SHARP ESTIMATES OF RADIAL MINIMIZERS 
of \( p \)-LAPLACE EQUATIONS

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Abstract. We study semi-stable, radially symmetric and decreasing solutions \( u \in W^{1,p}(B_1) \) of \( -\Delta_p u = g(u) \) in \( B_1 \setminus \{0\} \), where \( B_1 \) is the unit ball of \( \mathbb{R}^N \), \( p > 1 \), \( \Delta_p \) is the \( p \)-Laplace operator and \( g \) is a general locally Lipschitz function. We establish sharp pointwise estimates for such solutions, which do not depend on the nonlinearity \( g \). By applying these results, sharp pointwise estimates are obtained for the extremal solution and its derivatives (up to order three) of the equation \( -\Delta_p u = \lambda f(u) \), posed in \( B_1 \), with Dirichlet data \( u|_{\partial B_1} = 0 \), where the nonlinearity \( f \) is an increasing \( C^1 \) function with \( f(0) > 0 \) and \( \lim_{t \to +\infty} \frac{f(t)}{t^{p-1}} = +\infty \).

1. Introduction and main results

This paper is concerned with the semi-stability of radially symmetric and decreasing solutions \( u \in W^{1,p}(B_1) \) of

\[
-\Delta_p u = g(u) \quad \text{in} \quad B_1 \setminus \{0\},
\]

where \( p > 1 \), \( \Delta_p := \text{div}(|\nabla u|^{p-2} \nabla u) \) is the \( p \)-laplacian of \( u \), \( B_1 \) is the unit ball of \( \mathbb{R}^N \), and \( g : \mathbb{R} \to \mathbb{R} \) is a locally Lipschitz function.

By abuse of notation, we write \( u(r) \) instead of \( u(x) \), where \( r = |x| \) and \( x \in \mathbb{R}^N \). We denote by \( u_r \) the radial derivative of a radial function \( u \).

Since \( u \in W^{1,p}(B_1) \) is radial, by the Sobolev embedding in one dimension, we obtain \( u \in L^\infty_{\text{loc}}(B_1 \setminus \{0\}) \). Hence, by standard regularity results it is deduced that \( u \in C^{1,\beta}_{\text{loc}}(B_1 \setminus \{0\}) \) for some \( \beta \in (0,1) \).

A radial solution \( u \in W^{1,p}(B_1) \) of (1.1) such that \( u_r(r) < 0 \) for all \( r \in (0,1) \) is called semi-stable if

\[
\int_{B_1} (p-1) |u_r|^{p-2} |\xi_r|^2 - g'(u)\xi^2 \geq 0,
\]

for every radially symmetric function \( \xi \in C^1_c(B_1 \setminus \{0\}) \).

Note that the above expression is formally the second variation of the energy functional associated to (1.1):

\[
E_\Omega(u) := \frac{1}{p} \int_\Omega |\nabla u|^p \, dx - \int_\Omega G(u) \, dx,
\]

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where \( G' = g \) and \( \Omega \subset B_1 \). Thus, if \( u \) is a radial local minimizer of \( (1.2) \) with \( \Omega = B_1 \) (i.e., for every \( \delta \in (0, 1) \) there exists \( \varepsilon_\delta > 0 \) such that \( E_{B_1 \setminus \overline{B}_\delta}(u) \leq E_{B_1 \setminus \overline{B}_\delta}(u + \xi) \), for all radial functions \( \xi \in C^1_c(B_1 \setminus \overline{B}_\delta) \) satisfying \( \|\xi\|_{C^1} \leq \varepsilon_\delta \)), then \( u \) is a semi-stable solution of \( (1.1) \). Other general situations include semi-stable solutions: for instance, minimal solutions, extremal solutions, and also some solutions between a sub- and a supersolution (see [3] Remark 1.7 for more details). All the results obtained in this paper were obtained by the second author in [16] for the Laplace operator \((p = 2)\).

**Theorem 1.1** ([3]). Let \( g \) be a locally Lipschitz function and \( u \in W^{1,p}(B_1) \) be a semi-stable radial solution in \( B_1 \setminus \{0\} \) of \( (1.1) \) satisfying \( u_r(r) < 0 \) for all \( r \in (0, 1) \). Then:

- **a)** If \( N < p + 4p/(p - 1) \), then \( u \in L^\infty(B_1) \). Moreover,
  \[
  \|u\|_{L^\infty(B_1)} \leq C_{N,p} \|u\|_{W^{1,p}(B_1)},
  \]
  where \( C_{N,p} \) is a constant depending only on \( N \) and \( p \).
- **b)** If \( N = p + 4p/(p - 1) \), then \( u \in L^q(B_1) \) for all \( q < +\infty \). Moreover,
  \[
  |u(r)| \leq C_p \|u\|_{W^{1,p}(B_1)} (|\log r| + 1) \text{ in } B_1,
  \]
  where \( C_p \) is a constant depending only on \( p \).
- **c)** If \( N > p + 4p/(p - 1) \) and \( q < q_0 \), then \( u \in L^q(B_1) \) and
  \[
  \|u\|_{L^q(B_1)} \leq C_{N,p,q} \|u\|_{W^{1,p}(B_1)},
  \]
  where \( C_{N,p,q} \) is a constant depending only on \( N \), \( p \), and \( q \). Moreover,
  \[
  |u(r)| \leq C_{N,p} r^{-1/2} \left(N - 2 \sqrt{\frac{N - 1}{p - 1}} - 2\right) \left(|\log r|^{1/2} + 1\right) \text{ in } B_1,
  \]
  where \( C_{N,p} \) is a constant depending only on \( N \) and \( p \).
- **d)** Assume that \( g \) is nonnegative. Then:
  \[
  \begin{align*}
  \text{d1)} & \quad \text{We have}
  \\
  \|\nabla u\|_{L^p(B_1)} & \leq C_{N,p} \left\{ \|\left(u - u(1)\right)^{p-1}\|_{L^{1/p}(B_1)}^{1/p} + \|g(u)\|_{L^{1/p}(B_1)}^{1/p} \right\},
  \end{align*}
  \]
  for some constant \( C_{N,p} \) depending only on \( N \) and \( p \).
  \[
  \text{d2)} & \quad u \in W^{1,q}(B_1) \text{ for every } q < q_1, \text{ and}
  \\
  \|u\|_{W^{1,q}(B_1)} & \leq C \text{ if } q < q_1,
  \]
  where \( C \) is a constant depending only on \( N \), \( p \), \( q \), and on upper bounds for \( \|u\|_{L^1(B_1)} \) and \( g \).
  \[
  \text{d3)} & \quad \text{If } N \geq p + 4p/(p - 1), \text{ then}
  \\
  |u_r(r)| & \leq C_{N,p} \|u\|_{W^{1,p}(B_1)} r^{-1/2} \left(N - 2 \sqrt{\frac{N - 1}{p - 1}} - 2\right) |\log r|^{1/2} \text{ in } B_{1/4},
  \]
  where \( C_{N,p} \) is a constant depending only on \( N \) and \( p \).

The definition of \( q_k \) for \( k = 0, 1 \) is given by

\[
\begin{align*}
\frac{1}{q_k} := & \frac{1}{p} - \frac{2}{Np} \sqrt{\frac{N - 1}{p - 1}} + \frac{k-1}{N} - \frac{2}{Np} \quad \text{for } N \geq p + 4p/(p - 1), \\
q_k := & +\infty \quad \text{for } N < p + 4p/(p - 1).
\end{align*}
\]
In this paper we establish sharp pointwise estimates for semi-stable radially symmetric and decreasing solutions \(u \in W^{1,p}(B_1)\) of (1.1) and its derivatives (up to order three). We improve the above theorem, answering affirmatively an open question raised in [3] about the removal of the factor \(|\log r|^{\frac{1}{p}}\).

**Theorem 1.2.** Let \(N \geq p > 1\), \(g : \mathbb{R} \to \mathbb{R}\) be a locally Lipschitz function, and \(u \in W^{1,p}(B_1)\) be a semi-stable radial solution of (1.1) satisfying \(u_r(r) < 0\) for all \(r \in (0,1)\). Then there exists a constant \(C_{N,p}\) depending only on \(N\) and \(p\) such that:

i) If \(p \leq N < p + 4p/(p-1)\), then
\[
|u(r)| \leq C_{N,p} \|u\|_{W^{1,p}(B_1 \setminus \overline{B_{1/2}})} , \forall r \in (0,1].
\]

ii) If \(N = p + 4p/(p-1)\), then
\[
|u(r)| \leq C_{p} \|u\|_{W^{1,p}(B_1 \setminus \overline{B_{1/2}})} (|\log r| + 1) , \forall r \in (0,1].
\]

iii) If \(N > p + 4p/(p-1)\), then
\[
|u(r)| \leq C_{N,p} \|u\|_{W^{1,p}(B_1 \setminus \overline{B_{1/2}})} r^{-\frac{1}{p}}(N-2\sqrt{\frac{N-1}{p-1}}-p-2) , \forall r \in (0,1].
\]

**Remark 1.3.** For \(N < p\), since \(u \in W^{1,p}(B_1)\), the Sobolev embedding leads to \(u \in L^\infty(B_1)\) and \(\|u\|_{L^\infty(B_1)} \leq C_{N,p} \|u\|_{W^{1,p}(B_1)}\).

**Theorem 1.4.** Let \(N \geq p + 4p/(p-1)\), \(g : \mathbb{R} \to \mathbb{R}\) be a locally Lipschitz function, and \(u \in W^{1,p}(B_1)\) be a semi-stable radial solution of (1.1) satisfying \(u_{rr}(r) < 0\) for all \(r \in (0,1)\). Then there exists a constant \(C'_{N,p}\) depending only on \(N\) and \(p\) such that:

i) If \(g \geq 0\), then
\[
|u_{rr}(r)| \leq C'_{N,p} \|\nabla u\|_{L^p(B_1 \setminus \overline{B_{1/2}})} r^{-\frac{1}{p}}(N-2\sqrt{\frac{N-1}{p-1}}-2) , \forall r \in (0,1/2].
\]

ii) If \(g \geq 0\) is nondecreasing, then
\[
|u_{rrr}(r)| \leq C'_{N,p} \|\nabla u\|_{L^p(B_1 \setminus \overline{B_{1/2}})} r^{-\frac{1}{p}}(N-2\sqrt{\frac{N-1}{p-1}}+p-2) , \forall r \in (0,1/2].
\]

iii) If \(g \geq 0\) is nondecreasing and convex, then
\[
|u_{r(rr)}(r)| \leq C'_{N,p} \|\nabla u\|_{L^p(B_1 \setminus \overline{B_{1/2}})} r^{-\frac{1}{p}}(N-2\sqrt{\frac{N-1}{p-1}}+2p-2) , \forall r \in (0,1/2].
\]

**Remark 1.5.** Observe that the estimates obtained in Theorems 1.2 and 1.4 are stated in terms of the \(W^{1,p}\) norm of the annulus \(B_1 \setminus \overline{B_{1/2}}\), while \(u\) is required to belong to \(W^{1,p}(B_1)\). This requirement is essential to obtain our results, since we can easily find semi-stable radially decreasing solutions of (1.1) (for instance \(u(r) = r^s\), with \(s \ll 0\)), not in the energy class \(W^{1,p}(B_1)\), which the statements of Theorems 1.2 and 1.4 fail to satisfy.

**Remark 1.6.** To our knowledge there are no estimates of \(|u_{rr}|\) or \(|u_{rrr}|\) in the literature for these kinds of solutions.
As an application of some general results obtained in this paper for this class of solutions (for arbitrary \( g \in C^1(\mathbb{R}) \)), we consider the following problem:

\[
\begin{align*}
-\Delta_p u &= \lambda f(u) \quad \text{in } B_1, \\
 u &= 0 \quad \text{in } B_1,
\end{align*}
\]

(1.3\( \lambda, p \))

where \( \lambda > 0 \) and \( f \) is an increasing \( C^1 \) function with \( f(0) > 0 \) and

\[
\lim_{t \to +\infty} \frac{f(t)}{t^{p-1}} = +\infty.
\]

(1.4)

This problem is studied by Cabré and Sanchón in [4] for general smooth bounded domains \( \Omega \) of \( \mathbb{R}^N \). It is proved that there exists a positive parameter \( \lambda^* \) such that if \( \lambda \in (0, \lambda^*) \), then (1.3\( \lambda, p \)) admits a minimal (smallest) solution \( u_\lambda \in C^1(\overline{\Omega}) \) and if \( \lambda \in (\lambda^*, +\infty) \), then (1.3\( \lambda, p \)) admits no regular solution. In addition, for \( \lambda \in (0, \lambda^*) \) the minimal solution \( u_\lambda \) is semi-stable (in a similar sense of the definition when \( \Omega = B_1 \)). On the other hand, we may consider the increasing limit

\[
u^* := \lim_{\lambda \to \lambda^*} u_\lambda.
\]

In the case \( p = 2 \) it is well known that \( u^* \) is a weak solution of (1.3\( \lambda, p \)) for \( \lambda = \lambda^* \). It is called the extremal solution. For general \( p, \Omega \) and \( f \), it is not known if \( u^* \) is a weak solution of (1.3\( \lambda, p \)) for \( \lambda = \lambda^* \). In the case \( \Omega = B_1 \), Cabré, Capella and Sanchón [3] proved that \( u^* \) is actually a semi-stable radially decreasing energy solution (i.e., \( u^* \in W^{1,p}_0 \)) of (1.3\( \lambda, p \)). Hence we can apply to the extremal solution the results obtained in this paper for these kinds of solutions.

We refer to [2,5] for surveys on minimal and extremal solutions and to [1,6,10,12,15,17] for other interesting results in the topic of extremal solutions.

**Theorem 1.7.** Let \( N \geq p > 1 \). Suppose that \( f \) satisfies (1.3). Let \( u^* \) be the extremal solution of (1.3\( \lambda, p \)). We have that

i) If \( p \leq N < p + 4p/(p - 1) \), then \( u^*(r) \leq C(1 - r) \), \( \forall r \in (0, 1] \).

ii) If \( N = p + 4p/(p - 1) \), then \( u^*(r) \leq C \log r \), \( \forall r \in (0, 1] \).

iii) If \( N > p + 4p/(p - 1) \), then

\[
u^*(r) \leq C \left( r^{-\frac{1}{p}} (N - 2\sqrt{\frac{N - 1}{p - 1}} - p - 2) \right) \quad \forall r \in (0, 1],
\]

iv) If \( N \geq p + 4p/(p - 1) \), then

\[
\left| \partial_r^{(k)} u^*(r) \right| \leq Cr^{-\frac{1}{p}} \left( N - 2\sqrt{\frac{N - 1}{p - 1}} + (k - 1)p - 2 \right) \quad \forall r \in (0, 1], \forall k \in \{1, 2\},
\]

v) If \( N \geq p + 4p/(p - 1) \) and \( f \) is convex, then

\[
|u^*_{rr}(r)| \leq C r^{-\frac{1}{p}} \left( N - 2\sqrt{\frac{N - 1}{p - 1}} + 2p - 2 \right) \quad \forall r \in (0, 1],
\]

where \( C = C_{N, p} \min_{t \in [1/2, 1]} |u^*(t)| \), and \( C_{N, p} \) is a constant depending only on \( N \) and \( p \).

**Remark 1.8.** In [11] García-Azorero, Peral and Puel proved that if \( f(u) = e^u \) and \( N = p + 4p/(p - 1) \), then

\[
u^*(r) = -p \log r \quad \text{and} \quad \lambda^* = p^{p-1}(N - p).
\]

This shows that the pointwise estimates of Theorem 1.7 are optimal for \( N = p + 4p/(p - 1) \).
On the other hand, in [3] Cabrè and Sanchón proved that if \( N > p + 4p/(p-1) \) and \( f(u) = (1+u)^m \), where
\[
m := \frac{(p-1)N - 2\sqrt{(p-1)(N-1)} - p + 2}{N - 2\sqrt{\frac{N-1}{p-1}} - p - 2},
\]
then
\[
u^*(r) = r^{-\frac{1}{p}}(N - 2\sqrt{\frac{N-1}{p-1}} - p - 2) - 1,
\]
and
\[\lambda^* = \left(\frac{p}{m - (p-1)}\right)^{p-1}\left(N - \frac{mp}{m - (p-1)}\right).
\]
This also shows the optimality of the pointwise estimates of Theorem 1.7 for the case \( N \geq p + 4p/(p-1) \).

2. Proof of the main results

**Lemma 2.1.** Let \( N \geq p > 1 \), \( g : \mathbb{R} \rightarrow \mathbb{R} \) be a locally Lipschitz function, and \( u \in W^{1,p}(B_1) \) be a semi-stable radial solution of (1.1) satisfying \( u_r(r) < 0 \) for all \( r \in (0,1) \). Then there exists a constant \( K_{N,p} \) depending only on \( N \) and \( p \) such that
\[
\int_0^r |u_r(t)|^p t^{N-1} dt \leq K_{N,p} \left\| \nabla u \right\|_{L^p(B_1 \setminus B_{1/2})}^p r^{2\sqrt{\frac{N-1}{p-1}+2}}, \forall r \in [0,1].
\]

**Proof.** Let us use [3, Lem. 2.2] (see also the proof of [3, Lem. 2.3]) to assure that
\[
(N-1) \int_{B_1} |u_r|^p \eta^2 dx \leq (p-1) \int_{B_1} |u_r|^p |\nabla (|x| \eta)|^2 dx,
\]
for every radial Lipschitz function \( \eta \) vanishing on \( \partial B_1 \).

We now fix \( r \in (0,1/2) \) and consider the function
\[
\eta_\epsilon(t) = \begin{cases} 
  r^{-\frac{N-1}{p-1}} & \text{if } 0 \leq t \leq \epsilon, \\
  r^{-\frac{N-1}{p-1}} & \text{if } \epsilon < t \leq r, \\
  t^{-\frac{N-1}{p-1}-1} & \text{if } r < t \leq 1/2, \\
  2^{\frac{N-1}{p-1}+2}(1-t) & \text{if } 1/2 < t \leq 1.
\end{cases}
\]

Inequality (2.1) shows that
\[
(N-p) \left(\frac{r^{-\frac{N-1}{p-1}}}{\epsilon}\right)^2 \int_0^\epsilon |u_r(t)|^p t^{N-1} dt + (N-1)r^{-2\sqrt{\frac{N-1}{p-1}+2}} \int_\epsilon^r \left(\frac{r}{t}\right)^2 |u_r(t)|^p t^{N-1} dt \\
+ 2^{2\sqrt{\frac{N-1}{p-1}+4}} \int_{1/2}^1 ((N-1)(1-t)^2 - (p-1)(1-2t)^2) |u_r(t)|^p t^{N-1} dt \leq 0.
\]
Since $N \geq p$ and $r/t \geq 1$ for $0 < t \leq r$, letting $\epsilon \to 0$, we obtain
\[
\int_0^r |u_r(t)|^p t^{N-1} \, dt \leq \left( \frac{(p-1)2^\frac{N-1}{p-1}+4}{N-1} \right) r^{2\sqrt{\frac{N-1}{p-1}+2}} \int_{1/2}^1 |u_r(t)|^p t^{N-1} \, dt,
\]
and the lemma is proved for $0 < r \leq 1/2$.

If $r \in (1/2, 1]$, then, applying the above inequality for $r = 1/2$, we obtain
\[
\int_0^r |u_r(t)|^p t^{N-1} \, dt \leq \int_0^{1/2} |u_r(t)|^p t^{N-1} \, dt + \int_{1/2}^1 |u_r(t)|^p t^{N-1} \, dt
\]
\[
\leq \left[ \left( \frac{(p-1)2^\frac{N-1}{p-1}+4}{N-1} \right) \left( \frac{1}{2} \right)^{2\sqrt{\frac{N-1}{p-1}+2}} + 1 \right] \int_{1/2}^1 |u_r(t)|^p t^{N-1} \, dt
\]
\[
\leq (2r)^{2\sqrt{\frac{N-1}{p-1}+2}} \left( \frac{4(p-1)}{N-1} + 1 \right) \int_{1/2}^1 |u_r(t)|^p t^{N-1} \, dt,
\]
and the proof is completed for $1/2 < r \leq 1$.

\[\square\]

**Proposition 2.2.** Let $N \geq p > 1$, $g : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function, and $u \in W^{1,p}(B_1)$ be a semi-stable radial solution of \(\text{(1.1)}\) satisfying $u_r(r) < 0$ for all $r \in (0,1)$. Then there exists a constant $K_{N,p}'$ depending only on $N$ and $p$ such that
\[
(2.2) \quad |u(r) - u \left( \frac{r}{2} \right)| \leq K_{N,p}' \|\nabla u\|_{L^p(B_1 \setminus B_{1/2})} r^{-\frac{1}{p}} \left( N - 2 \sqrt{\frac{N-1}{p-1} - p - 2} \right), \quad \forall r \in (0,1].
\]

**Proof.** Fix $r \in (0,1]$. Applying Hölder’s inequality and Lemma 2.1 we deduce
\[
\left| u(r) - u \left( \frac{r}{2} \right) \right| = \int_{r/2}^r |u_r(t)| t^{\frac{N-1}{p}} t^{-\frac{N-1}{p}} \, dt
\]
\[
\leq \left( \int_{r/2}^r |u_r(t)|^p t^{N-1} \, dt \right)^{\frac{1}{p}} \left( \int_{r/2}^r t^{-\frac{N-1}{p-1}} \, dt \right)^{\frac{p-1}{p}}
\]
\[
\leq K_{N,p}^2 \|\nabla u\|_{L^p(B_1 \setminus B_{1/2})} r^{\frac{2}{p}} \sqrt{\frac{N-1}{p-1} + 2} \left( r^{-\frac{N-1-p}{p-1}} \int_{1/2}^1 t^{-\frac{N-1}{p-1}} \, dt \right)^{\frac{p-1}{p}},
\]
and (2.2) is proved with $K_{N,p}' = K_{N,p}^2 \left( \int_{1/2}^1 t^{-\frac{N-1}{p-1}} \, dt \right)^{\frac{p-1}{p}}$. \(\square\)

**Proof of Theorem 1.2.** Let $0 < r \leq 1$. Then, there exist $m \in \mathbb{N}$ and $1/2 < r_1 \leq 1$ such that $r = r_1/2^{m-1}$. Since $u$ is radial we have $|u(r_1)| \leq \|u\|_{L^\infty(B_1 \setminus B_{1/2})} \leq \gamma_{N,p} \|u\|_{W^{1,p}(B_1 \setminus B_{1/2})}$, where $\gamma_{N,p}$ depends only on $N$ and $p$. From this and Proposition 2.2 it follows that
\[
|u(r)| \leq |u(r) - u(r_1)| + |u(r_1)| = \sum_{i=1}^{m-1} \left| u \left( \frac{r_1}{2^{i-1}} \right) - u \left( \frac{r_1}{2^i} \right) \right| + |u(r_1)|
\]
\[
\leq \left( K_{N,p}' \sum_{i=1}^{m-1} \left( \frac{r_1}{2^i} \right)^{-\frac{1}{p}} \left( N - 2 \sqrt{\frac{N-1}{p-1} - p - 2} \right) + \gamma_{N,p} \right) \|u\|_{W^{1,p}(B_1 \setminus B_{1/2})}.
\]
Let $\delta_{N,p} = -\frac{1}{p} \left( N - 2 \sqrt{\frac{N-1}{p-1}} - p - 2 \right)$. We have

\begin{equation}
\sum_{i=1}^{m-1} \left( \frac{r_1}{2i-1} \right)^{\delta_{N,p}} \leq \alpha_{N,p} \begin{cases} \quad r^{\delta_{N,p}} & \text{if } \delta_{N,p} < 0, \\ \quad 1 & \text{if } \delta_{N,p} > 0, \\ \quad |\log r| & \text{if } \delta_{N,p} = 0, \end{cases}
\end{equation}

where $\alpha_{N,p}$ is a constant depending only on $N$ and $p$.

Then:

- If $p \leq N < p + 4p/(p-1)$, then $\delta_{N,p} > 0$. From (2.3) and (2.4) we obtain

  \[ |u(r)| \leq \left( K_{N,p} \alpha_{N,p} + \gamma_{N,p} \right) \|u\|_{W^{1,p}(B_1 \setminus \overline{B_{1/2}})}. \]

- If $N = p + 4p/(p-1)$, then $\delta_{N,p} = 0$. From (2.3) and (2.4) we obtain

  \[ |u(r)| \leq \left( K_{N,p} \alpha_{N,p} |\log r| + \gamma_{N,p} \right) \|u\|_{W^{1,p}(B_1 \setminus \overline{B_{1/2}})} \leq \left( K_{N,p} \alpha_{N,p} + \gamma_{N,p} \right) \|u\|_{W^{1,p}(B_1 \setminus \overline{B_{1/2}})} (|\log r| + 1). \]

- If $N > p + 4p/(p-1)$, we have $\delta_{N,p} < 0$ and $r^{\delta_{N,p}} \geq 1$. From (2.3) and (2.4) we obtain

  \[ |u(r)| \leq \left( K_{N,p} \alpha_{N,p} + \gamma_{N,p} \right) r^{\delta_{N,p}} \|u\|_{W^{1,p}(B_1 \setminus \overline{B_{1/2}})}, \]

which completes the proof.

\[ \square \]

**Lemma 2.3.** Let $N \geq 1$, $p > 1$, $g : \mathbb{R} \to \mathbb{R}$ be a nonnegative and nondecreasing locally Lipschitz function, and $u \in W^{1,p}(B_1)$ be a semi-stable radial solution of (1.1) such that $u_r < 0$ for all $r \in (0,1)$. Then

\begin{equation}
g(u(r)) \leq N \frac{|u_r(r)|^{p-1}}{r}, \quad \forall r \in (0,1].
\end{equation}

Moreover, if $g$ is convex, then

\begin{equation}
g'(u(r)) \leq M_{N,p} \frac{|u_r(r)|^{p-2}}{r^2}, \quad \forall r \in (0,1],
\end{equation}

where $M_{N,p}$ is a constant depending only on $N$ and $p$.

**Proof.** Consider the function

\[ \Psi(r) := N r^{1-1/N} \left| u_r \right|^{1/N} \left( r^{1/N} \right)^{p-1}, \quad r \in (0,1]. \]

It is easy to check that $\Psi'(r) = g \left( u \left( r^{1/N} \right) \right)$, $r \in (0,1]$. As $g$ is nonnegative and nondecreasing we have that $\Psi$ is a nonnegative nondecreasing concave function. It follows immediately that

\[ 0 \leq \Psi'(r) \leq \Psi(r)/r, \quad r \in (0,1], \]

and we obtain (2.5).

To obtain ii), we first observe that from (1.1) we obtain

\[ u_{rr} = - \frac{1}{p-1} \left( \frac{g(u)}{|u_r|^p} + \frac{N-1}{r} u_r \right), \quad \forall r \in (0,1]. \]
Therefore, using the nonnegativeness of $g$ and (2.5) we deduce that

\[(2.7) \quad |u_{rr}| \leq \frac{1}{p-1} \left( \frac{g(u)}{|u_r|^{p-2}} + \frac{N-1}{r} |u_r| \right) \leq \left( \frac{2N-1}{p-1} \right) \frac{|u_r|}{r}, \quad \forall r \in (0, 1].\]

For fixed $\alpha \in \mathbb{R}$ an easy computation shows that

\[
\partial_r \left( r^\alpha |u_r|^{p-2} \right) = r^\alpha |u_r|^{p-2} - (p-2)r^{\alpha-1}u_r |u_r|^{p-3}
\geq r^\alpha |u_r|^{p-2} \left( \alpha - \frac{|p-2| \left(2N-1\right)}{p-1} \right), \quad \forall r \in (0, 1].
\]

Thus $r^\alpha |u_r|^{p-2}$ is nondecreasing for $\alpha = \frac{|p-2| \left(2N-1\right)}{p-1}$. Using this, the monotocity of $g'(u(r))$ and the semi-stability of $u$, we deduce that

\[
g'(u(r)) \int_0^r s^{N-1} \xi(s)^2 ds \leq \int_0^r s^{N-1} g'(u(s)) \xi(s)^2 ds
\leq (p-1) \int_0^r |u_r(s)|^{p-2} s^\alpha s^{N-1-\alpha} \xi(s)^2 ds
\leq (p-1) |u_r(r)|^{p-2} r^\alpha \int_0^r s^{N-1-\alpha} \xi'(s)^2 ds,
\]

for every $r \in (0, 1)$ and every $\xi \in C^1$ with compact support in $(0, r)$.

Taking $\xi(s) = \zeta(s)$ for $s \in [0, r]$, where $\zeta \in C^1$ is any function with compact support in $(0, 1)$, we obtain (2.6).  \(\square\)

**Proof of Theorem 1.4**

i) We first observe that $\partial_r \left( r^{N-1} |u_r|^{p-1} \right) = r^{N-1} g(u)$. Hence $r^{N-1} |u_r|^{p-1}$ is a positive nondecreasing function and so is $\left( r^{N-1} |u_r|^{p-1} \right)^{\frac{p-1}{p}}$. Thus, for $0 < r \leq 1/2$, we have

\[
\int_0^{2r} |u_r(t)|^p t^{N-1} dt \geq \int_r^{2r} |u_r(t)|^p t^{N-1} dt
\geq r^{\frac{p(N-1)}{p-1}} |u_r(r)|^p \int_r^{2r} t^{-\frac{N-1}{p-1}} dt
\geq r^{N} |u_r(r)|^p \int_1^{2} t^{-\frac{N-1}{p-1}} dt.
\]

From this and Lemma [2.1] we obtain i).

ii) From (2.7) and i) we obtain ii).

iii) From (1.1) we obtain

\[
u_{rrr} = -\frac{1}{p-1} \left( \frac{g'(u)u_r}{|u_r|^{p-2}} - (p-2) \frac{u_r u_{rr} g(u)}{|u_r|^p} - \frac{N-1}{p^2} u_r + \frac{N-1}{r} u_{rrr} \right),
\]
for every \( r \in (0, 1) \). Therefore from (2.5), (2.6) and (2.7), we obtain

\[
|u_{rrr}| \leq \frac{1}{p-1} \left( \frac{g'(u) |u_r|}{|u_r|^{p-2}} + |p-2| \frac{|u_r| |u_{rrr}| g(u)}{|u_r|^p} \right.
\]
\[
+ \frac{N-1}{r^2} |u_r| + \frac{N-1}{r} |u_{rrr}| \right)
\]  
\[
\leq \frac{1}{p-1} \left( M_{N,p} + \frac{|p-2| N(2N-1)}{p-1} \right. 
\]
\[
+ (N-1) + \frac{(N-1)(2N-1)}{p-1} \right) \frac{|u_r|}{r^2}, \forall r \in (0, 1].
\]

Then iii) follows from i).

**Lemma 2.4.** Let \( N \geq 1, \ p > 1, \ g : \mathbb{R} \rightarrow \mathbb{R} \) be a locally Lipschitz nonnegative and nondecreasing function and \( u \) be a radial solution of (1.1) satisfying \( u_r(r) < 0 \) for all \( r \in (0, 1) \). Then:

i) \( r^{N-1} |u_r|^{p-1} \) is nondecreasing for \( r \in (0, 1] \).

ii) \( r^{-1} |u_r|^{p-1} \) is nonincreasing for \( r \in (0, 1] \).

iii) \( \max_{t \in [1/2, 1]} |u_r(t)| \leq 2^\frac{N}{p-1} \min_{t \in [1/2, 1]} |u_r(t)| \).

iv) \( \|\nabla u\|_{L^p(B_1 \setminus B_{1/2})} \leq q_{N,p} \min_{t \in [1/2, 1]} |u_r(t)| \) for a certain constant \( q_{N,p} \) depending only on \( N \) and \( p \).

**Proof.**

i) Since \( u_r < 0 \) we have \( \partial_r \left( r^{N-1} |u_r|^{p-1} \right) = r^{N-1} g(u) \geq 0 \).

ii) From (2.5) of Lemma 2.3, we have that

\[ N r^{N-2} |u_r|^{p-1} \geq \partial_r \left( r^{N-1} |u_r|^{p-1} \right) = N r^{N-2} |u_r|^{p-1} + r^N \partial_r \left( r^{-1} |u_r|^{p-1} \right), \]

and ii) follows immediately.

iii) Take \( r_1, r_2 \in [1/2, 1] \) such that \( |u_r(r_1)| = \min_{t \in [1/2, 1]} |u_r(t)| \) and \( |u_r(r_2)| = \max_{t \in [1/2, 1]} |u_r(t)| \).

- If \( r_2 \leq r_1 \), we deduce from i) that \( |u_r(r_2)|^{p-1} \leq (r_1/r_2)^{N-1} |u_r(r_1)|^{p-1} \leq 2^N |u_r(r_1)|^{p-1} .

- If \( r_2 > r_1 \), we deduce from ii) that \( |u_r(r_2)|^{p-1} \leq (r_2/r_1) |u_r(r_1)|^{p-1} \leq 2 |u_r(r_1)|^{p-1} \leq 2^N |u_r(r_1)|^{p-1} .

iv) We see at once that

\[ \|\nabla u\|_{L^p(B_1 \setminus B_{1/2})} \leq |B_1 \setminus B_{1/2}|^{1/p} \max_{t \in [1/2, 1]} |u_r(t)|, \]

and iv) follows from iii).

**Proof of Theorem 1.7** As we have mentioned, it is well known that \( u^* \) is a semistable radially decreasing \( W^{1,p} (B_1) \) solution of (1.1) for \( g(s) = \lambda^* f(s) \). Hence, we can apply to \( u^* \) the results obtained in Lemma 2.4 and Lemma 2.5.
Let $0 < s \leq 1$. From statement ii) of Lemma 2.4 and applying Hölder’s inequality, we deduce that

$$
|u^*_s(s)|^{p-1} \leq 2 \int_{s/2}^{s} t^{-1} |u^*_r(t)|^{p-1} dt = 2 \int_{s/2}^{s} t^{-(p-1)/(N-1)} |u^*_r(t)|^{p-1} t^{\frac{N-pN-1}{p}} dt
$$

$$
\leq 2 \left( \int_{s/2}^{s} t^{N-1} |u^*_r(t)|^p dt \right)^{\frac{p-1}{p}} \left( \int_{s/2}^{s} t^{N-pN-1} dt \right)^{\frac{1}{p}}.
$$

From this, Lemma 2.1 and statement iv) of Lemma 2.4, we have

$$
(2.9) \quad |u^*_s(s)| \leq C_{N,p} \min_{t \in [1/2,1]} |u^*_r(t)| \int_{r}^{1} s^{-\frac{1}{p}} (N-2\sqrt{\frac{N-1}{p-1}} - 2) ds,
$$

where $C_{N,p}$ depends only on $N$ and $p$.

Then as $u^*(1) = 0$, we use (2.9) to obtain

$$
(2.10) \quad |u^*(r)| = \int_{r}^{1} |u^*_r(s)| ds \leq C_{N,p} \min_{t \in [1/2,1]} |u^*_r(t)| \int_{r}^{1} s^{-\frac{1}{p}} (N-2\sqrt{\frac{N-1}{p-1}} - 2) ds,
$$

for all $0 < r \leq 1$. Next, we estimate the integral $\int_{r}^{1} s^{-\frac{1}{p}} (N-2\sqrt{\frac{N-1}{p-1}} - 2) ds$.

- If $p \leq N < p + 4p/(p-1)$, we have $-\frac{1}{p} \left( N - 2\sqrt{\frac{N-1}{p-1}} - 2 \right) > -1$. Then

$$
\int_{r}^{1} s^{-\frac{1}{p}} (N-2\sqrt{\frac{N-1}{p-1}} - 2) ds = \frac{1}{1 - \frac{1}{p} \left( N - 2\sqrt{\frac{N-1}{p-1}} - 2 \right)} (1 - r).
$$

- If $N = p + 4p/(p-1)$, we have $-\frac{1}{p} \left( N - 2\sqrt{\frac{N-1}{p-1}} - 2 \right) = -1$. Then

$$
\int_{r}^{1} s^{-\frac{1}{p}} (N-2\sqrt{\frac{N-1}{p-1}} - 2) ds = - \log r.
$$

- If $N > p + 4p/(p-1)$, we have $-\frac{1}{p} \left( N - 2\sqrt{\frac{N-1}{p-1}} - 2 \right) < -1$. Then

$$
\int_{r}^{1} s^{-\frac{1}{p}} (N-2\sqrt{\frac{N-1}{p-1}} - 2) ds = \frac{r^{-\frac{1}{p}} \left( N - 2\sqrt{\frac{N-1}{p-1}} - 2 \right) - 1}{\frac{1}{p} \left( N - 2\sqrt{\frac{N-1}{p-1}} - 2 \right)}.
$$

From this and (2.10), we conclude i), ii), and iii).

Finally, the proof of iv) and v) follows from (2.7), (2.8), and (2.9). $\square$

References


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