geq 1$ Schrijver constructs an induced subgraph G of $KG(n, 2n + k)$ obtained by deleting vertices such that $\chi(G) = \chi(KG(n, 2n + k))$ and deleting any vertex of G decreases the chromatic number [37]. Schrijver calls a subset of $\{1, \dots, 2n + k\}$ *stable* if it does not contain two successive elements in the cyclic order of $\{1, \dots, 2n + k\}$, that is, it neither contains i and $i + 1$ nor $2n + k$ and 1 . Define the *stable Kneser graph* $KG'(n, 2n + k)$ to be the subgraph of $KG(n, 2n + k)$ induced by the vertices corresponding to stable sets.

Theorem 5.1 (Schrijver [37]). *We have $\chi(KG'(n, 2n + k)) = k + 2$ and any proper induced subgraph of $KG'(n, 2n + k)$ is $(k + 1)$ -colorable.*

A combinatorial proof using cyclic matroids is due to Ziegler [42]. Here we will derive Theorem 5.1 as a consequence of Theorem 4.5 by observing that stable sets are the missing faces of even-dimensional cyclic polytopes. (The geometry of cyclic polytopes and chromatic numbers of certain graphs were already related to one another in [1].) The *cyclic d -polytope* $C_d(n)$ is the convex hull of $n \geq d + 1$ points on the moment curve $\gamma(t) = (t, t^2, \dots, t^d)$. By Gale's evenness criterion [15] the facets of a cyclic $2d$ -polytope are those $2d$ -element subsets σ of $\{1, \dots, n\}$ such that between any two elements in $\{1, \dots, n\} \setminus \sigma$ there are an even number of elements in σ . Thus cyclic $2d$ -polytopes are *neighborly*, that is, every d -element subset determines a face, and σ is a d -face if and only if it contains a pair $i, i + 1 \in \sigma$ or $n, 1 \in \sigma$. Moreover neighborly $2d$ -polytopes only have missing faces of dimension d :

Lemma 5.2 (Shemer [38, Theorem 2.4]). *Let P be a neighborly $2d$ -polytope, and let σ be a missing face of P . Then $\dim \sigma = d$.*

The stable $(d + 1)$ -element subsets of $\{1, \dots, n\}$ are thus exactly the missing faces of the cyclic $2d$ -polytope on $n \geq 2d + 2$ vertices. The first part of Theorem 5.1 is now a special case of the following Kneser theorem for missing faces of a sphere:

Theorem 5.3. *Let K be a triangulation of S^{d-1} on $N + 1$ vertices. Then we have that $\chi(KG^2(K, \Delta_N)) = N + 1 - d$.*

Proof. The cone C over K is a triangulation of the d -ball and hence embeds into \mathbb{R}^d . The complex C has the same missing faces as K . Thus $\chi(KG^2(K, \Delta_N)) = \chi(KG^2(C, \Delta_{N+1}))$. Moreover C has $N + 2$ vertices, so by Corollary 4.7 we have that $\chi(KG^2(C, \Delta_{N+1})) \geq N + 1 - d$. This bound is tight by Remark 4.10. \square

The stable Kneser graph $KG'(n, 2n + k)$ is equal to $KG^2(\partial C_{2n-2}(2n + k), \Delta_{2n+k-1})$. Applying Theorem 5.3 for $N = 2n + k - 1$ and $d = 2n - 2$ yields $\chi(KG'(n, 2n + k)) = k + 2$. More generally, any neighborly polytope yields a proper induced subgraph of a Kneser graph of the same chromatic number:

Corollary 5.4. *Let P be a neighborly $2d$ -polytope on $N + 1$ vertices distinct from the simplex. Then $KG^2(\partial P, \Delta_N)$ is a subgraph of the Kneser graph $KG(d + 1, N + 1)$ and*

$$\chi(KG^2(\partial P, \Delta_N)) = N + 1 - 2d = \chi(KG(d + 1, N + 1)).$$

This is an immediate consequence of Theorem 5.3 and Lemma 5.2. Since neighborly polytopes are plentiful, see Padrol [32], so are induced subgraphs of

$\text{KG}(n, 2n + k)$ with chromatic number $k + 2$ and the same number of vertices as $\text{KG}'(n, 2n + k)$. (They have the same number of vertices since neighborly $2d$ -polytopes with an equal number of vertices have the same number of d -faces.) We leave it as an open problem whether the subgraphs induced by the missing faces of neighborly polytopes are always vertex-critical.

6. POINTS IN CYCLIC POSITION AND COLORINGS OF KNESER HYPERGRAPHS

The approach of Section 5 can be generalized to the hypergraph setting. A subset σ of $\{1, \dots, n\}$ is called s -stable if any two elements of σ are at distance at least s in the cyclic ordering of $\{1, \dots, n\}$. In particular, 2-stable sets are the stable sets defined in Section 5. Denote by $\text{KG}^r(k, n)_{s\text{-stab}}$ the partial hypergraph of $\text{KG}^r(k, n)$ induced by those vertices corresponding to s -stable sets, that is, the vertices of $\text{KG}^r(k, n)_{s\text{-stab}}$ correspond to s -stable k -element subsets of $\{1, \dots, n\}$.

Conjecture 6.1 (Ziegler [42], Alon, Drewnowski, and Łuczak [3]). *Let k, r and n be positive integers where $n \geq rk$ and $r \geq 2$. Then we have that $\chi(\text{KG}^r(k, n)_{r\text{-stab}}) = \lceil \frac{n-r(k-1)}{r-1} \rceil$.*

Alon, Drewnowski, and Łuczak [3] showed that if Conjecture 6.1 holds for the hypergraphs $\text{KG}^q(k, n)_{q\text{-stab}}$ and $\text{KG}^p(k, n)_{p\text{-stab}}$, then it also holds for $\text{KG}^r(k, n)_{r\text{-stab}}$ with $r = pq$. Conjecture 6.1 holds for $r = 2$ by Theorem 5.1 and thus it holds if r is equal to any power of two. Moreover it suffices to prove Conjecture 6.1 for r a prime.

Meunier showed [29] that restricting $\text{KG}^r(k, n)$ to those vertices corresponding to almost stable sets does not decrease the chromatic number. A set $\sigma \subseteq \{1, \dots, n\}$ is called *almost stable* if for all $i, j \in \sigma$ we have that $|i - j| \geq 2$. More general is the following recent result:

Theorem 6.2 (Alishahi and Hajiabolhassan [2]). *Let k, n , and r be positive integers with r even or n and k not congruent mod $r - 1$. Then $\chi(\text{KG}^r(k, n)_{2\text{-stab}}) = \lceil \frac{n-r(k-1)}{r-1} \rceil$.*

Here we prove that certain small subhypergraphs of $\text{KG}^r(k, n)$ still have chromatic number $\lceil \frac{n-r(k-1)}{r-1} \rceil$ at least if $r - 1$ divides $n - 1$ where r is a prime power. The subsets of $\{1, \dots, n\}$ we restrict to are $(\frac{r(k-3)}{2(k-1)} + 1)$ -stable on average. We need the following definitions: A set $\sigma \subseteq \{1, \dots, n\}$ of cardinality k defines k possibly empty gaps $\{1, \dots, n\} \setminus \sigma$ in the cyclic ordering of $\{1, \dots, n\}$. We denote the cardinality of these gaps by g_1, \dots, g_k . The average distance among cyclically successive elements of σ is given by $\frac{1}{k} \sum g_i + 1$ and is of course equal to the constant $\frac{n}{k}$. Suppose that g_k is the maximum among g_1, \dots, g_k . We say that σ is t -stable on average if $t \leq \frac{1}{k-1} \sum_{i=1}^{k-1} g_i + 1$. Any t -stable set is t -stable on average. We do not require t to be an integer. Denote by $\text{KG}^r(k, n; t)$ the subhypergraph of $\text{KG}^r(k, n)$ induced by those vertices that correspond to sets that are t -stable on average.

The following lemma is simple to prove; see for example Oppermann and Thomas [30, Lemma 2.7]. As the proof there is only for the even-dimensional case we include a proof below.

Lemma 6.3. *Let X_1 and X_2 be sets of pairwise distinct points on the moment curve in \mathbb{R}^d such that $\text{conv } X_1 \cap \text{conv } X_2 \neq \emptyset$. Then there are subsets $Y_1 \subseteq X_1$ and $Y_2 \subseteq X_2$*

of cardinality $\lfloor \frac{d}{2} + 1 \rfloor$ and $\lceil \frac{d}{2} + 1 \rceil$, respectively, such that $\text{conv } Y_1 \cap \text{conv } Y_2 \neq \emptyset$ and such that the vertices of Y_1 and Y_2 alternate along the moment curve.

Proof. A finite set of pairwise distinct points on the moment curve is in general position. Find inclusion-minimal sets $Y_1 \subseteq X_1$ and $Y_2 \subseteq X_2$ such that $\text{conv } Y_1 \cap \text{conv } Y_2 \neq \emptyset$. Then the codimensions of these convex hulls add up to d , or equivalently $|Y_1| + |Y_2| = d + 2$. Since cyclic polytopes are neighborly and thus convex hulls of at most $\lfloor \frac{d}{2} \rfloor$ points are faces on the boundary of $\text{conv}(X_1 \cup X_2)$, we know that Y_1 and Y_2 have cardinality $\lfloor \frac{d}{2} + 1 \rfloor$ and $\lceil \frac{d}{2} + 1 \rceil$. Since those points are in general position we know that $\text{conv } Y_1 \cap \text{conv } Y_2$ consists of a single point x which is in the relative interior of both convex hulls.

Continuously moving the points in $Y_1 \cup Y_2$ along the moment curve while keeping them distinct continuously varies the intersection point x that remains in the relative interior of both convex hulls. If two points y, y' of Y_1 , say, were adjacent along the moment curve with no point of Y_2 between them, we could continuously move y towards y' . By continuity the intersection $\text{conv } Y_1 \cap \text{conv } Y_2$ is nonempty when $y = y'$, which is in contradiction to these $d + 1$ points being in general position. \square

We say that a finite point set $X \subseteq \mathbb{R}^d$ is in *strong general position* if for any pairwise disjoint subsets $X_1, \dots, X_r \subseteq X$ of cardinality at most $d + 1$ the codimension of $\text{aff } X_1 \cap \dots \cap \text{aff } X_r$ is the sum of the codimensions of the affine hulls $\text{aff } X_i$ provided that this sum is at most $d + 1$. Here we define the empty set to have codimension $d + 1$. See Perles and Sigron [33] for this stronger notion of general position and the fact that points generically are in strong general position.

Lemma 6.4. *Let $r \geq 2$, $k \geq 2$, and $d \geq 1$ be integers such that $(r - 1)d > r(k - 2)$ and let $t < \frac{r-1}{k-1} \lfloor \frac{d}{2} \rfloor + 1$. Let X be the set system consisting of those subsets of $\{1, \dots, n\}$ that have cardinality k and are t -stable on average. Let K be the simplicial complex that contains $\sigma \subseteq \{1, \dots, n\}$ as a face if no subset of σ is contained in X . Then there is an affine map $f: K \rightarrow \mathbb{R}^d$ such that for any r pairwise disjoint faces $\sigma_1, \dots, \sigma_r$ of K we have that $f(\sigma_1) \cap \dots \cap f(\sigma_r) = \emptyset$.*

Proof. We spread the vertices $1, \dots, n$ in cyclic order along the moment curve in \mathbb{R}^d such that they are in strong general position. Suppose there were r pairwise disjoint faces $\sigma_1, \dots, \sigma_r$ of K such that $f(\sigma_1) \cap \dots \cap f(\sigma_r) \neq \emptyset$. W.l.o.g. we can assume that each σ_i has dimension at most d . Then the map f preserves dimensions: $\dim f(\sigma_i) = \dim \sigma_i$. Since the sum of codimensions of the σ_i is at most d and $r(d - k + 2) > d$, we know that at least one σ_j has dimension at least $k - 1$ and thus cardinality $\ell \geq k$.

Now $f(\sigma_j)$ intersects each $f(\sigma_i)$, and thus there are subsets $\tau_i \subseteq \sigma_i$ and $\tau'_i \subseteq \sigma_j$ of cardinality $\geq \lfloor \frac{d}{2} + 1 \rfloor$ that are intertwined in the sense of Lemma 6.3, that is, their vertices alternate in $\{1, \dots, n\}$. Thus at least $\lfloor \frac{d}{2} \rfloor$ elements of τ_i are not contained in the largest gap of σ_j . The largest gap of σ_j has cardinality at most $n - \ell - (r - 1) \lfloor \frac{d}{2} \rfloor$. Since the sum of the gap sizes is $n - \ell$, we deduce that the sum of the cardinalities of all but the largest gap is at least $(r - 1) \lfloor \frac{d}{2} \rfloor$. Thus σ_j is t -stable on average for $t = \frac{r-1}{k-1} \lfloor \frac{d}{2} \rfloor + 1$. This is a contradiction since some k -element subset of σ_j must be t -stable on average too, but then σ_j is not a face of K . \square

Theorem 6.5. *Suppose $r \geq 2$ is a prime power and that $r - 1$ divides $n - 1$. Let $k \geq 4$ and let $t = \frac{r(k - 3)}{2(k - 1)} + 1$. Then $\chi(\text{KG}^r(k, n; t)) = \lceil \frac{n - r(k - 1)}{r - 1} \rceil$.*

Proof. Choose the integer d such that $r(k-1) - 1 \geq (r-1)d > r(k-2)$. Then we can verify that $t = \frac{r(k-3)}{2(k-1)} + 1 = \frac{r(k-2)-r}{2(k-1)} + 1 < \frac{r-1}{k-1} \frac{d-1}{2} + 1 \leq \frac{r-1}{k-1} \lfloor \frac{d}{2} \rfloor + 1$. Thus with the simplicial complex K chosen as in Lemma 6.4 we have that there is an affine map $f: K \rightarrow \mathbb{R}^d$ such that for any r pairwise disjoint faces $\sigma_1, \dots, \sigma_r$ of K we have that $f(\sigma_1) \cap \dots \cap f(\sigma_r) = \emptyset$. Moreover the k -element sets that are t -stable on average are the missing faces of K by Lemma 4.2: $\text{KG}^r(K, \Delta_{n-1}) = \text{KG}^r(k, n; t)$. By Corollary 4.7 we get the lower bound $\chi(\text{KG}^r(k, n; t)) \geq \frac{n-1}{r-1} - d \geq \frac{n-r(k-1)}{r-1}$. Then $\chi(\text{KG}^r(k, n; t)) = \lceil \frac{n-r(k-1)}{r-1} \rceil$ since $\frac{n-1}{r-1} - d$ is an integer. \square

One could hope for a proof of Conjecture 6.1 in this way, at least if $r-1$ divides $n-1$. Let X be the set system consisting of k -element subsets of $\{1, \dots, n\}$ that are r -stable. Let K be the simplicial complex that contains $\sigma \subseteq \{1, \dots, n\}$ as a face if no subset of σ is contained in X . Let $d \leq \frac{r}{r-1}(k-1) - \frac{1}{r-1}$. Find a continuous map $f: K \rightarrow \mathbb{R}^d$ such that for any r pairwise disjoint faces $\sigma_1, \dots, \sigma_r$ of K we have that $f(\sigma_1) \cap \dots \cap f(\sigma_r) = \emptyset$. By work of Mabillard and Wagner [25], see Theorem 4.14 and Corollary 4.15, it is sufficient to exhibit an \mathfrak{S}_r -equivariant map $K_{\Delta}^r \rightarrow S(W_r^{\oplus d})$ provided that $(r+1)(k-1) \leq rd - 3$.

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