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THE SCALE FUNCTION AND LATTICES

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ABSTRACT. It is shown that, given a lattice H in a totally disconnected, locally compact group G, the contraction subgroups in G and the values of the scale function on G are determined by their restrictions to H. Group theoretic properties intrinsic to the lattice, such as being periodic or infinitely divisible, are then seen to imply corresponding properties of G.

1. Introduction

Lattices in locally compact groups have been studied extensively. In the connected case this reduces to the study of lattices in Lie groups; strong results have been obtained, including the Margulis superrigidity theorem for lattices in higher rank semisimple Lie groups and some of its extensions; see [17, 19, 20].

Lattices in specific types of totally disconnected, locally compact (t.d.l.c.) groups have been studied in [2, 8, 16], and that a uniform lattice which happens to be a free group controls the scale on an ambient t.d.l.c. group was seen in [3]. It has also been shown recently that, in contrast to the connected case, compactly generated, simple t.d.l.c. groups need not have lattices, [1, 7]. This note gives further information about how lattices control an ambient t.d.l.c. group G by showing that the contraction subgroups in G and the values of the scale function on G are determined by their restrictions to the lattice.

We begin by recalling a few key facts about lattices and the scale function.

Definition 1.1. The closed subgroup H of the locally compact group G has *finite co-volume* if G/H supports a finite G-invariant measure. If H has finite co-volume and is discrete with the subspace topology, it is a *lattice* in G.

The discrete subgroup H has finite co-volume if G/H is compact, that is, if H is co-compact; see [19, Remark 1.11].

Definition 1.2. The *scale* on the t.d.l.c. group G is the function $s_G : G \to \mathbb{Z}^+$ defined by

$$s_G(x) := \min\left\{[xVx^{-1} : xVx^{-1} \cap V] : V \leqslant G,\ V \text{ compact and open}\right\}.$$

A compact open subgroup $V \leq G$ is *tidy for x in G* if, setting $V_\pm = \bigcap_{n\geq 0} x^{\pm n} V x^{\mp n}$,

$$V = V_+V_- \text{ and } V_{++} := \bigcup_{n\geq 0} x^n V_+ x^{-n} \text{ is closed.}$$

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Every t.d.l.c. group has a base of identity neighbourhoods consisting of compact open subgroups, by [12], [18, Theorem II.2.3] or [13, Theorem II.7.7]. The value $s_G(x)$ is then well-defined because $[xVx^{-1} : xVx^{-1} \cap V]$ is a positive integer. That subgroups tidy for x always exist is implied by the following.

Theorem 1.3 ([21, Definition, p. 343; 22, Theorem 3.1])**.** *The compact open subgroup $V \leq G$ is tidy for x if and only if*

$$s_G(x) = [xVx^{-1} : xVx^{-1} \cap V].$$

The scale function on G is related to the modular function $\Delta : G \to (\mathbb{Q}^+, \times)$ by the formula $\Delta(x) = s_G(x)/s_G(x^{-1})$. It is well known that the modular function is identically equal to 1 if G contains a lattice (see [19, Remark 1.9]), and the motivation for this work is to understand how properties of a lattice influence the scale function and related properties of G. The relevant properties of G are as follows.

Definition 1.4. Let G be a t.d.l.c. group.

(1) G is *uniscalar* if the scale function is identically equal to 1.
(2) The *contraction subgroup* for $x \in G$ is

$$\mathsf{con}(x) := \left\{ g \in G \mid x^n g x^{-n} \to e \text{ as } n \to \infty \right\}.$$

(3) G is *anisotropic* if $\mathsf{con}(x) = \{e\}$ for every $x \in G$.

These features of G are related by the fact that triviality of $\mathsf{con}(x)$ implies that $s_G(x^{-1}) = 1$; see [5, Proposition 3.24] for the case when G is metrizable and [14] for the non-metrizable case. Hence anisotropic groups are uniscalar; the converse does not hold however. Triviality of the *Tits core* is equivalent to G being anisotropic; see [10, Proposition 3.1].

The observation that $\mathsf{con}(x^n) = \mathsf{con}(x)$ for every $n \geq 1$ will be useful later.

2. Subgroups with finite co-volume

The proof of the main theorem relies on two results which may be found in [4] and [10] respectively but are restated here. The proofs of these results involve an iterative argument that uses the factoring $V = V_+V_-$ of tidy subgroups and the easily verified containments $xV_+x^{-1} \geq V_+$ and $xV_-x^{-1} \leq V_-$.

Lemma 2.1 ([4, Lemma 2.4])**.** *Suppose that V is a compact, open subgroup of G that is tidy for $x \in G$, and let $n \geq 1$. Then $(VxV)^n = V_-x^nV_+$ and $s_G(y) = s_G(x)^n$ for every $y \in (VxV)^n$.*

The following result is established in [10] for all y in xV. The proof that it holds for all y in VxV, as claimed here, is the same except that at one point a certain product that is observed to belong to V has an extra factor of v_1, where $y = v_1xv_2$.

Lemma 2.2 (*cf.* [10, Lemma 4.1])**.** *Suppose that V is a compact, open subgroup of G that is tidy for x. Then for every $y \in VxV$ there is $t \in V_+ \cap \mathsf{con}(x^{-1})$ such that $t^{-1}y^ktx^{-k} \in V$ for every $k \geq 0$.*

Remark 2.3. It may happen in Lemma 2.2 that $\mathsf{con}(x^{-1})$ is trivial, in which case necessarily $t = e$. However any subgroup tidy for x then satisfies $xVx^{-1} \leq V$ and it follows that $y^kx^{-k} \in V$.

Lemma 2.2 implies the following by the same argument as given in [10].

Proposition 2.4 (*cf.* [10, Corollary 4.2]). *Suppose that V is a compact, open subgroup of G that is tidy for x. Then for every $y \in VxV$ there is $t \in V_+ \cap \mathsf{con}(x^{-1})$ such that $t\mathsf{con}(x)t^{-1} = \mathsf{con}(y)$.*

We are now ready to prove the main result of this note.

Theorem 2.5. *Let G be a totally disconnected, locally compact group and H be a closed subgroup of G that either*

(1) *has finite co-volume* *or* (2) *is co-compact.*

Then, for every $x \in G$ there are $n \geq 1$ and $h \in H$ such that

$$s_G(x)^n = s_G(h).$$

Moreover, h may be chosen so that $\mathsf{con}(h)$ is conjugate to $\mathsf{con}(x)$.

Proof. (1) Let V be a compact, open subgroup of G that is tidy for x and let μ be a finite G-invariant measure on G/H. Then $\pi_H(V)$ is an open subset of G/H and hence $\mu(\pi_H(V)) > 0$. For each $n \geq 1$ we have that

$$\mu(\pi_H(x^n V)) = \mu(x^n.\pi_H(V)) = \mu(\pi_H(V))$$

because μ is G-invariant. Since μ is finite, finite additivity of μ implies that there exists $n > 0$ such that

$$\pi_H(x^n V) \cap \pi_H(V) \neq \emptyset.$$

The definition of π_H then implies that there is $h \in H$ with $x^n V \cap V h \neq \emptyset$. Therefore h belongs to $V x^n V$. Hence Lemma 2.1 implies that $s_G(h) = s_G(x)^n$, and Proposition 2.4 implies that $\mathsf{con}(h)$ is conjugate to $\mathsf{con}(x^n)$, which, as remarked in the introduction, is equal to $\mathsf{con}(x)$.

(2) Let V be a compact, open subgroup of G that is tidy for x. Then, since H is co-compact and V is open, G partitions as the disjoint union of finitely many double co-sets $Vy_1H, \ldots, Vy_kH$. Hence there are $m, n > 0$ such that $Vx^mH = Vx^{m+n}H$, and it follows that there exists $h \in H$ with

$$h \in x^{-m}Vx^{m+n} = \left(x^{-m}Vx^m\right)x^n.$$

The subgroup $x^{-m}Vx^m$ is tidy for x. Therefore $s(h) = s(x^n)$ and $\mathsf{con}(h)$ is conjugate to $\mathsf{con}(x^n)$ for the same reasons as in part (1). □

It is not true in general that $s_G(H) = s_G(G)$. For example, the group $G = \mathbb{Q}_p \rtimes_p \mathbb{Z}$ has the co-compact subgroup $H = \mathbb{Q}_p \rtimes_p n\mathbb{Z}$ and

$$s_G(G) = \left\{p^k \mid k \geq 0\right\}, \text{ whereas } s_G(H) = \left\{p^{nk} \mid k \geq 0\right\}.$$

Several conclusions may be drawn immediately from the theorem.

Corollary 2.6. *Let G be a t.d.l.c. group and H be a closed subgroup that either has finite co-volume or is co-compact.*

1. *If $s_G(h) = 1$ for every $h \in H$, then G is uniscalar.*
2. *If $\mathsf{con}(h)$ is closed for every $h \in H$, then $\mathsf{con}(x)$ is closed for every $x \in G$.*
3. *If every element of H has trivial contraction group, then G is anisotropic.*

Here are two situations in which Corollary 2.6 may be used to show that group theoretic properties of a lattice imply properties of its ambient group.

Proposition 2.7. *Suppose that the t.d.l.c. group G has a closed co-compact or finite co-volume subgroup H with $\langle h\rangle$ pre-compact for every $h \in H$. Then G is anisotropic and $\langle x\rangle$ is pre-compact for every $x \in G$. However, G need not have a normal compact open subgroup.*

Proof. Every element of G which generates a pre-compact subgroup has trivial contraction group. Hence H is anisotropic, and it follows by Corollary 2.6 that G is anisotropic. Suppose that x is in G and let $V \leq G$ be tidy for x. Then V is normalised by x and, by the argument in the proof of Theorem 2.5, there is $n \geq 1$ such that $x^n V \cap H \neq \emptyset$. Choose $h \in x^n V \cap H$. Then $x^n \in \langle h, V\rangle$, which is compact because $\langle h\rangle$ is pre-compact and normalises V. Therefore $\langle x\rangle$ has compact closure.

To see the final claim, observe that the group

$$G = \big\{x = (x_n)_{n\in\mathbb{N}} \in S_3^{\mathbb{N}} \mid x_n \in \{e, (\,1\;2\,)\} \text{ for almost all } n\big\}$$

contains the co-compact lattice

$$H = \{x \in G \mid x_n \in \{e, (\,1\;2\;3\,), (\,1\;3\;2\,)\} \text{ for all } n\}$$

but has no compact open normal subgroup. □

In the next proposition, *infinite divisibility* of the element x in G means that there are increasing sequences, n_k, of positive integers and, x_k, of elements of G such that $x = x_k^{n_k}$ for every k.

Proposition 2.8. *Suppose that G has a closed co-compact or finite co-volume subgroup H in which every element is infinitely divisible. Then G is uniscalar but need not be anisotropic, even if it contains an infinitely divisible co-compact lattice.*

Proof. Since $s_G(x^n) = s_G(x)^n$ for every $n \geq 0$, infinite divisibility of x implies that $s_G(x) = 1$. Hence H is uniscalar, and it follows by Corollary 2.6 that G is uniscalar.

The group $G = C_2^{\mathbb{Q}} \rtimes \mathbb{Q}$, where C_2 is the group of order 2 and $\mathbb{Q}$ acts on $C_2^{\mathbb{Q}}$ by translation, has the infinitely divisible lattice $\mathbb{Q}$, but $\overline{\mathsf{con}(x)} = C_2^{\mathbb{Q}}$ for every $x \neq 0$ in $\mathbb{Q}$. Hence G is not anisotropic, thus justifying the last claim. □

This note is partly motivated by the problem of whether there are non-discrete topologically simple t.d.l.c. groups that are compactly generated and uniscalar. It follows from the results here that one way in which such a group G might be found would be to exploit group-theoretic constructions of finitely generated groups that are periodic or infinitely divisible by embedding them into G as a lattice. Then, by [9, Theorem A], G would have either an infinite discrete quotient or a non-discrete but uniscalar and topologically simple subquotient. It is therefore of interest to know whether a t.d.l.c. group with a periodic or infinitely divisible lattice must have an open normal subgroup with infinite index. The group with no normal compact open subgroup given in the proof of Proposition 2.7 is not compactly generated. On the other hand, while there is a compactly generated, uniscalar group that does not have a compact, open normal subgroup (see [15] and [6]), that group does not contain a lattice. The following question therefore remains open.

Question 1. Suppose that G is a compactly generated t.d.l.c. group with a lattice that either is a periodic group or is infinitely divisible. Must G have a compact open normal subgroup?

The paper concludes with another question about how properties of a group might depend on a lattice. It is shown in [3] that if the t.d.l.c. group G has a uniform lattice isomorphic to the free group of rank k, then the set of prime divisors of $s_G(G)$ is bounded by a number that depends on k. Note, too, that by [11, Theorem 1], if G is a compactly generated t.d.l.c. group having no compact normal subgroup, there is a finite set $\eta = \eta(G)$ of prime numbers such that the open pro-η subgroups of G form a base of identity neighbourhoods.

Question 2. Suppose that G is a t.d.l.c. group that has no compact normal subgroup and with a lattice having k generators. Is there a bound on the set of prime divisors of $s_G(G)$ and on $\eta(G)$ that depends only on k?

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References

[1] Uri Bader, Pierre-Emmanuel Caprace, Tsachik Gelander, and Shahar Mozes, *Simple groups without lattices*, Bull. Lond. Math. Soc. **44** (2012), no. 1, 55–67, DOI 10.1112/blms/bdr061. MR2881324

[2] Hyman Bass and Alexander Lubotzky, *Tree lattices,* with appendices by H. Bass, L. Carbone, A. Lubotzky, G. Rosenberg and J. Tits, Progress in Mathematics, vol. 176, Birkhäuser Boston, Inc., Boston, MA, 2001. MR1794898

[3] Udo Baumgartner, *Scales for co-compact embeddings of virtually free groups*, Geom. Dedicata **130** (2007), 163–175, DOI 10.1007/s10711-007-9212-2. MR2365784

[4] Udo Baumgartner, Jacqui Ramagge, and George A. Willis, *Scale-multiplicative semigroups and geometry: automorphism groups of trees*, Groups Geom. Dyn. **10** (2016), no. 3, 1051–1075, DOI 10.4171/GGD/376. MR3551189

[5] Udo Baumgartner and George A. Willis, *Contraction groups and scales of automorphisms of totally disconnected locally compact groups*, Israel J. Math. **142** (2004), 221–248, DOI 10.1007/BF02771534. MR2085717

[6] Meenaxi Bhattacharjee and Dugald MacPherson, *Strange permutation representations of free groups*, J. Aust. Math. Soc. **74** (2003), no. 2, 267–285, DOI 10.1017/S1446788700003293. MR1957883

[7] Adrien Le Boudec, *Groups acting on trees with almost prescribed local action*, Comment. Math. Helv. **91** (2016), no. 2, 253–293, DOI 10.4171/CMH/385. MR3493371

[8] Marc Burger and Shahar Mozes, *Lattices in product of trees*, Inst. Hautes Études Sci. Publ. Math. **92** (2000), 151–194 (2001). MR1839489

[9] Pierre-Emmanuel Caprace and Nicolas Monod, *Decomposing locally compact groups into simple pieces*, Math. Proc. Cambridge Philos. Soc. **150** (2011), no. 1, 97–128, DOI 10.1017/S0305004110000368. MR2739075

[10] Pierre-Emmanuel Caprace, Colin D. Reid, and George A. Willis, *Limits of contraction groups and the Tits core*, J. Lie Theory **24** (2014), no. 4, 957–967. MR3328731

[11] Pierre-Emmanuel Caprace, Colin D. Reid, and George A. Willis, *Locally normal subgroups of simple locally compact groups* (English, with English and French summaries), C. R. Math. Acad. Sci. Paris **351** (2013), no. 17-18, 657–661, DOI 10.1016/j.crma.2013.09.010. MR3124321

[12] D. van Dantzig, *Zur topologischen Algebra* (German), Math. Ann. **107** (1933), no. 1, 587–626, DOI 10.1007/BF01448911. MR1512818

[13] Edwin Hewitt and Kenneth A. Ross, *Abstract harmonic analysis. Vol. I: Structure of topological groups. Integration theory, group representations*, Die Grundlehren der mathematischen Wissenschaften, Bd. 115, Academic Press, Inc., Publishers, New York; Springer-Verlag, Berlin-Göttingen-Heidelberg, 1963. MR0156915

[14] Wojciech Jaworski, *On contraction groups of automorphisms of totally disconnected locally compact groups*, Israel J. Math. **172** (2009), 1–8, DOI 10.1007/s11856-009-0059-0. MR2534235

[15] A. Kepert and G. Willis, *Scale functions and tree ends*, J. Aust. Math. Soc. **70** (2001), no. 2, 273–292, DOI 10.1017/S1446788700002640. MR1815284

[16] Alexander Lubotzky, *Lattices in rank one Lie groups over local fields*, Geom. Funct. Anal. **1** (1991), no. 4, 406–431, DOI 10.1007/BF01895641. MR1132296

[17] G. A. Margulis, *Discrete subgroups of semisimple Lie groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 17, Springer-Verlag, Berlin, 1991. MR1090825

[18] Deane Montgomery and Leo Zippin, *Topological transformation groups*, Interscience Publishers, New York-London, 1955. MR0073104

[19] M. S. Raghunathan, *Discrete subgroups of Lie groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 68, Springer-Verlag, New York-Heidelberg, 1972. MR0507234

[20] Yehuda Shalom, *Rigidity of commensurators and irreducible lattices*, Invent. Math. **141** (2000), no. 1, 1–54, DOI 10.1007/s002220000064. MR1767270

[21] G. Willis, *The structure of totally disconnected, locally compact groups*, Math. Ann. **300** (1994), no. 2, 341–363, DOI 10.1007/BF01450491. MR1299067

[22] George A. Willis, *Further properties of the scale function on a totally disconnected group*, J. Algebra **237** (2001), no. 1, 142–164, DOI 10.1006/jabr.2000.8584. MR1813900

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