ON A COUNTEREXAMPLE RELATED TO WEIGHTED WEAK TYPE ESTIMATES FOR SINGULAR INTEGRALS

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Abstract. We show that the Hilbert transform does not map $L^1(M_{\Phi} w)$ to $L^{1,\infty}(w)$ for every Young function $\Phi$ growing more slowly than $t \log \log (e^e + t)$. Our proof is based on a construction of M.C. Reguera and C. Thiele.

1. Introduction

Let $H$ be the Hilbert transform. One of the open questions in the one-weighted theory of singular integrals is about the optimal Young function $\Phi$ for which the weak type inequality

\[(1.1) \quad w\{x \in \mathbb{R} : |Hf(x)| > \lambda\} \leq \frac{c}{\lambda} \int_{\mathbb{R}} |f| M_{\Phi} w \, dx \quad (\lambda > 0)\]

holds for every weight (i.e., non-negative measurable function) $w$ and any $f \in L^1(M_{\Phi} w)$, where $M_{\Phi}$ is the Orlicz maximal operator defined by

\[M_{\Phi} f(x) = \sup_{I \ni x} \inf_{\lambda > 0} \left\{ \lambda > 0 : \frac{1}{|I|} \int_I \Phi \left( \frac{|f(y)|}{\lambda} \right) dy \leq 1 \right\}.\]

If $\Phi(t) = t$, then $M_{\Phi} = M$ is the standard Hardy-Littlewood maximal operator. If $\Phi(t) = t^r, r > 1$, denote $M_{\Phi} f = M_r f$.

C. Fefferman and E.M. Stein [6] showed that if $H$ is replaced by the maximal operator $M$, then the corresponding inequality holds with $\Phi(t) = t$. Next, A. Córdoba and C. Fefferman [1] proved (1.1) with $\Phi(t) = t^r, r > 1$. This result was improved by C. Pérez [8], who showed that (1.1) holds with $\Phi(t) = t \log^\varepsilon (e^e + t), \varepsilon > 0$ (see also [7] for a different proof of this result).

Very recently, C. Domingo-Salazar, M.T. Lacey and G. Rey [5] obtained a further improvement; their result states that (1.1) holds whenever $\Phi$ satisfies

\[\int_1^\infty \frac{\Phi^{-1}(t)}{t^2 \log (e + t)} \, dt < \infty.\]

For example, one can take $\Phi(t) = t \log \log^\alpha (e^e + t), \alpha > 1$, or

\[\Phi(t) = t \log \log (e^e + t) \log \log \log^\alpha (e^e + t) \quad (\alpha > 1),\]

etc.
A question whether (1.1) is true with $\Phi(t) = t$ has become known as the Muckenhoupt-Wheeden conjecture. This conjecture was disproved by M.C. Reguera and C. Thiele [10] (see also [9] and [2] for dyadic and multidimensional versions of this result).

Denote $\Psi(t) = t \log \log (e^e + t)$. It was conjectured in [7] that (1.1) holds with $\Phi = \Psi$. The above-mentioned result in [5] establishes (1.1) for essentially every $\Phi$ growing faster than $\Psi$.

The main result of this note is the observation that the Reguera-Thiele example [10] actually shows that (1.1) does not hold for every $\Phi$ growing more slowly than $\Psi$.

**Theorem 1.1.** Let $\Phi$ be a Young function such that

$$\lim_{t \to \infty} \frac{\Phi(t)}{t \log \log (e^e + t)} = 0.$$ 

Then for every $c > 0$, there exist $f, w$ and $\lambda > 0$ such that

$$w \{ x \in \mathbb{R} : |Hf(x)| > \lambda \} > \frac{c}{\lambda} \int_{\mathbb{R}} |f| \mathcal{M}_{\Phi} w \, dx.$$ 

This theorem along with the main result in [5] emphasizes that the case of $\Phi = \Psi$ is critical for (1.1). However, the question whether (1.1) holds with $\Phi = \Psi$ remains open.

We mention briefly the main ideas of the Reguera-Thiele example [10] and, in parallel, our novel points. First, it was shown in [10] that given $k \in \mathbb{N}$ sufficiently large, there is a weight $w_k$ supported on $[0, 1]$ satisfying $H w_k \geq c w_k$ and $M w_k \leq c w_k$ on some subset $E \subset [0, 1]$. In Section 2, we show that the latter “$A_1$ property” can be slightly improved until $M_r w_k \leq c w_k$ with $r > 1$ depending on $k$. The second ingredient in [10] was the extrapolation argument of D. Cruz-Uribe and C. Pérez [3]. This argument says that assuming (1.1) with $M w$ on the right-hand side, one can deduce a certain weighted $L^2$ inequality for $H$. It is not clear how to extrapolate in a similar way starting with a general Orlicz maximal function $M_{\Phi}$ in (1.1). In Section 3, we obtain a substitute of the argument in [3] for $M_r w, r > 1$, instead of $M w$.

2. The Reguera-Thiele construction

We describe below the main parts of the example constructed by M.C. Reguera and C. Thiele [10].

An interval $I$ of the form $[3^j n, 3^j (n + 1)), j, n \in \mathbb{Z}$, is called a triadic interval.

Fix $k \in \mathbb{N}$ large enough. Given a triadic interval $I_0 \subset [0, 1)$, denote $I^\Delta = \frac{1}{3} I$; namely, $I^\Delta$ is the interval with the same center as $I$ and one third its length. Further, denote by $P(I)$ a triadic interval adjacent to $I^\Delta$ and such that $|P(I)| = \frac{1}{3} |I|$. Observe that $P(I)$ can be situated either on the left or on the right of $I^\Delta$; we will return to this point a bit later.

Now set $J^1 = [0, 1)$ and $I_{1,1} = P(J^1)$. Next, we subdivide $(J^1)^\Delta$ into $3^{k-1}$ triadic intervals of equal length and denote them by $J^2_m, m = 1, 2, \ldots, 3^{k-1}$. Set correspondingly $I_{2,m} = P(J^2_m)$. Notice that $|J^2_m| = \frac{1}{3^2}$ and $|I_{2,m}| = \frac{1}{3^2}$ for $m = 1, 2, \ldots, 3^{k-1}$. Observe also that the intervals $I_{1,1}$ and $I_{2,m}$ are pairwise disjoint.

Proceeding by induction, at the $l$-th stage, we subdivide each interval $(J^{l-1}_m)^\Delta$ into $3^{k-1}$ triadic intervals of equal length and denote all obtained intervals by
Similarly, $J_{m}^l, m = 1, 2, \ldots, 3^{(k-1)(l-1)}$. Set $I_{l,m} = P(J_{m}^l)$. Then $|J_{m}^l| = \frac{1}{3^{k(l-1)}}$ and $|I_{l,m}| = \frac{1}{3^l}$, and the intervals $\{I_{l,m}\}$ are pairwise disjoint.

Denote by $\mathcal{I}_l$ and $\mathcal{J}_l$ the families of all intervals $\{I_{l,m}\}$ and $\{J_{m}^l\}$, respectively, and set $\Omega_l = \bigcup_{I \in \mathcal{I}_l} I$. Define the weight $w_k$ such that $w_k([0, 1]) = 1$, $w_k$ is a constant on $\Omega_l$, and for every $I \in \mathcal{I}_l$ and $J \in \mathcal{J}_{l+1}$, $w_k(I) = w_k(J)$ (we use the standard notation $w_k(E) = \int_E w_k$).

It was proved in [10] that one can specify the situation of the intervals $\{I_{l,m}\}$ such that if $k > 3000$ and $x \in \bigcup_{l,m} I_{l,m}^\alpha$, then
\begin{equation}
|H w_k(x)| \geq (k/3) w_k(x);
\end{equation}
moreover,
\begin{equation}
M w_k(x) \leq 7 w_k(x) \quad (x \in \bigcup_{l,m} I_{l,m}^\alpha),
\end{equation}
irrespective of the precise configuration of $\{I_{l,m}\}$.

We will show that the latter estimate can be improved by means of replacing $M w_k$ on the left-hand side by a larger operator $M_r w_k$, with $r > 1$ depending on $k$.

In order to do that, we need a more constructive description of $w_k$.

**Lemma 2.1.** We have
\begin{equation}
w_k(x) = \sum_{l=1}^{\infty} \left( \frac{3^k}{3^{k-1}+1} \right)^l \chi_{\Omega_l}(x).
\end{equation}

**Proof.** Assume that $w_k = \alpha_l$ on $\Omega_l$. Let $J \in \mathcal{J}_l$ and take $I \in \mathcal{I}_l$ such that $I \subset J$. Then
\begin{equation}
w_k(J) = w_k(I) + w_k(J^\Delta) = w_k(I) + \sum_{J' \in \mathcal{J}_{l+1} : J' \subset J^\Delta} w_k(J').
\end{equation}

Let $I' \in \mathcal{I}_{l-1}$. Then
\begin{equation}
w_k(J) = w_k(I') = \alpha_{l-1}|I'| = \alpha_{l-1}|J|.
\end{equation}

Similarly, $w_k(J') = \alpha_l|J'|$, and also $w_k(I) = \alpha_l|I| = \alpha_l \frac{|J|}{3^k}$. Hence, (2.3) implies that
\begin{equation}
\alpha_{l-1}|J| = \alpha_l \frac{|J|}{3^k} + \alpha_l \sum_{J' \in \mathcal{J}_{l+1} : J' \subset J^\Delta} |J'| = \alpha_l \frac{|J|}{3^k} + \alpha_l \frac{|J|}{3^k}.
\end{equation}

From this, $\alpha_l = \frac{3^k}{3^{k-1} + 1} \alpha_{l-1}$, and therefore $\alpha_l = \left( \frac{3^k}{3^{k-1}+1} \right)^l \gamma$ for some $\gamma > 0$.

From the condition $w_k([0, 1]) = 1$, we obtain
\begin{align*}
1 &= w_k([0, 1]) = \gamma \sum_{l=1}^{\infty} \left( \frac{3^k}{3^{k-1}+1} \right)^l |\Omega_l| \\
&= \gamma \sum_{l=1}^{\infty} \left( \frac{3^k}{3^{k-1}+1} \right)^l \frac{3^{(k-1)(l-1)}}{3^l} = \gamma \frac{1}{3^k-1} \sum_{l=1}^{\infty} \left( \frac{3^{k-1}}{3^{k-1}+1} \right)^k = \gamma,
\end{align*}
and therefore the lemma is proved. \qed

**Lemma 2.2.** Let $r = 1 + \frac{1}{3^k-1}$. Then for every $I \in \mathcal{I}_l, l \in \mathbb{N}$, and for all $x \in I^\Delta$, $M_r w_k(x) \leq 21 w_k(x)$. 

Proof. Let $I \in \mathcal{I}_l$ and let $x \in I^A$. Take an arbitrary interval $R$ containing $x$. If $R \subset I$, then
\[
\left( \frac{1}{|R|} \int_R w^r_k(y)dy \right)^{1/r} = \left( \frac{3^k}{3^{k-1} + 1} \right)^l = w_k(x).
\]
Assume that $R \not\subset I$. Then $|R| \geq |I|/3$. Denote by $\mathcal{F}$ the family of all triadic intervals $I' \subset [0,1)$ such that $|I'| = |I|$ and $I' \cap R \neq \emptyset$. There are at most two intervals $I' \in \mathcal{F}$ not contained in $R$, and therefore,
\[
(2.4) \quad \sum_{I' \in \mathcal{F}} |I'| \leq |R| + \sum_{I' \in \mathcal{F} : I' \not\subset R} |I'| \leq |R| + 2|I| \leq 7|R|.
\]
We claim that if $r = 1 + \frac{1}{3^{k+1}}$, then for every $I' \in \mathcal{F}$,
\[
(2.5) \quad \left( \frac{1}{|I'|} \int_{I'} w^r_k(y)dy \right)^{1/r} \leq 3w_k(x).
\]
This property would conclude the proof since then, by (2.4),
\[
\frac{1}{|R|} \int_R w^r_k(y)dy \leq \sum_{I' \in \mathcal{F}} \frac{|I'|}{|R|} \frac{1}{|I'|} \int_{I'} w^r_k(y)dy \leq 7(3w_k(x))^r.
\]
To show (2.5), one can assume that $I'$ has a non-empty intersection with the support of $w_k$. If $I' \neq J$ for some $J \in \mathcal{J}_{l+1}$, then $I' \subset L$, where $L \in \mathcal{I}_\nu$, $\nu \leq l$, and hence
\[
\left( \frac{1}{|I'|} \int_{I'} w^r_k(y)dy \right)^{1/r} = \left( \frac{3^k}{3^{k-1} + 1} \right)^r \leq w_k(x).
\]
It remains to consider the case when $I' = J$ for some $J \in \mathcal{J}_{l+1}$. Using that for every $j \geq l + 1$, $J \in \mathcal{J}_{l+1}$ contains $3(j-l-1)$ triadic intervals $I \in \mathcal{I}_j$, we obtain
\[
\frac{1}{|I'|} \int_{I'} w^r_k(y)dy = 3^{l_j} \sum_{j=l+1}^{\infty} \sum_{I \in \mathcal{I}_j : I \subset I'} \int_I w^r_k(y)dy
\]
\[
= \sum_{j=l+1}^{\infty} 3^{j-l-1} 3^{j-l} \left( \frac{3^k}{3^{k-1} + 1} \right)^{j/r}
\]
\[
= \frac{1}{3^{k-1}} \sum_{j=1}^{\infty} 3^{-j} \left( \frac{3^k}{3^{k-1} + 1} \right)^{(j+l)r}.
\]
Therefore,
\[
\frac{1}{|I'|} \int_{I'} w^r_k(y)dy \leq \frac{1}{3^{k-1}} \left( \sum_{j=1}^{\infty} 3^{-j} \left( \frac{3^k}{3^{k-1} + 1} \right)^{(j+l)r} \right) w_k(x)^r
\]
\[
\leq \frac{1}{3^{k-1}} \left( \frac{3^k}{3^{k-1} + 1} \right)^{r} w_k(x)^r,
\]
whenever $\left( \frac{3^k}{3^{k-1} + 1} \right)^{r} < 3$.
If $r = 1 + \frac{1}{3^{k+1}}$, then
\[
\left( \frac{3^k}{3^{k-1} + 1} \right)^{1+\frac{1}{3^{k+1}}} \leq 3^{\frac{1}{3^{k+1}}} \frac{3^k}{3^{k-1} + 1} \leq \left( 1 + \frac{1}{3^k} \right) \frac{3^k}{3^{k-1} + 1} = 3 - \frac{3^k}{3^{k-1} + 1}.
\]
Hence,
\[
\frac{1}{|I|} \int_I w_k^r(y)dy \leq \frac{3^k-1+1}{2} w_k(x)^r \leq 3w_k(x)^r,
\]
which completes the proof. \qed

3. Extrapolation

Here we follow the extrapolation argument of D. Cruz-Uribe and C. Pérez [3], with some modifications.

Denote by \( M_c^e \) the centered weighted maximal operator with respect to a weight \( v \), and skip the index \( v \) in the unweighted case.

**Lemma 3.1.** Assume that for every weight \( w \) and for all \( f \in L^1(M_v,w) \),
\[
\|Hf\|_{L^{1,\infty}(w)} \leq A_r \|f\|_{L^1(M_v(w))} \quad (1 < r < 2).
\]
Let \( \alpha_r = \frac{r}{2-r} \). There is \( c > 0 \) such that for any weight \( w \) supported in \([0,1]\) one has
\[
\int_0^1 \left( \frac{|Hw|}{(M\alpha, w)^{\alpha_r/r}} \right)^2 w^{\alpha_r} dx \leq c A_r^2 \int_0^1 w dx \quad (1 < r < 2).
\]
**Proof.** Denote \( \beta_r = \frac{r(r-1)}{2-r} \). The numbers \( \alpha_r \) and \( \beta_r \) are chosen in such a way that they satisfy \( \alpha_r - \beta_r = r \) and \( \alpha_r - \frac{2\beta_r}{r} = 1 \).

For \( \varepsilon > 0 \) set \( w_{\varepsilon} = \max(w, \varepsilon) \). Let \( g \geq 0 \). Since
\[
\frac{1}{|I|} \int_I (gw_{\varepsilon})^r = \left( \frac{1}{w_{\varepsilon}^{\alpha_r}(I)} \right) \int_I (g^r/w_{\varepsilon}^{\beta_r}) w_{\varepsilon}^{\alpha_r} \frac{w_{\varepsilon}^{\alpha_r}(I)}{|I|},
\]
using that \( Mf \leq 2M^c f \), we get
\[
(3.1) \quad M_r(gw_{\varepsilon})(x) \leq 2 \left( M^c_{w_{\varepsilon}^{\alpha_r}}(g^r/w_{\varepsilon}^{\beta_r})(x) M_{\alpha_r}(w_{\varepsilon})(x)^{\alpha_r} \right)^{1/r}.
\]

Using the initial assumption on \( H \) for the weight \( gw_{\varepsilon} \) along with \( M^c \), and applying Hölder’s inequality along with the boundedness of \( M^c_{w_{\varepsilon}^{\alpha_r}} \) on \( L^p(v) \), for \( p = \frac{2}{r} > 1 \), we obtain
\[
\int_{\{|Hf| > 1\}} gw_{\varepsilon} \leq A_r \|f\|_{L^1(M_v(gw_{\varepsilon}))} \leq 2A_r \int \left( |f| M_{\alpha_r}(w_{\varepsilon}) w_{\varepsilon}^{\frac{\alpha_r}{\alpha_r/2}} \right) \left( M^c_{w_{\varepsilon}^{\alpha_r}}(g^r/w_{\varepsilon}^{\beta_r}) w_{\varepsilon}^{\alpha_r/2} \right) dx \leq 2A_r \|f\|_{L^2(M_{\alpha_r}(w_{\varepsilon})^{\frac{2\alpha_r}{\alpha_r/2}}/w_{\varepsilon}^{\alpha_r}))} \left( M^c_{w_{\varepsilon}^{\alpha_r}}(g^r/w_{\varepsilon}^{\beta_r}) \right)^{1/2} \|L^2(w_{\varepsilon}^{\alpha_r}) \leq cA_r \|f\|_{L^2(M_{\alpha_r}(w_{\varepsilon})^{\frac{2\alpha_r}{\alpha_r/2}}/w_{\varepsilon}^{\alpha_r})} \|g\|_{L^2(w_{\varepsilon})}.
\]
Taking here the supremum over all \( g \geq 0 \) with \( \|g\|_{L^2(w_{\varepsilon})} = 1 \) yields
\[
\|Hf\|_{L^{2,\infty}(w_{\varepsilon})} \leq cA_r \|f\|_{L^2(M_{\alpha_r}(w_{\varepsilon})^{\frac{2\alpha_r}{\alpha_r/2}}/w_{\varepsilon}^{\alpha_r})}.
\]
By duality, the latter inequality is equivalent to
\[
\|Hf\|_{L^2(w_{\varepsilon}^{\alpha_r}/(M_{\alpha_r}(w_{\varepsilon})^{\frac{2\alpha_r}{\alpha_r/2}})} \leq cA_r \|f/w_{\varepsilon}\|_{L^2,1(w_{\varepsilon})},
\]
where \( L^2,1(w_{\varepsilon}) \) is the weighted Lorentz space. Take here \( f = w \) and use that
\[
\|w/w_{\varepsilon}\|_{L^2,1(w_{\varepsilon})} \leq \|\chi_{[0,1]}\|_{L^{2,1}(w_{\varepsilon})} = \int_0^{w_{\varepsilon}(0,1)} t^{-1/2} dt = 2w_{\varepsilon}(0,1)^{1/2}.
\]
We obtain
\[ \|Hw\|_{L^2}\left(w^{\alpha_r}/(M_{\alpha_r}, w)\right) \leq 2cA_r w_\varepsilon([0,1])^{1/2}. \]
It remains to let \( \varepsilon \to 0 \) and to use the Fatou convergence theorem. \( \square \)

4. Proof of Theorem 1.1

Our goal is to use the extrapolation Lemma 3.1 assuming (1.1) with a general Orlicz maximal function \( M_\Phi \). Hence, we need a relation between \( M_\Phi \) and \( M_r \) with possibly good dependence of the corresponding constant on \( r \) when \( r \to 1 \). Such a relation was recently obtained in [4] (see Lemma 6.2 and inequality (6.4) there). For the reader’s convenience we include a proof here.

Lemma 4.1. For all \( x \in \mathbb{R} \),
\[ M_\Phi f(x) \leq \left( 2 \sup_{t \geq \Phi^{-1}(1/2)} \frac{\Phi(t)}{t^r} \right)^{1/r} M_r f(x) \quad (r > 1). \]

Proof. For any interval \( I \subset \mathbb{R} \),
\[ \int_I \Phi \left( \frac{|f|}{\lambda} \right) = \int_{\{x \in I : |f| \leq \Phi^{-1}(1/2)\lambda\}} \Phi \left( \frac{|f|}{\lambda} \right) + \int_{\{x \in I : |f| \geq \Phi^{-1}(1/2)\lambda\}} \Phi \left( \frac{|f|}{\lambda} \right) \leq \frac{|I|}{2} + c_r \int_I (|f|/\lambda)^r \, dx, \]
where \( c_r = \sup_{t \geq \Phi^{-1}(1/2)} \frac{\Phi(t)}{t^r} \). Therefore, setting \( \lambda_0 = \left( \frac{2c_r}{|I|} \int_I |f|^r \right)^{1/r} \), we obtain
\[ \frac{1}{|I|} \int_I \Phi(|f|/\lambda_0) \, dx \leq 1, \]
which proves (4.1). \( \square \)

It follows easily from (4.1) that
\[ M_\Phi f(x) \leq c \left( \sup_{t \geq 1} \frac{\Phi(t)^{1/r}}{t} \right) M_r f(x) \quad (r > 1), \]
where \( c \) may depend on \( \Phi \), but it does not depend on \( r \).

Proof of Theorem 1.1 Suppose, by contrast, that (1.1) holds. Then combining (4.2) with Lemma 3.1, we obtain
\[ \int_0^1 \left( \frac{|Hw|}[(M_{\alpha_r}, w)^{\alpha_r/r}] \right)^2 w^{\alpha_r} \, dx \leq c \left( \sup_{t \geq 1} \frac{\Phi(t)^{1/r}}{t} \right)^2 \int_0^1 w \, dx \quad (1 < r < 2). \]

Set here \( r = r_k = 1 + \frac{1}{3^k + 1}, \) and \( w = w_k \) as constructed in Section 2. Then \( \alpha_k = r_k r = 1 + \frac{1}{3^{k+1}} \). Applying (2.1) along with Lemma 2.2 yields
\[ \int_0^1 \left( \frac{|Hw_k|}[(M_{\alpha_r}, w_k)^{\alpha_r/r}] \right)^2 w_k^{\alpha_k} \, dx \geq \frac{k^2}{9 \cdot 27^{2-r_k}} \int_{[k \notin I \cup \ell \in \mathbb{N} \cup \ell \in I_i \ell \Delta} w_k \]
\[ = \frac{k^2}{27^{1+2-r_k}} \int_0^1 w_k, \]
and we obtain
\[ k \leq c \sup_{t \geq 1} \frac{\Phi(t)^{1/r_k}}{t}. \]
It remains to estimate the right-hand side of (4.3). Write
\[ \Phi(t) = t \log \log(e^t + t) \phi(t), \]
where \( \lim_{t \to \infty} \phi(t) = 0 \). Since
\[ \log \log t = \log(r') + \log t^{1/r'} \leq \log(r') + t^{1/r'}, \]
for \( t > e^{r'} \) we obtain
\[ \frac{\Phi(t)^{1/r}}{t} = \left( \frac{\log \log(e^t + t) \phi(t)}{t^{1/r'}} \right)^{1/r} \leq c \left( \frac{\log(r')^{1/r} (\sup_{t \geq e^{r'}} \phi(t))^{1/r}}{t^{1/r'}} \right) \]
(here \( r' \) is the dual exponent to \( r \)).

On the other hand, if \( 0 < \delta < 1 \), then
\[ \sup_{1 \leq t \leq e^{r'}} \frac{\Phi(t)^{1/r}}{t} \leq \sup_{1 \leq t \leq e^{(\log r')^\delta}} \left( \frac{\log \log(e^t + t) \phi(t)}{t^{1/r'}} \right)^{1/r} \]
\[ + \sup_{e^{(\log r')^\delta} \leq t \leq e^{r'}} \left( \frac{\log \log(e^t + t) \phi(t)}{t^{1/r'}} \right)^{1/r} \]
\[ \leq c \left( (\log r')^{\delta/r} + (\log r')^{1/r} \sup_{t \geq e^{(\log r')^\delta}} \phi(t)^{1/r} \right). \]

Setting \( \beta_k = \sup_{t \geq e^{(\log r')^\delta}} \phi(t)^{1/r} \) and combining both cases, we obtain
\[ \sup_{t \geq 1} \frac{\Phi(t)^{1/r_k}}{t} \leq c \left( (\log r')^{\delta/r_k} + \beta_k (\log r')^{1/r_k} \right) \]
\[ \leq c(k^{\delta} + \beta_k k). \]

Since \( \beta_k \to 0 \) as \( k \to \infty \), we arrive at a contradiction with (4.3), and therefore the theorem is proved. \( \square \)

**Remark 4.2.** The following inequality is contained implicitly in [7]:
\[ \lambda \{ x \in \mathbb{R} : |Hf(x)| > \lambda \} \leq c \log(r') \|f\|_{L^1(M,w)} \quad (r > 1). \]
The proof of Theorem [11] shows that \( \log(r') \) here is optimal; namely, it cannot be replaced by \( \varphi(r') \) for any increasing \( \varphi \) such that \( \lim_{t \to \infty} \frac{\varphi(t)}{\log t} = 0 \).

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