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AN ARC GRAPH DISTANCE FORMULA FOR THE FLIP GRAPH

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Abstract. Using existing technology, we prove a Masur-Minsky style distance formula for flip-graph distance between two triangulations, expressed as a sum of the distances of the projections of these triangulations into arc graphs of the suitable subsurfaces of S.

1. Introduction

Let S be a surface with at least one puncture and $\chi(S) < 0$, and write $\mathcal{F}(S)$ for the *flip graph of* S. This is the graph whose vertices are in a one-to-one correspondence with ideal triangulations and whose edges connect triangulations that differ by a *flip*; see [DP14] and Figure 1. The purpose of this note is to prove the following formula estimating distance in $\mathcal{F}(S)$.

Theorem 1.1. *Fix S, a connected, orientable, finite type, surface of non-positive Euler characteristic, with at least one puncture, and not a pair of pants. For any $k > 0$ sufficiently large, there exist $K \geq 1, C \geq 0$ so that for any two triangulations $T_1, T_2 \in \mathcal{F}(S)$ we have*

$$d_{\mathcal{F}}(T_1, T_2) \stackrel{K,C}{\asymp} \sum_{Y \subseteq S} [d_{\mathcal{A}(Y)}(T_1, T_2)]_k.$$

The distances on the right are *arc graph* distances in subsurfaces, $[x]_k$ is the cut-off function giving value x if $x \geq k$ and 0 otherwise, and $x \stackrel{K,C}{\asymp} y$ is shorthand for the condition $\frac{1}{K}(x - C) \leq y \leq Kx + C$. See the next section for a precise statement.

Our theorem follows more or less directly from the Masur-Minsky distance formula [MM00] and the Masur-Schleimer distance formula [MS13], but seems worth making explicit since $\mathcal{F}(S)$ is an important, particularly tractable, geometric model for the mapping class group of S (see e.g. [Har85, Har86, Hat91, Mos95, DP14, Bel14]), while on the other side, the geometry of the arc graph has been greatly simplified in [HPW15]. Various distance formulas [MM99, MS13, Raf07] have been used extensively to understand the geometry of mapping class group, Teichmüller space, and homomorphisms (see e.g. [Bro03, Beh06, KL08, Bow09, BDS11, BKMM12, CLM12, Tao13, EMR14, BBF15]) and have motivated research in related areas (see e.g. [SS12, CP12, Tay13, Sis13, KK14, BF14, Tay14, HH15, Vog15]).

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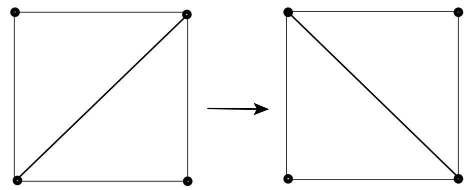

FIGURE 1. An example of a flip in the flip graph

It would be interesting to find a proof of Theorem 1.1 that does not appeal to the previous distance formulas.

2. The proof

For a surface S of genus g with n punctures, we write $\xi(S) = 3g-3+n$ (we do not distinguish between a puncture and a hole, and will only refer to punctures to avoid confusion later). All surfaces we consider are connected, orientable, have at least one puncture, and have $\xi > 0$, with one exception: we allow annuli (which have $\xi = -1$). In particular, we exclude three-punctured spheres in all of what follows. Arcs, curves, multiarcs, and multicurves are assumed essential and are considered up to isotopy. Multiarcs and multicurves have pairwise non-isotopic components. Ideal triangulations are multiarcs with a maximal number of components. Markings are complete clean markings (see [MM00]).

We write $\mathcal{C}(Y)$ for the arc-and-curve graph of a surface Y, which is quasi-isometric to the curve graph (more precisely, the inclusion of the curve graph into the arc-and-curve graph is a quasi-isometry). Given any multiarc, multicurve, marking, or triangulation, α on a surface S and subsurface $Y \subseteq S$ which is not an annulus, we let $\pi_Y(\alpha)$ denote the arc-and-curve projection: This is the union of the isotopy classes of arcs and curves of intersection of α with Y (assuming they are in minimal position). For Y an annulus, we use the usual projection to $\mathcal{A}(Y)$ via the cover corresponding to Y; see [MM00] for details. We will write

$$d_{\mathcal{C}(Y)}(\alpha, \beta) = diam(\pi_Y(\alpha) \cup \pi_Y(\beta))$$

where the diameter is taken in $\mathcal{C}(Y)$. When the projections are non-empty, for example if α is a marking or a triangulation, then $d_{\mathcal{C}(Y)}$ satisfies a triangle inequality. If α is an arc or a triangulation, then $\pi_Y(\alpha)$ is in the arc graph, $\mathcal{A}(Y)$, and so we can define $d_{\mathcal{A}(Y)}(\alpha, \beta)$ similarly. We note that using the arc-and-curve graph projection, it follows that for any $X \subseteq Y \subseteq S$, we have $\pi_X \circ \pi_Y = \pi_X$, unless X is an annulus.

As stated in the introduction, the flip graph $\mathcal{F}(S)$ is the graph whose vertex set is the set isotopy classes of (ideal) triangulations. Two vertices in the graph share an edge if they are related by a *flip*, in other words, if they differ at most by an arc; see [DP14] and Figure 1.

For markings μ_1, μ_2 on S, we let $d_{\mathcal{M}}(\mu_1, \mu_2)$ denote the distance in the marking graph $\mathcal{M}(S)$; see [MM00]. The first distance formula we will need is due to Masur and Minsky.

Theorem 2.1 ([MM00]). *Fix S, a connected, orientable surface with $\xi(S) > 0$. For any $k > 0$ sufficiently large, there exists $K, C \geq 1$ so that for any two markings μ_1, μ_2 we have*

$$d_{\mathcal{M}}(\mu_1, \mu_2) \overset{K,C}{\asymp} \sum_{Y \subseteq S} [d_{\mathcal{C}(Y)}(\mu_1, \mu_2)]_k.$$

In this theorem, we note that K, C can be chosen to depend monotonically on k. Indeed, the right-hand side becomes less efficient at estimating the left-hand side as k increases, so at least coarsely, this monotonicity is necessary.

There is a distance formula for arc graphs due to Masur and Schleimer (see Lemma 7.2 and Theorems 5.10 and 13.1 of [MS13]). To state this formula, we recall that given a surface Y, a *hole* for $\mathcal{A}(Y)$ is an essential subsurface $X \subseteq Y$ such that the punctures of Y are also punctures of X, which we write as $\partial Y \subseteq \partial X$. We let $H(\mathcal{A}(Y))$ denote the set of holes for $\mathcal{A}(Y)$. For Y an annulus, the only hole for $\mathcal{A}(Y)$ is Y, and Y is not a hole for $\mathcal{A}(X)$, for any other surface X.

Theorem 2.2 ([MS13]). *Fix S, a connected, orientable surface with at least one puncture and $\xi(S) > 0$. Then for any $k > 0$ sufficiently large, there exist $K \geq 1, C \geq 0$ so that for any two arcs α_1, α_2,*

$$d_{\mathcal{A}(S)}(\alpha_1, \alpha_2) \overset{K,C}{\asymp} \sum_{X \in H(\mathcal{A}(S))} [d_{\mathcal{C}(X)}(\alpha_1, \alpha_2)]_k.$$

The proof of Theorem 1.1 also requires the following elementary observation.

Lemma 2.3. *Fix a surface S. For any essential subsurface $X \subseteq S$, there are at most $2^{\xi(S)}$ subsurfaces Y such that X is a hole for $\mathcal{A}(Y)$.*

Proof. An essential subsurface X is a component of the complement of an essential multicurve that we denote $\partial_0 X$. If X is a hole for $\mathcal{A}(Y)$, then observe that Y is the component of the complement of $\partial_0 Y$ containing X. Therefore Y is determined by X and the multicurve $\partial_0 Y \subseteq \partial_0 X$. There are $2^{|\partial_0 X|}$ submulticurves of $\partial_0 X$, and $|\partial_0 X| \leq \xi(S)$, and hence at most this many $Y \subseteq S$ such that X is a hole for $\mathcal{A}(Y)$. □

Proof of Theorem 1.1. Fix S. For every ideal triangulation T, we choose a marking $\mu(T)$ so that $i(T, \mu(T))$ is minimized (here we simply take the sum of intersection numbers of components of T and $\mu(T)$). Because the mapping class group $\mathrm{Mod}(S)$ has only finitely many orbits on $\mathcal{F}(S)$, this intersection number is uniformly bounded, independent of T. Consequently, there exists $\delta_0 > 0$ such that for each triangulation T of S and every subsurface $Y \subseteq S$ we have

$$d_{\mathcal{C}(Y)}(\mu(T), T) < \delta_0. \tag{1}$$

Furthermore, we claim that $T \mapsto \mu(T)$ is coarsely $\mathrm{Mod}(S)$–equivariant. More precisely, for every $g \in \mathrm{Mod}(S)$ and $T \in \mathcal{F}(S)$, we claim that $d_{\mathcal{M}}(\mu(gT), g\mu(T))$ is uniformly bounded. This follows from Theorem 2.1 since (1) and the triangle inequality imply that

$$\begin{aligned} d_{\mathcal{C}(Y)}(\mu(gT), g\mu(T)) &\leq d_{\mathcal{C}(Y)}(\mu(gT), gT) + d_{\mathcal{C}(Y)}(gT, g\mu(T)) \\ &= d_{\mathcal{C}(Y)}(\mu(gT), gT) + d_{\mathcal{C}(g^{-1}Y)}(T, \mu(T)) \leq 2\delta_0. \end{aligned}$$

Since $\mathrm{Mod}(S)$ acts cocompactly by isometries on the proper geodesic spaces $\mathcal{F}(S)$ and $\mathcal{M}(S)$, the Milnor-Švarc Lemma implies $T \mapsto \mu(T)$ is a quasi-isometry.

Thus, for $T_1, T_2 \in \mathcal{F}(S)$ and $\mu_i = \mu(T_i)$, for $i = 1, 2$, we have

$$d_{\mathcal{F}}(T_1, T_2) \asymp d_{\mathcal{M}}(\mu_1, \mu_2). \tag{2}$$

Let (K_0, C_0) be the implicit constants in this coarse equation.

Next, we choose constants $0 < k_1 < k_2 < k_3 < \infty$ large enough so that for all $T_1, T_2 \in \mathcal{F}(S)$:
(i) If X is a hole for $\mathcal{A}(Y)$ and $d_{\mathcal{C}(X)}(T_1, T_2) \geq k_3$, then $d_{\mathcal{A}(Y)}(T_1, T_2) \geq k_2$; and
(ii) if $d_{\mathcal{A}(Y)}(T_1, T_2) \geq k_2$, then

$$d_{\mathcal{A}(Y)}(T_1, T_2) \asymp \sum_{X \in H(\mathcal{A}(Y))} [d_{\mathcal{C}(X)}(T_1, T_2)]_{k_1}$$

where the implicit constants in this coarse equation are $(K_1, 0)$. For **(ii)**, this means that when the arc graph distance is at least k_2, the sum with cut-off function k_1 is correct with only a multiplicative error. To see that we can find such k_1, k_2, k_3 and K_1, we first appeal to Theorem 2.2 to find k_1, k_2, K_1 so that **(ii)** holds. This is possible since once the arc-graph distance is bigger than twice the additive constant, say, by doubling the multiplicative constant, we may remove the additive error. Appealing to Theorem 2.2 again guarantees that for k_3 sufficiently large **(i)** also holds. For reasons that will become clear later, we will also assume that $k_1 \geq 10\delta$ and that $k_1 - 2\delta_0$ is above the threshold for Theorem 2.1 to hold.

For $T_1, T_2 \in \mathcal{F}(S)$, let $\Omega(T_1, T_2, k_2)$ be the set of subsurfaces $Y \subseteq S$ so that $d_{\mathcal{A}(Y)}(T_1, T_2) \geq k_2$. Then we have

$$\begin{aligned} \sum_{Y \subseteq S} [d_{\mathcal{A}(Y)}(T_1, T_2)]_{k_2} &= \sum_{Y \in \Omega(T_1, T_2, k_2)} d_{\mathcal{A}(Y)}(T_1, T_2) \\ &\asymp \sum_{Y \in \Omega(T_1, T_2, k_2)} \sum_{X \in H(\mathcal{A}(Y))} [d_{\mathcal{C}(X)}(T_1, T_2)]_{k_1}. \end{aligned}$$

The implicit constants in the coarse equation are again $(K_0, 0)$ by **(ii)**.

Let $\mathcal{H} = \mathcal{H}(T_1, T_2, k_1, k_2, k_3)$ be the set of all X which appear with non-zero contribution in the sum on the right-hand side of the above coarse equation. We note that $\mathcal{H}$ does not keep track of how many times such an X appears. By Lemma 2.3, any $X \in \mathcal{H}$ appears at most $2^{\xi(S)}$ times in the sum. Therefore we have

$$\sum_{X \in \mathcal{H}} d_{\mathcal{C}(X)}(T_1, T_2) \asymp \sum_{Y \subseteq S} [d_{\mathcal{A}(Y)}(T_1, T_2)]_{k_2}. \tag{3}$$

Here the implicit constants can be taken to be $(2^{\xi(S)} K_0, 0)$.

By definition, for each $X \in \mathcal{H}$, $d_{\mathcal{C}(X)}(T_1, T_2) \geq k_1$. On the other hand, if $d_{\mathcal{C}(X)}(T_1, T_2) \geq k_3$, then $X \in \mathcal{H}$. Thus $\mathcal{H}$ contains *all* subsurfaces with distance at least k_3 and *some* subsurfaces with distance at least k_1. Since $d_{\mathcal{C}(X)}(\mu_i, T_i) \leq \delta_0$, it follows that if $X \in \mathcal{H}$, then $d_{\mathcal{C}(X)}(\mu_1, \mu_2) \geq k_1 - 2\delta_0$, and if $d_{\mathcal{C}(X)}(\mu_1, \mu_2) \geq k_3 + 2\delta_0$, then $X \in \mathcal{H}$. By the monotonicity of the constants in Theorem 2.1, we have

$$d_{\mathcal{M}}(\mu_1, \mu_2) \asymp \sum_{X \in \mathcal{H}} d_{\mathcal{C}(X)}(\mu_1, \mu_2). \tag{4}$$

Here the implicit constants (K_2, C_2) in the coarse equation are the same as those in Theorem 2.1 for threshold $k_3 + 2\delta_0$. Finally, since $k_1 \geq 10\delta_0$, we have

$$\sum_{X \in \mathcal{H}} d_{\mathcal{C}(X)}(\mu_1, \mu_2) \asymp \sum_{X \in \mathcal{H}} d_{\mathcal{C}(X)}(T_1, T_2), \tag{5}$$

and one can check that the implicit constant is $(\frac{9}{8}, 0)$ (since each term on the left differs from the corresponding term on the right by an additive error which is small compared to its size).

Setting $k = k_2$, and combining (2), (4), (5), and (3), we get

$$d_{\mathcal{F}}(T_1, T_2) \asymp \sum_{Y \subseteq S} [d_{\mathcal{A}(Y)}(T_1, T_2)]_{k_2}$$

where the implicit constants in the coarse equation depend on all the above constants. This completes the proof. □

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