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STATISTICAL STABILITY FOR MULTIDIMENSIONAL PIECEWISE EXPANDING MAPS

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Abstract. We present sufficient conditions for the (strong) statistical stability of some classes of multidimensional piecewise expanding maps. As a consequence we get that a certain natural two-dimensional extension of the classical one-dimensional family of tent maps is statistically stable.

1. Introduction

Within this paper we deal with multidimensional piecewise expanding maps defined in some compact subset of an Euclidean space. A first approach to this topic has been made in dimension one by Lasota and Yorke in [12], where they proved the existence of absolutely continuous invariant probability measures for a class of piecewise C^2 expanding maps of the interval. The extension to higher dimensions in general is a very delicate question, mostly because of the intricate geometry of the domains of smoothness and their images under iterations. In the last decades many results have appeared in the literature with several different approaches. A first result in dimension two was obtained by Keller in [11]. In the multidimensional case, Góra and Boyarski in [9] proved the existence of absolutely continuous invariant probability measures for maps with a finite number of domains of smoothness under some condition of no cusps in the domains of smoothness. This result was later extended by Adl-Zarabi in [1] to piecewise expanding maps allowing cusps in the domains of smoothness, and by Alves in [2] to piecewise expanding maps with countably many domains of smoothness. Similar results were drawn in the particular case of piecewise linear maps by Buzzi and Tsujii in [5, 20], and by the same authors in the case of piecewise real analytic expanding maps of the plane in [6, 19]. A general result on the existence of absolutely continuous invariant probability measures in any finite dimension was given by Saussol in [17] for C^{1+} piecewise expanding maps with infinitely many domains of smoothness under some control on the accumulation of discontinuities under iterations of the map.

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Many concrete situations lead to the appearance of families of piecewise expanding maps under conditions that guarantee the existence of absolutely continuous invariant probability measures, in many cases this probability measure being unique. It is then a natural question trying to decide whether these measures depend continuously on the dynamics, i.e., the statistical stability of those maps. This question was addressed in [4] for certain robust classes of maps with non-uniform expansion; see also [3]. In [13], the authors consider a one-parameter family $(\Lambda_t)_t$ of two-dimensional piecewise linear maps defined on a triangle in $\mathbb{R}^2$. This family $(\Lambda_t)_t$ is closely related to the family of limit return maps arising when certain three-dimensional homoclinic bifurcations take place; see [18]. It was shown in [13, Proposition 6.1] and the comments following it that Λ_t becomes the best choice in the piecewise linear setting for describing the dynamics of the original limit return maps given in [18] when the unstable manifold of the saddle point has dimension two.

The results in the present paper have a twofold aim: to give sufficient conditions for the statistical stability of some general classes of multidimensional piecewise expanding maps; and to prove the statistical stability of the family of maps introduced in [13]. This last result will be obtained as an application of our general result.

1.1. **Statistical stability.** In this work we consider discrete-time dynamical systems defined in a compact region $R \subset \mathbb{R}^d$, for some $d \geq 1$. Given a measurable map $\phi : R \to R$, we say that a probability measure μ on the Borel sets of R is *ϕ-invariant* if $\mu(\phi^{-1}(A)) = \mu(A)$, for any Borel set $A \subset R$. If $\mu(A) = 0$ whenever $m(A) = 0$, where m denotes the Lebesgue measure on the Borel sets of $\mathbb{R}^d$, then μ is called *absolutely continuous*. In this case, there exists an m-integrable function $h \geq 0$, usually denoted $d\mu/dm$ and called the density of μ with respect to m, such that for any Borel set $A \subset R$ we have $\mu(A) = \int_A h dm$. A ϕ-invariant probability measure μ is called ergodic if $\mu(A)\mu(R \setminus A) = 0$, whenever $\phi^{-1}(A) = A$. As a consequence of Birkhoff's Ergodic Theorem we have that any ϕ-invariant absolutely continuous ergodic probability measure μ is a *physical measure*, meaning that for a subset of points $x \in R$ with positive Lebesgue measure we have

$$\lim_{n\to\infty} \sum_{j=0}^{n-1} f(\phi^j(x)) = \int f d\mu$$

for all continuous $f : R \to \mathbb{R}$.

Let I be a metric space and $(\phi_t)_{t\in I}$ a family of maps $\phi_t : R \to R$. We say that the family $(\phi_t)_{t\in I}$ is *statistically stable* if:

(1) each ϕ_t has some absolutely continuous ϕ_t-invariant probability measure;
(2) given any sequence $(t_n)_n$ in I converging to $t_0 \in I$ and $(\mu_n)_n$ a sequence of absolutely continuous ϕ_{t_n}-invariant probability measures, there is a subsequence $(t_{n_k})_k$ such that the densities $d\mu_{t_{n_k}}/dm$ converge in the L^1-norm to the density of an absolutely continuous ϕ_{t_0}-invariant probability measure.

Of course, when each ϕ_t has a unique absolutely continuous invariant probability measure μ_t, then statistical stability means that $d\mu_t/dm$ converges in the L^1-norm to $d\mu_{t_0}/dm$ when $t \to t_0$. A strictly weaker notion of statistical stability may be given if we assume only weak* convergence of the measures μ_t to μ_{t_0} when $t \to t_0$.

1.2. **Piecewise expanding maps.** Here we state precisely sufficient conditions for the statistical stability of certain higher dimensional families of C^2 piecewise expanding maps with countably many domains of smoothness. We follow the approach in [2] which, in turn, was inspired by [9].

Let R be a compact set in $\mathbb{R}^d$, for some $d \geq 1$. For each $1 \leq p \leq \infty$ we denote by $L^p(R)$ the Banach space of functions in $L^p(m)$ with support contained in R, endowed with the usual norm $\| \quad \|_p$. Let $\phi : R \to R$ be a map for which there is a (Lebesgue mod 0) partition $\{R_i\}_{i=1}^{\infty}$ of R such that each R_i is a closed domain with piecewise C^2 boundary of finite $(d-1)$-dimensional measure and $\phi_i = \phi|_{R_i}$ is a C^2 bijection from $\operatorname{int}(R_i)$, the interior of R_i, onto its image with a C^2 extension to R_i. We say that ϕ is *piecewise expanding* if

(P$_1$) there is $0 < \sigma < 1$ such that for every $i \geq 1$ and $x \in \operatorname{int}(\phi(R_i))$
$$\|D\phi_i^{-1}(x)\| < \sigma.$$

We say that ϕ has *bounded distortion* if

(P$_2$) there is $D \geq 0$ such that for every $i \geq 1$ and $x \in \operatorname{int}(\phi(R_i))$
$$\frac{\left\|D\left(J \circ \phi_i^{-1}\right)(x)\right\|}{\left|J \circ \phi_i^{-1}(x)\right|} \leq D,$$
where J denotes the Jacobian of ϕ.

Finally, we say that ϕ has *long branches* if

(P$_3$) there are $\beta, \rho > 0$ and for each $i \geq 1$ there is a C^1 unitary vector field X_i in $\partial\phi(R_i)$ such that:
  (a) the segments joining each $x \in \partial\phi(R_i)$ to $x + \rho X_i(x)$ are pairwise disjoint and contained in $\phi(R_i)$, and their union forms a neighborhood of $\partial\phi(R_i)$ in $\phi(R_i)$;
  (b) for every $x \in \partial\phi(R_i)$ and $v \in T_x\partial\phi(R_i) \setminus \{0\}$ the angle $\angle(v, X_i(x))$ between v and $X_i(x)$ satisfies $|\sin\angle(v, X_i(x))| \geq \beta$.

Here we assume that at the singular points $x \in \partial\phi(R_i)$ where $\partial\phi(R_i)$ is not smooth the vector $X_i(x)$ is a common C^1 extension of X_i restricted to each $(d-1)$-dimensional smooth component of $\partial\phi(R_i)$ having x in its boundary. We also assume that the tangent space of any such singular point x is the union of the tangent spaces to the $(d-1)$-dimensional smooth components it belongs to.

In the one-dimensional case $d = 1$, condition (P$_3$)(a) is clearly satisfied once we take the sets in the partition of R as being intervals whose images $\phi(R_i)$ have sizes uniformly bounded away from zero. Additionally, condition (P$_3$)(b) always holds in dimension one, since $\partial\phi(R_i)$ is a zero-dimensional manifold and so $T_x\partial\phi(R_i) = \{0\}$ for any $x \in \phi(R_i)$. In this case we can even take the optimal value $\beta = 1$; see Remark 3.2.

Theorem A. *Let I be a metric space and $(\phi_t)_{t\in I}$ a family of C^2 piecewise expanding maps $\phi_t : R \to R$ with bounded distortion and long branches. Assume that there exist $0 < \lambda < 1$ and $K > 0$ such that for each $t \in I$*

1. *for each continuous $f : R \to \mathbb{R}$ we have $\|f \circ \phi_{t'} - f \circ \phi_t\|_d \to 0$ when $t' \to t$;*
2. *$\sigma_t \left(1 + \dfrac{1}{\beta_t}\right) \leq \lambda$ and $D_t + \dfrac{1}{\beta_t\rho_t} + \dfrac{D_t}{\beta_t} \leq K$, where $\sigma_t, D_t, \beta_t, \rho_t$ are constants for which* (P$_1$), (P$_2$) *and* (P$_3$) *hold for ϕ_t.*

Then $(\phi_t)_{t\in I}$ is statistically stable.

It follows from [2, Section 5] that under the assumptions above each ϕ_t has a finite number of ergodic absolutely continuous invariant probability measures. The proof of this result uses the space of functions of bounded variation in $\mathbb{R}^d$, which are known to belong to the space $L^p(R)$, with $p = d/(d-1)$; see (3) below. Observing that $1/p + 1/d = 1$, this makes the choice of the norm $\| \quad \|_d$ in condition (1) less mysterious; see the proof of Lemma 3.5. Notice that condition (1) in Theorem A holds whenever the maps ϕ_t are continuous and ϕ_t depends continuously (in the C^0-norm) on $t \in I$.

1.3. **Two-dimensional tent maps.** Here we present the family of maps introduced in [13] and give some results on its statistical stability. We define the family of maps $\Lambda_t : T \to T$ on the triangle $T = T_0 \cup T_1$, where

$$T_0 = \{(x, y) : 0 \le x \le 1,\ 0 \le y \le x\}, \quad T_1 = \{(x, y) : 1 \le x \le 2,\ 0 \le y \le 2 - x\}, \tag{1}$$

and

$$\Lambda_t(x, y) = \begin{cases} (t(x + y), t(x - y)), & \text{if } (x, y) \in T_0; \\ (t(2 - x + y), t(2 - x - y)), & \text{if } (x, y) \in T_1. \end{cases} \tag{2}$$

The domains T_0 and T_1 are separated by a straight line segment $\mathcal{C} = \{(x_1, x_2) \in T : x_1 = 1\}$ that we call the *critical set* of Λ_t.

As shown in [15], the map Λ_1 displays the same properties of the one-dimensional tent map $\lambda_2(x) = 1 - 2|x|$. Among them, the consecutive pre-images $\{\Lambda_1^{-n}(\mathcal{C})\}_{n \in \mathbb{N}}$ of the critical line $\mathcal{C}$ define a sequence of partitions (whose diameter tends to zero as n goes to infinity) of T leading them to conjugate Λ_1 to a one-sided shift with two symbols. Hence, it easily follows that Λ_1 is transitive in T. Furthermore, for every point $(x_0, y_0) \in T$ whose orbit never hits the critical line the Lyapunov exponent of Λ_1 along the orbit of (x_0, y_0) is positive (and coincides with $\frac{1}{2}\log 2$) in all nonzero directions. Finally, it can be constructed as an absolutely continuous ergodic invariant probability measure for Λ_1; see [15]. Because of this, Λ_1 was called the *two-dimensional tent map.* Since the parameter t in (2) essentially gives the rate of expansion for Λ_t (playing the same roll of the parameter a for $\lambda_a(x) = 1 - a|x|$), the family $(\Lambda_t)_t$ can be considered as a natural extension of the one-dimensional family of tent maps and naturally called a *family of two-dimensional tent maps.*

The results obtained in [15] for $t = 1$ were extended to a larger set of parameters. More precisely, it was proved in [14] that for each $t \in [\tau, 1]$, with $\tau = \frac{1}{\sqrt{2}}(\sqrt{2} + 1)^{\frac{1}{4}} \approx 0.882$, the map Λ_t exhibits a *strange attractor* $A_t \subset T$: Λ_t is (strongly) transitive in A_t, the periodic orbits are dense in A_t, and there exists a dense orbit in A_t with two positive Lyapunov exponents. Furthermore, A_t supports a unique absolutely continuous Λ_t-invariant ergodic probability measure μ_t. As an application of Theorem A we shall obtain the following result.

Theorem B. *The family* $(\Lambda_t)_{t \in [\tau, 1]}$ *is statistically stable.*

As each Λ_t has a unique absolutely continuous invariant probability measure μ_t, the statistical stability means in this case that $d\mu_t/dm$ converges in the L^1-norm to $d\mu_{t_0}/dm$ when $t \to t_0$, for each $t_0 \in [\tau, 1]$.

2. FUNCTIONS OF BOUNDED VARIATION

The main ingredient for the proof of Theorem A is the notion of variation for functions in multidimensional spaces. We adopt the definition given in [8].

Given $f \in L^1(\mathbb{R}^d)$ with compact support we define the *variation* of f as

$$V(f) = \sup\left\{ \int_{\mathbb{R}^d} f \operatorname{div}(g) dm \,:\, g \in C_0^1(\mathbb{R}^d, \mathbb{R}^d) \text{ and } \|g\| \le 1 \right\},$$

where $C_0^1(\mathbb{R}^d, \mathbb{R}^d)$ is the set of C^1 functions from $\mathbb{R}^d$ to $\mathbb{R}^d$ with compact support, $\operatorname{div}(g)$ is the divergence of g and $\| \quad \|$ is the sup norm in $C_0^1(\mathbb{R}^d, \mathbb{R}^d)$. Given a bounded set $R \subset \mathbb{R}^d$ we consider the space of *bounded variation* functions in $L^1(R)$

$$BV(R) = \left\{ f \in L^1(R) : V(f) < +\infty \right\}.$$

Contrary to the classical one-dimensional definition of bounded variation, a multidimensional bounded variation function need not be bounded; see [10]. However, by Sobolev's Inequality (see e.g. [8, Theorem 1.28]) there is some constant $C > 0$ (only depending on the dimension d) such that for any $f \in BV(R)$

$$\left(\int |f|^p dm_d \right)^{1/p} \le C\, V(f), \quad \text{with} \quad p = \frac{d}{d-1}. \tag{3}$$

This in particular gives $BV(R) \subset L^p(R)$. We shall use the following properties of bounded variation functions whose proofs may be found in [7] or [8]:

(B_1) $BV(R)$ is dense in $L^1(R)$;

(B_2) if $(f_k)_k$ is a sequence in $BV(R)$ converging to f in the L^1-norm, then $V(f) \le \liminf_k V(f_k)$;

(B_3) if $(f_k)_k$ is a sequence in $BV(R)$ such that $\big(\|f_k\|_1\big)_k$ and $\big(V(f_k)\big)_k$ are bounded, then $(f_k)_k$ has some subsequence converging in the L^1-norm to a function in $BV(R)$.

3. Piecewise expanding maps

In this section we prove Theorem A. Let $\{R_i^t\}_{i=1}^\infty$ be the domains of smoothness of ϕ_t with $t \in I$ satisfying the assumptions of Theorem A and define $\phi_{t,i} = \phi_t|_{R_i^t}$ for all $i \ge 1$. For each $t \in I$ we consider the *Perron-Frobenius operator*

$$P_t : L^1(R) \longrightarrow L^1(R)$$

defined for $f \in L^1(R)$ as

$$P_t f = \sum_{i=1}^\infty \frac{f \circ \phi_{t,i}^{-1}}{|J \circ \phi_{t,i}^{-1}|} \chi_{\phi_t(R_i^t)}.$$

It is well known that the following two properties hold for each P_t:

(C_1) $\|P_t f\|_1 \le \|f\|_1$ for every $f \in L^1(R)$;

(C_2) $P_t f = f$ if and only if f is the density of an absolutely continuous ϕ_t-invariant measure.

Considering $0 < \lambda < 1$ and $K > 0$ as in the statement of Theorem A, the proof of the next lemma follows immediately from [2, Lemma 5.4 & Lemma 5.5] with

$$K_1 = K \sum_{j=0}^\infty \lambda^j.$$

Lemma 3.1. *Given $t \in I$ and $j \ge 1$ we have for each $f \in BV(R)$,*

$$V(P_t^j f) \le \lambda^j V(f) + K_1 \|f\|_1.$$

Remark 3.2. The proof of [2, Lemma 5.4] uses [9, Lemma 3] applied to the sets $S = \phi(R_i)$, which gives for a function $f \in C^1(S)$,

$$\int_{\partial S} |f| dm \leq \frac{1}{\beta}\left(\frac{1}{\rho}\int_S |f| dm + \int_S \|Df\| dm\right). \tag{4}$$

In the one-dimensional case we have for any interval S and $x \in S$,

$$f(x) \leq \frac{1}{|S|}\int_S |f| dm + \int_S |Df| dm,$$

which yields a formula similar to (4) in the one-dimensional case with $\beta = 1$.

In the proof of the result below we follow some standard arguments with functions of bounded variation, namely those used in [12] for the one-dimensional case.

Proposition 3.3. *Given $t \in I$ and $f \in L^1(R)$, the sequence $1/n\sum_{j=0}^{n-1} P_t^j f$ has some accumulation point in the L^1-norm. Moreover, any such accumulation point belongs to $BV(R)$ and has variation bounded by $4K_1\|f\|_1$.*

Proof. Given $f \in L^1(R)$, by property (B_1) we may consider a sequence of functions $(f_k)_k$ in $BV(R)$ converging to f in the L^1-norm. With no loss of generality we may assume that $\|f_k\|_1 \leq 2\|f\|_1$ for every $k \geq 1$. It follows from Lemma 3.1 that for each $k \geq 1$ and large j we have

$$V(P_t^j f_k) \leq \lambda^j V(f_k) + K_1\|f_k\|_1 \leq 3K_1\|f\|_1.$$

So, for large n we have

$$V\left(\frac{1}{n}\sum_{j=0}^{n-1} P_t^j f_k\right) \leq 4K_1\|f\|_1.$$

Using that $\|f_k\|_1 \leq 2\|f\|_1$ for every $k \geq 1$, it easily follows from (C_1) that

$$\left\|\frac{1}{n}\sum_{j=0}^{n-1} P_t^j f_k\right\|_1 \leq 2\|f\|_1.$$

Then it follows from (B_3) that there exists some $g_k \in BV(R)$ and a sequence $(n_i)_i$ such that $1/n_i\sum_{j=0}^{n_i-1} P_t^j f_k$ converges in the L^1-norm to g_k as i goes to $+\infty$. Moreover, by (B_2) we have $V(g_k) \leq 4K_1\|f\|_1$ for every $k \geq 1$. Hence, we may apply the same argument to the sequence $(g_k)_k$ and obtain a subsequence $(k_i)_i$ such that $(g_{k_i})_i$ converges in the L^1-norm to some $g \in BV(R)$ with $V(g) \leq 4K_1\|f\|_1$. Hence, there must be some sequence $(n_\ell)_\ell$ converging to $+\infty$ for which $1/n_\ell\sum_{j=0}^{n_\ell-1} P_t^j f_{k_\ell}$ converges to g in the L^1-norm as $\ell \to +\infty$. On the other hand,

$$\left\|\frac{1}{n_\ell}\sum_{j=0}^{n_\ell-1}\left(P_t^j f_{k_\ell} - P_t^j f\right)\right\|_1 \leq \frac{1}{n_\ell}\sum_{j=0}^{n_\ell-1}\|f_{k_\ell} - f\|_1 = \|f_{k_\ell} - f\|_1$$

and this last term goes to 0 as $\ell \to +\infty$. This clearly gives that $1/n_\ell\sum_{j=0}^{n_\ell-1} P_t^j f$ converges to g in the L^1-norm.

To prove the second part of the lemma, consider some subsequence of $1/n\sum_{j=0}^{n-1} P_t^j f$ converging to f_0 in the L^1-norm. Taking that subsequence playing the role of the whole sequence in the argument above we easily see that f_0 satisfies the conclusion by uniqueness of the limit. ☐

Corollary 3.4. *If h_t is the density of an absolutely continuous ϕ_t-invariant probability measure, then $h_t \in BV(R)$ and $V(h_t) \leq 4K_1$.*

Proof. Take h_t the density of an absolutely continuous ϕ_t-invariant probability measure. We have from property (C$_2$) that $P_t^j h_t = h_t$ for all $j \geq 1$. This implies that the sequence $1/n \sum_{j=0}^{n-1} P_t^j h_t$ is constant and equal to h_t, and so the result follows. ☐

Let $(t_n)_n$ be a sequence in I converging to some $t_0 \in I$. Assume that for each $n \geq 1$ we have an absolutely continuous ϕ_{t_n}-invariant probability measure μ_n and consider

$$h_n = \frac{d\mu_n}{dm}.$$

Using the fact that each h_n is the density of a probability measure and Corollary 3.4 we have for all $n \geq 1$,

$$\|h_n\|_1 = 1 \quad \text{and} \quad V(h_n) \leq 4K_1.$$

Hence, by (B$_2$) and (B$_3$) there exists $h_0 \in BV$ with $V(h_0) \leq 4K_1$ such that the sequence $(h_n)_n$ converges to h_0 in the L^1-norm. Let μ_0 be the probability measure in R whose density with respect to m is h_0. Theorem A is now a consequence of the following lemma.

Lemma 3.5. *μ_0 is a ϕ_{t_0}-invariant measure.*

Proof. Since the sequence $(h_n)_n$ converges to h_0 in the L^1-norm, it easily follows that $(\mu_n)_n$ converges to μ_0 in the weak* topology. Thus, given any $f\colon R \to \mathbb{R}$ continuous we have

$$\int f d\mu_n \longrightarrow \int f d\mu_0, \quad \text{when } n \to \infty.$$

On the other hand, since μ_n is ϕ_{t_n}-invariant we have

$$\int f d\mu_n = \int (f \circ \phi_{t_n}) d\mu_n, \quad \text{for every } n.$$

It is enough to prove that

$$\int (f \circ \phi_{t_n}) d\mu_n \longrightarrow \int (f \circ \phi_{t_0}) d\mu_0, \quad \text{when } n \to \infty.$$

We have

$$\begin{aligned}
&\left| \int (f \circ \phi_{t_n}) d\mu_n - \int (f \circ \phi_{t_0}) d\mu_0 \right| \\
&\qquad \leq \left| \int (f \circ \phi_{t_n}) d\mu_n - \int (f \circ \phi_{t_0}) d\mu_n \right| + \left| \int (f \circ \phi_{t_0}) d\mu_n - \int (f \circ \phi_{t_0}) d\mu_0 \right| \\
&\qquad \leq \int |f \circ \phi_{t_n} - f \circ \phi_{t_0}| \, d\mu_n + \left| \int (f \circ \phi_{t_0}) d\mu_n - \int (f \circ \phi_{t_0}) d\mu_0 \right| \\
&\qquad = \int |f \circ \phi_{t_n} - f \circ \phi_{t_0}| \, h_n \, dm + \left| \int (f \circ \phi_{t_0})(h_n - h_0) \, dm \right|.
\end{aligned}$$

Using (3) we easily get that each $h_n \in L^p(R)$ with $p = d/(d-1)$ and

$$\|h_n\|_p \leq C V(h_n) \leq 4CK_1.$$

Observing that $1/p + 1/d = 1$, then by Hölder's Inequality we get

$$\int |f \circ \phi_{t_n} - f \circ \phi_{t_0}|\, h_n \, dm \le \|f \circ \phi_{t_n} - f \circ \phi_{t_0}\|_d \cdot \|h_n\|_p \le 4CK_1 \|f \circ \phi_{t_n} - f \circ \phi_{t_0}\|_d,$$

and this clearly converges to zero, when $n \to +\infty$, by assumption (1) in the statement of Theorem A. On the other hand, as f is bounded we have

$$\left| \int (f \circ \phi_{t_0})(h_n - h_0)\, dm \right| \le \|f \circ \phi_{t_0}\|_\infty \cdot \|h_n - h_0\|_1$$

and this clearly converges to 0 when $n \to +\infty$ as well. □

4. TWO-DIMENSIONAL TENT MAPS

In this section we shall prove Theorem B. The idea is to obtain it as a corollary of Theorem A. As observed before, each Λ_t is *strongly transitive*: any nonempty open set becomes the whole space under a finite number of iterations by Λ_t. In our setting, this implies that the absolutely continuous Λ_t-invariant ergodic probability measure μ_t must be unique. Moreover, any power of Λ_t has a unique absolutely continuous invariant ergodic probability measure as well, which must necessarily coincide with μ_t. Thus, it is enough to obtain the statistical stability for some power of the maps in our family.

We are going to see that the family $(\Lambda_t^6)_{t\in[\tau,1]}$ is in the conditions of Theorem A. Namely, each $\Lambda_t^6 : T \to T$ is a C^2 piecewise expanding map with bounded distortion and long branches with constants $\sigma_t, D_t, \beta_t, \rho_t$ satisfying (P_1), (P_2) and (P_3), and

$$\sigma_t\left(1 + \frac{1}{\beta_t}\right) \le \lambda \quad \text{and} \quad D_t + \frac{1}{\beta_t \rho_t} + \frac{D_t}{\beta_t} \le K, \tag{5}$$

for some choice of uniform constants $0 < \lambda < 1$ and $K > 0$; observe that as the maps Λ_t^6 are continuous, then the first condition in Theorem A is trivially satisfied.

From the definition of Λ_t in (1) and (2) we obviously have that T_0 and T_1 are the domains of smoothness of Λ_t. The map Λ_t is piecewise linear with

$$D\Lambda_t(x) = \begin{pmatrix} t & t \\ t & -t \end{pmatrix}$$

for $x \in T_0 \setminus \mathcal{C}$, and

$$D\Lambda_t(x) = \begin{pmatrix} -t & t \\ -t & -t \end{pmatrix}$$

for $x \in T_1 \setminus \mathcal{C}$. From here we deduce that for all $x \in T \setminus \mathcal{C}$ we have

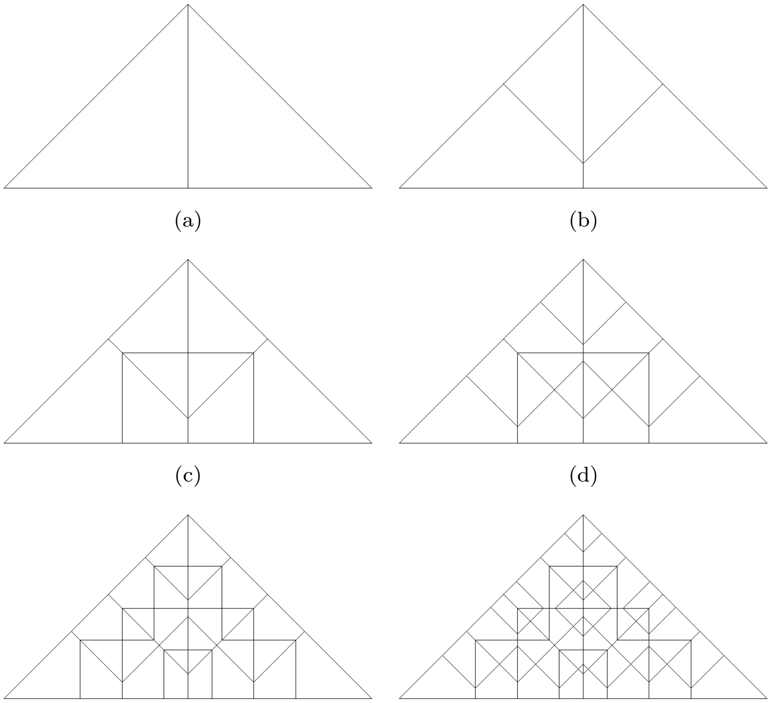
$$\|D\Lambda_t^{-1}(x)\| = \frac{1}{\sqrt{2}t}, \tag{6}$$

where $\| \quad \|$ stands for the Euclidian norm. Now take $R = T$ and $\{R_i^t\}_{i=1}^{64}$ the (Lebesgue mod 0) partition of R given by the domains of smoothness of Λ_t^6.

From (6) we easily deduce that

$$\|(D\Lambda_t^6)^{-1}(x)\| = \frac{1}{8t^6} := \sigma_t < 1$$

and so property (P_1) holds for each $t \in [\tau, 1]$; recall that $\tau \approx 0.88$. Since Λ_t^6 is linear on each domain of smoothness, then it has zero distortion. Thus we obtain property (P_2) with $D_t = 0$, for each $t \in [\tau, 1]$.



(a) (b)

(c) (d)

(e) (f)

FIGURE 1. Smoothness domains: (a) for Λ_t, (b) for Λ_t^2, (c) for Λ_t^3, (d) for Λ_t^4, (e) for Λ_t^5, (f) for Λ_t^6.

Let us now check (P_3). As each Λ_t^6 is linear on each R_i^t and preserves angles, it is enough to obtain the geometric property (P_3) for the domains R_i^t's instead of their images. Since the pre-image of the critical set $\mathcal{C}$ delimits the boundary of the domains of smoothness, it easily follows that the boundary of each R_i^t is formed by at most five straight line segments with slope -1, 0, 1 or ∞ meeting at an angle at least $\pi/4$; see Figure 1(c).

Then, it is not hard to check that for every $t \in [\tau, 1]$ and $i = 1, \dots, 64$ there is a piecewise C^1 unitary vector field X_i^t in ∂R_i^t such that

$$|\sin \angle(v, X_i^t(x))| \geq \sin \frac{\pi}{8} := \beta_t$$

for every $x \in \partial R_i^t$ and $v \in T_x \partial R_i^t \setminus \{0\}$; see Figure 2. To prove the existence of ρ_t it is enough to observe that the domains of smoothness of Λ_t^6 depend continuously on the parameter t as illustrated in Figure 3, and so it is possible to choose a uniform value of ρ such that (P_3) holds for each $t \in [\tau, 1]$.

Altogether this shows that there are $0 < \lambda < 1$ and $K > 0$ such that

$$\sigma_t \left(1 + \frac{1}{\beta_t}\right) = \frac{1}{8t^6} \left(1 + \frac{1}{\sin(\pi/8)}\right) \leq \lambda$$

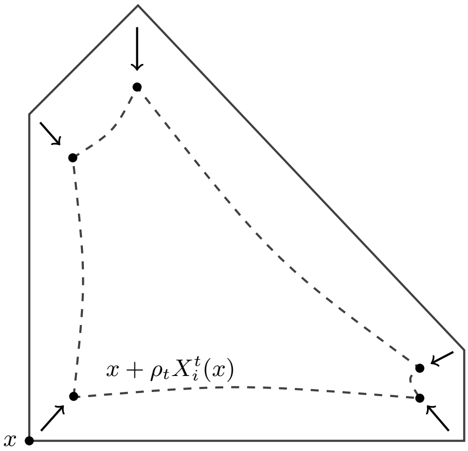


FIGURE 2. A long branch for Λ_t^6.

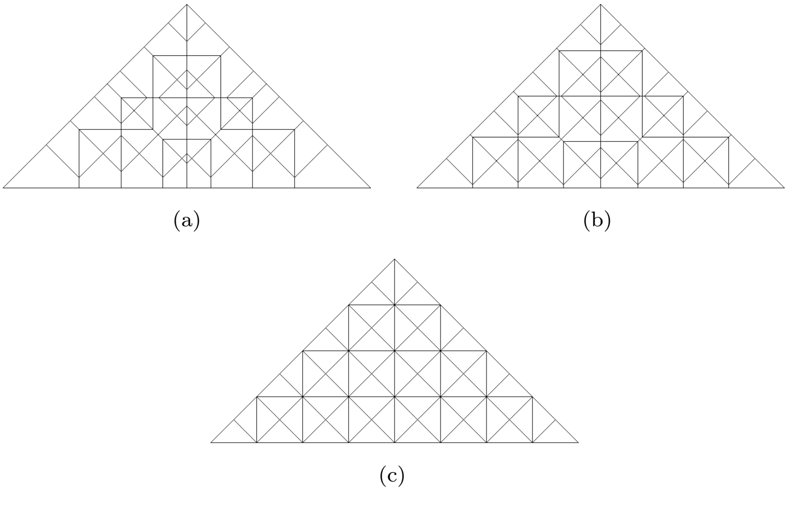


FIGURE 3. Domains of smoothness for Λ_t^6: (a) $t = \tau$, (b) $t = 0.95$, (c) $t = 1$.

for every $t \in [\tau, 1]$ (recall that $\tau \approx 0.88$) and

$$D_t + \frac{1}{\beta_t \rho_t} + \frac{D_t}{\beta_t} = \frac{1}{\rho \sin(\pi/8)} := K,$$

thus having proved (5) and hence Theorem B.

References

[1] Kourosh Adl-Zarabi, *Absolutely continuous invariant measures for piecewise expanding* C^2 *transformations in* $\mathbf{R}^n$ *on domains with cusps on the boundaries*, Ergodic Theory Dynam. Systems **16** (1996), no. 1, 1–18, DOI 10.1017/S0143385700008683. MR1375124

[2] José Ferreira Alves, *SRB measures for non-hyperbolic systems with multidimensional expansion* (English, with English and French summaries), Ann. Sci. École Norm. Sup. (4) **33** (2000), no. 1, 1–32, DOI 10.1016/S0012-9593(00)00101-4. MR1743717

[3] José F. Alves, *Strong statistical stability of non-uniformly expanding maps*, Nonlinearity **17** (2004), no. 4, 1193–1215, DOI 10.1088/0951-7715/17/4/004. MR2069701

[4] José F. Alves and Marcelo Viana, *Statistical stability for robust classes of maps with non-uniform expansion*, Ergodic Theory Dynam. Systems **22** (2002), no. 1, 1–32, DOI 10.1017/S0143385702000019. MR1889563

[5] J. Buzzi, *Absolutely continuous invariant measures for generic multi-dimensional piecewise affine expanding maps*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **9** (1999), no. 9, 1743–1750, DOI 10.1142/S021812749900122X. Discrete dynamical systems. MR1728733

[6] Jérôme Buzzi, *Absolutely continuous invariant probability measures for arbitrary expanding piecewise* $\mathbf{R}$*-analytic mappings of the plane*, Ergodic Theory Dynam. Systems **20** (2000), no. 3, 697–708, DOI 10.1017/S0143385700000377. MR1764923

[7] Lawrence C. Evans and Ronald F. Gariepy, *Measure theory and fine properties of functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992. MR1158660

[8] Enrico Giusti, *Minimal surfaces and functions of bounded variation*, Monographs in Mathematics, vol. 80, Birkhäuser Verlag, Basel, 1984. MR775682

[9] P. Góra and A. Boyarsky, *Absolutely continuous invariant measures for piecewise expanding* C^2 *transformation in* $\mathbf{R}^N$, Israel J. Math. **67** (1989), no. 3, 272–286, DOI 10.1007/BF02764946. MR1029902

[10] Paweł Góra and Abraham Boyarsky, *On functions of bounded variation in higher dimensions*, Amer. Math. Monthly **99** (1992), no. 2, 159–160, DOI 10.2307/2324185. MR1144357

[11] Gerhard Keller, *Ergodicité et mesures invariantes pour les transformations dilatantes par morceaux d'une région bornée du plan* (French, with English summary), C. R. Acad. Sci. Paris Sér. A-B **289** (1979), no. 12, A625–A627. MR556443

[12] A. Lasota and James A. Yorke, *On the existence of invariant measures for piecewise monotonic transformations*, Trans. Amer. Math. Soc. **186** (1973), 481–488 (1974). MR0335758

[13] Antonio Pumariño, José Ángel Rodríguez, Joan Carles Tatjer, and Enrique Vigil, *Expanding Baker maps as models for the dynamics emerging from 3D-homoclinic bifurcations*, Discrete Contin. Dyn. Syst. Ser. B **19** (2014), no. 2, 523–541, DOI 10.3934/dcdsb.2014.19.523. MR3170197

[14] Antonio Pumariño, José Ángel Rodríguez, Joan Carles Tatjer, and Enrique Vigil, *Chaotic dynamics for two-dimensional tent maps*, Nonlinearity **28** (2015), no. 2, 407–434, DOI 10.1088/0951-7715/28/2/407. MR3303174

[15] Antonio Pumariño and Joan Carles Tatjer, *Dynamics near homoclinic bifurcations of three-dimensional dissipative diffeomorphisms*, Nonlinearity **19** (2006), no. 12, 2833–2852, DOI 10.1088/0951-7715/19/12/006. MR2273761

[16] Antonio Pumariño and Joan Carles Tatjer, *Attractors for return maps near homoclinic tangencies of three-dimensional dissipative diffeomorphisms*, Discrete Contin. Dyn. Syst. Ser. B **8** (2007), no. 4, 971–1005 (electronic), DOI 10.3934/dcdsb.2007.8.971. MR2342131

[17] Benoît Saussol, *Absolutely continuous invariant measures for multidimensional expanding maps*, Israel J. Math. **116** (2000), 223–248, DOI 10.1007/BF02773219. MR1759406

[18] Joan Carles Tatjer, *Three-dimensional dissipative diffeomorphisms with homoclinic tangencies*, Ergodic Theory Dynam. Systems **21** (2001), no. 1, 249–302, DOI 10.1017/S0143385701001146. MR1826668

[19] Masato Tsujii, *Absolutely continuous invariant measures for piecewise real-analytic expanding maps on the plane*, Comm. Math. Phys. **208** (2000), no. 3, 605–622, DOI 10.1007/s002200050003. MR1736328

[20] Masato Tsujii, *Absolutely continuous invariant measures for expanding piecewise linear maps*, Invent. Math. **143** (2001), no. 2, 349–373, DOI 10.1007/PL00005797. MR1835391

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