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THE MONOTONICITY AND CONVEXITY FOR THE RATIOS OF MODIFIED BESSEL FUNCTIONS OF THE SECOND KIND AND APPLICATIONS

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This paper is dedicated to the 60th anniversary of Zhejiang Normal University

Abstract. Let $K_v(x)$ be the modified Bessel functions of the second kind of order v. We prove that the function $x \mapsto K_u(x) K_v(x) / K_{(u+v)/2}(x)^2$ is strictly decreasing on $(0, \infty)$. Our study not only involves the Turán type inequalities, log-convexity or log-concavity of $K_v(x)$, and the conjecture posed by Baricz, but also yields various new results concerning the monotonicity and convexity of the ratios of the modified Bessel functions of the second kind. As applications of our main theorems, some new sharp inequalities involving $K_v(x)$ are presented, which contain sharp estimates for $K_v(x)$ and sharp bounds for the ratios $K_v'(x)/K_v(x)$ and $K_{v+1}(x)/K_v(x)$.

1. Introduction

It would be difficult to overestimate the importance of the Bessel equation and its solutions, the Bessel functions, which arise in a huge variety of problems widely used in mathematics, engineering, and physics. It is an essential ingredient under many circumstances to investigate some related properties of Bessel functions. It is well-known that the modified Bessel function of the first kind of order v, denoted by $I_v(x)$, is a particular solution of the second-order differential equation [1, p. 77]

$$x^2 y''(x) + x y'(x) - \left(x^2 + v^2\right) y(x) = 0, \tag{1.1}$$

which is explicitly represented by

$$I_v(x) = \sum_{n=0}^{\infty} \frac{(x/2)^{2n+v}}{n! \Gamma(v+n+1)}, \quad x \in \mathbb{R}, \ v \in \mathbb{R} \backslash \{-1, -2, \cdots\}. \tag{1.2}$$

The modified Bessel functions of the second kind K_v, also called the MacDonald function [1, p. 78], is defined by

$$K_v(x) = \frac{\pi}{2} \frac{I_{-v}(x) - I_v(x)}{\sin v\pi},$$

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where the right-hand side of the formula above is replaced by its limiting value if v is an integer. Note that $K_v(x)$ has the integral representation ([1, p. 181])

$$K_v(x) = \int_0^\infty e^{-x\cosh t} \cosh(vt)\, dt.$$

It is easy to verify that $K_{-v}(x) = K_v(x)$, $K_v(x) > 0$, and $K_v'(x) < 0$ for all $v \in \mathbb{R}, x > 0$.

In recent decades, Turán type inequalities for special functions, including modified Bessel functions, have attracted the interest of many mathematicians and were rediscovered many times by many authors in different forms; for details see [2–12]. Specifically, it is worth mentioning that existing Turán type inequalities for the modified Bessel functions appeared in many problems concerning probability and statistics [13–15], chemistry [16], physics [17,18], engineering sciences [19] and references therein.

The Turán type inequalities for the modified Bessel functions of the second kind state that, for $v > 1$, the double inequality

$$\frac{1}{1-v} K_v(x)^2 < K_v(x)^2 - K_{v-1}(x) K_{v+1}(x) < 0 \tag{1.3}$$

holds for $x > 0$. The right-hand side of (1.3) for $v \in \mathbb{R}$ was first proved independently by Ismail and Muldoon [20, Lemma 2.2] and van Haeringen [17], respectively. For $v > 1/2$ it was deduced by Laforgia and Natalini [3, Eq. (2.18)] in 2006. Another proof of the right-hand side of (1.3) for all $v \in \mathbb{R}$ was given in [10]. Moreover, Baricz [21] and Segura [22] proved the double inequality (1.3) by using rather different approaches, respectively. Recently, Segura [22, Theorem 10] and Baricz in [23, Theorem 2] obtained some further bounds of the double inequality (1.3) for certain $v \in \mathbb{R}$.

The Turán type inequalities are closely related to the log-convexity or log-concavity. Ismail and Muldoon [20, Lemma 2.2] in 1978 achieved that, for all fixed $x > 0$ and $\beta > 0$, the function $v \mapsto K_{v+\beta}(x)/K_v(x)$ is increasing on $\mathbb{R}$. This actually implies that $v \mapsto K_v(x)$ is log-convex on $\mathbb{R}$. Later, by the classical Hölder-Rogers inequality, Baricz [10] pointed out that the function $v \mapsto K_v(x)$ is strictly log-convex on $\mathbb{R}$ for all $x > 0$. Furthermore, Baricz [21, Conjecture 3.2] conjectured that the function

$$v \mapsto \frac{2^{v-1}\Gamma(v)\, x^{-v}}{K_v(x)}$$

is strictly log-convex on $(0, \infty)$ for all $x > 0$. For more details related to Turán type inequalities and log-convexity or log-concavity see [23,24] and references therein.

Clearly, the fact that $v \mapsto K_v(x)$ is strictly log-convex on $\mathbb{R}$ is equivalent to

$$1 < \frac{K_u(x) K_v(x)}{K_{(u+v)/2}(x)^2} \quad \text{for all } u, v \in \mathbb{R}.$$

If Bacricz's conjecture is true, then for any $u, v > 0$ we get

$$\frac{2^{u-1}\Gamma(u)\, x^{-u}}{K_u(x)} \frac{2^{v-1}\Gamma(v)\, x^{-v}}{K_v(x)} > \left[\frac{2^{(u+v)/2-1}\Gamma((u+v)/2)\, x^{-(u+v)/2}}{K_{(u+v)/2}(x)}\right]^2,$$

which is equivalent to

$$\frac{K_u(x)K_v(x)}{K_{(u+v)/2}(x)^2}<\frac{\Gamma(u)\Gamma(v)}{\Gamma((u+v)/2)^2}\qquad \text{for all } u,v>0.$$

By the asymptotic formulas [25, p. 375 and p. 378]

$$K_v(x)\sim\frac{1}{2}\Gamma(v)\left(\frac{x}{2}\right)^{-v},\qquad \text{as } x\to 0 \text{ for all } v>0, \tag{1.4}$$

$$K_v(x)\sim\sqrt{\frac{\pi}{2x}}e^{-x},\qquad \text{as } x\to\infty, \tag{1.5}$$

we obtain

$$\lim_{x\to 0}\frac{K_u(x)K_v(x)}{K_{(u+v)/2}(x)^2}=\frac{\Gamma(u)\Gamma(v)}{\Gamma((u+v)/2)^2}\qquad \text{for all } u,v>0, \tag{1.6}$$

$$\lim_{x\to\infty}\frac{K_u(x)K_v(x)}{K_{(u+v)/2}(x)^2}=1\qquad \text{for all } u,v\in\mathbb{R}. \tag{1.7}$$

On the basis of the considerations above, we further make a conjecture that the ratio

$$x\mapsto\frac{K_u(x)K_v(x)}{K_{(u+v)/2}(x)^2}:=\mathcal{K}_{u,v}(x) \tag{1.8}$$

is strictly decreasing on $(0,\infty)$. Obviously, if our conjecture holds, then it yields the log-convexity of functions $v\mapsto K_v(x)$ on $\mathbb{R}$ and $v\mapsto 2^{v-1}\Gamma(v)x^{-v}/K_v(x)$ on $(0,\infty)$, respectively. As a direct consequence, Bacricz's conjecture is also true. Furthermore, by replacing (u,v) with $(v-1,v+1)$ we see that for $v>1$ the function $x\mapsto K_{v-1}(x)K_{v+1}(x)/K_v(x)^2$ is strictly decreasing from $(0,\infty)$ onto $(1,v/(v-1))$. Consequently, we have

$$1<\frac{K_{v-1}(x)K_{v+1}(x)}{K_v(x)^2}<\frac{v}{v-1}, \tag{1.9}$$

which is equivalent to Turán type inequalities (1.3) with the best constants 0 and $1/(1-v)$.

Motivated by the observations mentioned above, the main aim of this paper is to prove our conjecture. From this, we can deduce the monotonicity and convexity (log-convexity) of the ratios

$$v\mapsto\frac{K_v'(x)}{K_v(x)},\quad\frac{K_{v-1}(x)}{K_v(x)},\quad\frac{K_{v+1}(x)}{K_v(x)},\quad\frac{K_v(x)}{K_v(y)} \tag{1.10}$$

and monotonicity of the ratios

$$x\mapsto\frac{K_u(x)^pK_v(x)^q}{K_{pu+qv}(x)},\quad\frac{K_{u+a}(x)}{K_u(x)}\frac{K_v(x)}{K_{v+a}(x)}, \tag{1.11}$$

respectively.

The rest of this paper is organized as follows. Our main results are stated and proved in Section 2. As applications, some sharp estimates for $K_v(x)$ are presented in Section 3. In the last section, some sharp bounds for certain ratios listed in (1.10) are given.

2. Main results and proofs

In this section, we are devoted to giving and proving our main results. To this objective, let us recall the following useful lemmas.

Lemma 2.1 ([1, p. 440]). *Let $K_v(x)$ be the modified Bessel functions of the second kind. Then $K_u(x)K_v(x)$ has the integral representation*

$$K_u(x)K_v(x) = 2\int_0^\infty K_{u\pm v}(2x\cosh t)\cosh[(u\mp v)t]\,dt. \tag{2.1}$$

Lemma 2.2 ([13, (1.3)], [27, (2.1)]). *Let $K_v(x)$ be the modified Bessel functions of the second kind. Then for $x>0$ and $v\geq 0$, we have the integral representation*

$$\frac{K_{v-1}(x)}{K_v(x)} = \frac{4}{\pi^2}\int_0^\infty \frac{x\,dt}{t(x^2+t^2)\left(J_v^2(t)+Y_v^2(t)\right)}, \tag{2.2}$$

where J_v and Y_v stand for the Bessel functions of the first and second kind, respectively.

Remark 2.3. The above lemma is crucial to prove the next lemma. More integral representations for ratios of Bessel functions of the second kind can be found in [28].

Lemma 2.4. *Let $K_v(x)$ be the modified Bessel functions of the second kind. Then, for fixed $v\in\mathbb{R}$, the function*

$$x\mapsto \frac{xK_v'(x)}{K_v(x)} \tag{2.3}$$

is strictly decreasing on $(0,\infty)$.

Proof. Making use of the recurrence formula [1, p. 79]

$$xK_v'(x)+vK_v(x) = -xK_{v-1}(x) \tag{2.4}$$

and Lemma 2.2, we get

$$\frac{xK_v'(x)}{K_v(x)} = -v-\frac{xK_{v-1}(x)}{K_v(x)} = -v-\frac{4}{\pi^2}\int_0^\infty \frac{x^2\,dt}{t(x^2+t^2)\left(J_v^2(t)+Y_v^2(t)\right)}.$$

Differentiation yields

$$\left(\frac{xK_v'(x)}{K_v(x)}\right)' = -\frac{8}{\pi^2}\int_0^\infty \frac{xt\,dt}{(x^2+t^2)^2\left(J_v^2(t)+Y_v^2(t)\right)}<0,$$

which proves this lemma. □

Now, we are in a position to state and prove our main results.

Theorem 2.5. *For fixed $u,v\in\mathbb{R}$, the function $x\mapsto \mathcal{K}_{u,v}(x)$ defined by (1.8) is strictly decreasing on $(0,\infty)$. Moreover, for any $x>0$, we have the inequalities*

$$1<\frac{K_u(x)K_v(x)}{K_{(u+v)/2}(x)^2} \quad \text{if } u,v\in\mathbb{R}, \tag{2.5}$$

$$1<\frac{K_u(x)K_v(x)}{K_{(u+v)/2}(x)^2}<\frac{\Gamma(u)\Gamma(v)}{\Gamma\Big((u+v)/2\Big)^2} \quad \text{if } u,v>0, \tag{2.6}$$

where the lower and upper bounds are sharp.

Proof. By the formula (2.1) we see that

$$K_{(u+v)/2}(x)^2 = 2\int_0^\infty K_{u+v}(2x\cosh t)\,dt$$

and

$$\mathcal{K}_{u,v}(x) = \frac{K_u(x)K_v(x)}{K_{(u+v)/2}(x)^2} = \frac{\int_0^\infty K_{u+v}(2x\cosh t)\cosh[(u-v)t]\,dt}{\int_0^\infty K_{u+v}(2x\cosh t)\,dt}.$$

Differentiation yields

$$\mathcal{K}'_{u,v}(x) = \frac{F(x)}{\left(\int_0^\infty K_{u+v}(2x\cosh t)\,dt\right)^2},$$

where

$$\begin{aligned} F(x) &= \int_0^\infty (2\cosh t)\,K'_{u+v}(2x\cosh t)\cosh[(u-v)t]\,dt \int_0^\infty K_{u+v}(2x\cosh t)\,dt \\ &\quad - \int_0^\infty K_{u+v}(2x\cosh t)\cosh[(u-v)t]\,dt \int_0^\infty (2\cosh t)\,K'_{u+v}(2x\cosh t)\,dt. \end{aligned}$$

Let

$$x_t = 2x\cosh t \quad \text{and} \quad h(t) = \cosh[(u-v)t] = \cosh(|u-v|t).$$

Then we get

(2.7)
$$F(x) = \frac{1}{x}\int_0^\infty\int_0^\infty \Big(x_sK'_{u+v}(x_s)K_{u+v}(x_t) - x_tK'_{u+v}(x_t)K_{u+v}(x_s)\Big)h(s)\,dsdt.$$

According to the symmetry of variables of the double integral s and t, by exchanging s and t it yields

(2.8)
$$F(x) = \frac{1}{x}\int_0^\infty\int_0^\infty \Big(x_tK'_{u+v}(x_t)K_{u+v}(x_s) - x_sK'_{u+v}(x_s)K_{u+v}(x_t)\Big)h(t)\,dsdt.$$

Adding (2.7) and (2.8) leads to

$$\begin{aligned} &2F(x) \\ &= \frac{1}{x}\int_0^\infty\int_0^\infty \Big(K_{u+v}(x_s)K_{u+v}(x_t)\left(\tfrac{x_sK'_{u+v}(x_s)}{K_{u+v}(x_s)} - \tfrac{x_tK'_{u+v}(x_t)}{K_{u+v}(x_t)}\right)(h(s)-h(t))\Big)dsdt \end{aligned}$$

or, equivalently,

$$\begin{aligned} F(x) &= \int_0^\infty\int_0^\infty \Big(K_{u+v}(x_s)K_{u+v}(x_t)\left(\frac{x_sK'_{u+v}(x_s)}{K_{u+v}(x_s)} - \frac{x_tK'_{u+v}(x_t)}{K_{u+v}(x_t)}\right)\frac{1}{x_s-x_t} \\ &\quad \times(\cosh s - \cosh t)(h(s)-h(t))\Big)dsdt. \end{aligned}$$

Since $K_v(x) > 0$ for $x > 0$, both the functions $t \mapsto \cosh t$, $h(t)$ are strictly increasing on $(0,\infty)$, and $x \mapsto xK'_v(x)/K_v(x)$ is strictly decreasing on $(0,\infty)$ by Lemma 2.4, we get that $F(x) < 0$ for all $x > 0$, which proves the decreasing property of $x \mapsto \mathcal{K}_{u,v}(x)$ on $(0,\infty)$ for fixed $u, v \in \mathbb{R}$.

Furthermore, by taking into account (1.6) and (1.7), Theorem 2.5 is proved. $\square$

The following theorem is a direct consequence of Theorem 2.5.

Theorem 2.6. *The following statements are true:*

(i) The function $v \mapsto K_v'(x)/K_v(x)$ *is strictly concave on* $\mathbb{R}$ *and decreasing on* $[0,\infty)$*, while both* $v \mapsto K_{v-1}(x)/K_v(x)$ *and* $v \mapsto K_{v+1}(x)/K_v(x)$ *are convex on* $\mathbb{R}$.

(ii) For fixed $a>0$*, the function* $v \mapsto K_{v+a}'(x)/K_{v+a}(x) - K_v'(x)/K_v(x)$ *is strictly decreasing on* $\mathbb{R}$*, while both* $v \mapsto K_{v+a-1}(x)/K_{v+a}(x) - K_{v-1}(x)/K_v(x)$ *and* $v \mapsto K_{v+a+1}(x)/K_{v+a}(x) - K_{v+1}(x)/K_v(x)$ *are strictly increasing on* $\mathbb{R}$.

(iii) For $y>x>0$*, the function* $v \mapsto K_v(x)/K_v(y)$ *is log-convex on* $\mathbb{R}$.

Proof. (i) Theorem 2.5 tells us that $x \mapsto \ln \mathcal{K}_{u,v}(x)$ is strictly decreasing on $(0,\infty)$, which implies that

$$\Big(\ln \mathcal{K}_{u,v}(x)\Big)' = \frac{K_u'(x)}{K_u(x)} + \frac{K_v'(x)}{K_v(x)} - 2\frac{K_{(u+v)/2}'(x)}{K_{(u+v)/2}(x)} < 0.$$

This also indicates that the function $v \mapsto K_v'(x)/K_v(x)$ is strictly concave on $\mathbb{R}$.

Suppose that $u>v>0$. By the property of concave functions we get

$$\frac{1}{u-v}\left[\frac{K_u'(x)}{K_u(x)} - \frac{K_v'(x)}{K_v(x)}\right] < \frac{1}{-v-v}\left[\frac{K_{-v}'(x)}{K_{-v}(x)} - \frac{K_v'(x)}{K_v(x)}\right] = 0,$$

which implies that $v \mapsto K_v'(x)/K_v(x)$ is strictly decreasing on $[0,\infty)$.

By the recurrence formula (2.4) and the following formula [1, p. 79],

$$xK_v'(x) - vK_v(x) = -xK_{v+1}(x), \tag{2.9}$$

we get

$$\frac{K_v'(x)}{K_v(x)} = -\frac{K_{v-1}(x)}{K_v(x)} - \frac{v}{x} = -\frac{K_{v+1}(x)}{K_v(x)} + \frac{v}{x}, \tag{2.10}$$

which shows that $v \mapsto K_{v-1}(x)/K_v(x)$ and $v \mapsto K_{v+1}(x)/K_v(x)$ have the opposite convexity in accordance with the convexity of $v \mapsto K_v'(x)/K_v(x)$ on $\mathbb{R}$.

(ii) Obviously the decreasing property of $v \mapsto K_{v+a}'(x)/K_{v+a}(x) - K_v'(x)/K_v(x)$ on $\mathbb{R}$ follows from the concavity of the function $v \mapsto K_v'(x)/K_v(x)$ on $\mathbb{R}$, while the increasing properties of $v \mapsto K_{v+a-1}(x)/K_{v+a}(x) - K_{v-1}(x)/K_v(x)$ and $v \mapsto K_{v+a+1}(x)/K_{v+a}(x) - K_{v+1}(x)/K_v(x)$ are direct consequences of the convexity of $v \mapsto K_{v-1}(x)/K_v(x)$ and $v \mapsto K_{v+1}(x)/K_v(x)$ on $\mathbb{R}$, respectively.

(iii) It follows from Theorem 2.5 that for $u,v \in \mathbb{R}$ the inequality $\mathcal{K}_{u,v}(x) > \mathcal{K}_{u,v}(y)$ holds if $y>x>0$, which is equivalent to

$$\frac{K_u(x)}{K_u(y)}\frac{K_v(x)}{K_v(y)} > \left[\frac{K_{(u+v)/2}(x)}{K_{(u+v)/2}(y)}\right]^2 \quad \text{for } y>x>0.$$

This means that $v \mapsto K_v(x)/K_v(y)$ is log-convex on $\mathbb{R}$, and the proof is complete. □

Theorem 2.7. *For fixed* $u,v,p,q \in \mathbb{R}$ *with* $p+q=1$*, the function*

$$x \mapsto \frac{K_u(x)^p K_v(x)^q}{K_{pu+qv}(x)} := \Psi_{u,v}(x) \tag{2.11}$$

is strictly decreasing (increasing) on $(0,\infty)$ *for* $pq > \Big(< \Big) 0$. *Consequently, for any* $x>0$*, we have the following inequalities:*

$$1 < \Big(> \Big) \frac{K_u(x)^p K_v(x)^q}{K_{pu+qv}(x)} \quad \textit{if } u,v \in \mathbb{R},$$
$$1 < \Big(> \Big) \frac{K_u(x)^p K_v(x)^q}{K_{pu+qv}(x)} < \Big(> \Big) \frac{\Gamma(u)^p \Gamma(v)^q}{\Gamma(pu+qv)} \quad \textit{if } u,v > 0,$$

where the lower and upper bounds are sharp.

Proof. From Theorem 2.6 we see that the function $v \mapsto K_v'(x)/K_v(x)$ is strictly concave on $\mathbb{R}$. Hence, by the property of concave functions, for $u, v, pu+qv \in \mathbb{R}$ and $pq > 0$ with $p+q = 1$ we get

$$(\ln \Psi_{u,v}(x))' = p\frac{K_u'(x)}{K_u(x)} + q\frac{K_v'(x)}{K_v(x)} - \frac{K_{pu+qv}'(x)}{K_{pu+qv}(x)} < 0,$$

which implies that the function $x \mapsto \Psi_{u,v}(x)$ is strictly decreasing on $(0,\infty)$ for $pq > 0$.

For $pq < 0$, without loss of generality we can assume that $p > 0$ and $q < 0$. By making a change of variable $p^* = -q/p$ and $q^* = 1/p$, we see that p^* and q^* satisfy $p^*, q^* > 0$ with $p^* + q^* = 1$ and $p^* v + q^*(pu+qv) = u$. Therefore, we have

$$\begin{aligned}(\ln \Psi_{u,v}(x))' &= -p\left(-\frac{K_u'(x)}{K_u(x)} - \frac{q}{p}\frac{K_v'(x)}{K_v(x)} + \frac{1}{p}\frac{K_{pu+qv}'(x)}{K_{pu+qv}(x)}\right) \\ &= -p\left(p^*\frac{K_v'(x)}{K_v(x)} + q^*\frac{K_{pu+qv}'(x)}{K_{pu+qv}(x)} - \frac{K_u'(x)}{K_u(x)}\right) > 0,\end{aligned}$$

which means that the function $x \mapsto \Psi_{u,v}(x)$ is strictly increasing on $(0,\infty)$ for $pq < 0$.

Using the asymptotic formulas (1.4) and (1.5) yields

$$\Psi_{u,v}(0) = \frac{\Gamma(u)^p \Gamma(v)^q}{\Gamma(pu+qv)} \quad \text{for } u,v > 0 \;\text{ and }\; \Psi_{u,v}(\infty) = 1 \text{ for } u,v \in \mathbb{R},$$

which completes the proof. □

Undoubtedly, Theorem 2.7 can be further generalized as follows.

Theorem 2.8. *Let* $v_k \in \mathbb{R}$ *and* $p_k > 0$ *for* $k = 1, 2, \cdots, n$ *with* $\sum_{k=1}^n p_k = 1$ *and* $\bar{v} = \sum_{k=1}^n p_k v_k$. *Then the function*

$$x \mapsto \frac{\prod_{k=1}^n K_{v_k}(x)^{p_k}}{K_{\bar{v}}(x)}$$

is decreasing on $(0,\infty)$. *Moreover, for* $x \in (0,\infty)$ *we have*

$$1 \le \frac{\prod_{k=1}^n K_{v_k}(x)^{p_k}}{K_{\bar{v}}(x)} \quad \textit{for all } v_k \in \mathbb{R},$$
$$1 \le \frac{\prod_{k=1}^n K_{v_k}(x)^{p_k}}{K_{\bar{v}}(x)} \le \frac{\prod_{k=1}^n \Gamma(v_k)^{p_k}}{\Gamma(\bar{v})} \quad \textit{for all } v_k > 0,$$

where the lower and upper bounds are sharp. The equalities are valid if and only if all v_k *are equal.*

Theorem 2.9. *Let $u, v \in \mathbb{R}$ with $u > v$ and $a > 0$. Then the function*

$$x \mapsto \frac{K_{u+a}(x)}{K_u(x)} \frac{K_v(x)}{K_{v+a}(x)} := \Xi_{u,v}(x)$$

is strictly decreasing on $(0, \infty)$. Moreover, for $u, v > 0$ with $u > v$, we have

$$1 < \frac{K_{u+a}(x)}{K_u(x)} \frac{K_v(x)}{K_{v+a}(x)} < \frac{\Gamma(a+u)\Gamma(v)}{\Gamma(a+v)\Gamma(u)}$$

for $x > 0$, where both the lower and upper bounds are sharp too.

Proof. By part (ii) of Theorem 2.6, we get

$$\Big(\ln \Xi_{u,v}(x)\Big)' = \frac{K'_{u+a}(x)}{K_{u+a}(x)} - \frac{K'_u(x)}{K_u(x)} - \left[\frac{K'_{v+a}(x)}{K_{v+a}(x)} - \frac{K'_v(x)}{K_v(x)}\right] < 0,$$

which implies that the function $x \mapsto \Xi_{u,v}(x)$ is strictly decreasing on $(0, \infty)$. A direct computation gives

$$\Xi_{u,v}(0) = \frac{\Gamma(a+u)\Gamma(v)}{\Gamma(a+v)\Gamma(u)} \quad \text{for } u, v > 0 \text{ and } \ \Xi_{u,v}(\infty) = 1 \ \text{ for } u, v \in \mathbb{R},$$

which completes the proof. □

Remark 2.10. Note that

$$\lim_{a \to 0^+} \frac{1}{a} \ln\left(\frac{K_{u+a}(x)}{K_u(x)} \frac{K_v(x)}{K_{v+a}(x)}\right) = \frac{\partial \ln K_u(x)}{\partial u} - \frac{\partial \ln K_v(x)}{\partial v},$$
$$\lim_{a \to 0^+} \frac{1}{a} \ln \frac{\Gamma(a+u)\Gamma(v)}{\Gamma(a+v)\Gamma(u)} = \psi(u) - \psi(v),$$

where ψ is the psi function. Hence, Theorem 2.9 implies that for $u, v \in \mathbb{R}$ with $u > v$, both the functions

$$x \mapsto \frac{\partial \ln K_u(x)}{\partial u} - \frac{\partial \ln K_v(x)}{\partial v} \quad \text{and} \quad x \mapsto \frac{\partial^2 \ln K_v(x)}{\partial v^2}$$

are strictly decreasing on $(0, \infty)$. Therefore, for $u > v > 0$ and $x > 0$ we have

$$0 < \frac{\partial \ln K_u(x)}{\partial u} - \frac{\partial \ln K_v(x)}{\partial v} < \psi(u) - \psi(v),$$
$$0 < \frac{\partial^2 \ln K_v(x)}{\partial v^2} < \psi'(v).$$

3. Sharp estimates for $K_v(x)$

By Rayleigh type formulas [25, p. 445, (10.2.24), (10.2.25)], we obtain that for $n = 0, 1, 2, \cdots$,

$$I_{n-1/2}(x) = \sqrt{\frac{2}{\pi}} x^{n-1/2} \left(\frac{1}{x}\frac{d}{dx}\right)^n (\cosh x),$$
$$I_{-n+1/2}(x) = \sqrt{\frac{2}{\pi}} x^{n-1/2} \left(\frac{1}{x}\frac{d}{dx}\right)^n (\sinh x),$$

which imply that

$$\begin{aligned} K_{n-1/2}(x) &= \frac{\pi}{2}\frac{I_{-n+1/2}(x)-I_{n-1/2}(x)}{\sin(n\pi-\pi/2)} \\ &= \sqrt{\frac{\pi}{2}}(-1)^{n-1}x^{n-1/2}\left(\frac{1}{x}\frac{d}{dx}\right)^n(\sinh x-\cosh x) \\ &= \sqrt{\frac{\pi}{2}}(-1)^{n}x^{n-1/2}\left(\frac{1}{x}\frac{d}{dx}\right)^n\left(e^{-x}\right). \end{aligned}$$

In particular,

$$K_{1/2}(x)=\sqrt{\frac{\pi}{2x}}e^{-x},\qquad K_{3/2}(x)=\sqrt{\frac{\pi}{2x}}\frac{x+1}{x}e^{-x}. \tag{3.1}$$

These show that the modified Bessel functions of half-integer order are elementary functions. The main aim of this section is to present some sharp bounds for $K_v(x)$ by using $K_{n-1/2}(x)$ and $K_{n+1/2}(x)$.

Now, putting $(u,v)=(n-1/2,n+1/2)$ and $(p,q)=(n-v+1/2,v-n+1/2)$ into Theorem 2.7 we obtain

Proposition 3.1. *For $n=0,1,2,\cdots$, the function*

$$x\mapsto\left(\frac{K_{n+1/2}(x)}{K_{n-1/2}(x)}\right)^{n-v}\frac{K_v(x)}{\sqrt{K_{n-1/2}(x)K_{n+1/2}(x)}}$$

is strictly increasing on $(0,\infty)$ for $v\in(n-1/2,n+1/2)$ and decreasing on $(0,\infty)$ for $v\in(-\infty,n-1/2)\cup(n+1/2,\infty)$. Furthermore, while $v\in(n-1/2,n+1/2)$ with $n\in\mathbb{N}$, the double inequality

$$\begin{aligned} &\frac{(n-1/2)^{n-v+1/2}\Gamma(v)}{\Gamma(n+1/2)}\left(\frac{K_{n-1/2}(x)}{K_{n+1/2}(x)}\right)^{n-v}\sqrt{K_{n-1/2}(x)K_{n+1/2}(x)} \\ <\;& K_v(x)<\left(\frac{K_{n-1/2}(x)}{K_{n+1/2}(x)}\right)^{n-v}\sqrt{K_{n-1/2}(x)K_{n+1/2}(x)} \end{aligned}$$

holds for $x\in(0,\infty)$, where both the lower and upper bounds are the best. It is reversed while $v\in(0,n-1/2)\cup(n+1/2,\infty)$ with $n\in\mathbb{N}$.

By taking $n=0,1$ in Proposition 3.1 we arrive at the following assertion.

Corollary 3.2. *For $v\in(-1/2,1/2)$, the function $x\mapsto\sqrt{x}e^xK_v(x)$ is strictly increasing on $(0,\infty)$, while for $v\in(-\infty,-1/2)\cup(1/2,\infty)$ it is strictly decreasing on $(0,\infty)$. Moreover, if $v\in(-1/2,1/2)$, then the inequality*

$$K_v(x)<\sqrt{\frac{\pi}{2x}}e^{-x}$$

holds for $x\in(0,\infty)$. The inequality above is reversed if $v\in(-\infty,-1/2)\cup(1/2,\infty)$.

Remark 3.3. It was first proved in [29] that $x\mapsto\sqrt{x}e^xK_v(x)$ is strictly monotonic on $(0,\infty)$ for all $|v|>1/2$. In particular, we would like to remark that Baricz in [30] pointed out that for all $|v|>1/2$ and $0<x<y$ it holds that

$$\frac{K_v(x)}{K_v(y)}>e^{y-x}\left(\frac{y}{x}\right)^{1/2}, \tag{3.2}$$

which is better than the previous results [31–34]. Furthermore, Baricz conjectured that $x\mapsto\sqrt{x}e^xK_v(x)$ for all $|v|<1/2$ is a Bernstein function, which means that its derivative with respect to x is strictly completely monotonic on $(0,\infty)$ for all $|v|<1/2$, and therefore, the inequality (3.2) is reversed if $|v|<1/2$. Obviously, Corollary 3.2 gives a positive answer to the latter conjecture posted by Baricz.

By the way, we would like to point out that the former inequality of Baricz's conjecture is realized by way of the following formula [35]:

$$K_v(x)=\frac{\sqrt{x}e^{-x}}{\Gamma(v+1/2)\Gamma(1/2-v)}\int_0^\infty \frac{e^{-t}}{(x+t)\sqrt{t}}K_v(t)\,dt,$$

where $x>0$ and $v\in(0,1/2)$. Indeed, we have

$$\sqrt{x}e^{x}K_v(x)=\frac{1}{\Gamma(v+1/2)\Gamma(1/2-v)}\int_0^\infty \frac{x}{x+t}\frac{e^{-t}K_v(t)}{\sqrt{t}}dt,$$

$$\begin{aligned}\left[\sqrt{x}e^{x}K_v(x)\right]' &= \frac{1}{\Gamma(v+1/2)\Gamma(1/2-v)}\int_0^\infty \frac{\sqrt{t}e^{-t}K_v(t)}{(x+t)^2}dt>0,\\ (-1)^n\left[\left(\sqrt{x}e^{x}K_v(x)\right)'\right]^{(n)} &= \frac{(n+1)!}{\Gamma(v+1/2)\Gamma(1/2-v)}\int_0^\infty \frac{\sqrt{t}e^{-t}K_v(t)}{(x+t)^{n+2}}dt>0.\end{aligned}$$

Corollary 3.4. *For $v\in(1/2,3/2)$, the function*

$$x\mapsto\left(\frac{x}{x+1}\right)^v\sqrt{x+1}e^{x}K_v(x)$$

is strictly increasing on $(0,\infty)$, while for $v\in(-\infty,1/2)\cup(3/2,\infty)$ it is strictly decreasing on $(0,\infty)$. Consequently, if $v\in(1/2,3/2)$, then for $x\in(0,\infty)$ we have

$$2^{v-1}\Gamma(v)\frac{(1+1/x)^v}{\sqrt{x+1}}e^{-x}<K_v(x)<\sqrt{\frac{\pi}{2}}\frac{(1+1/x)^v}{\sqrt{x+1}}e^{-x}$$

with the best coefficients $2^{v-1}\Gamma(v)$ and $\sqrt{\pi/2}$. If $v\in(0,1/2)\cup(3/2,\infty)$, then the double inequality above is reversed.

Remark 3.5. Letting $v=0$ in Corollary 3.4, we see that $x\mapsto\sqrt{x+1}e^{x}K_0(x)$ is strictly decreasing on $(0,\infty)$ and for $x>0$ with the estimate

$$K_0(x)>\sqrt{\frac{\pi}{2}}\frac{e^{-x}}{\sqrt{x+1}}. \tag{3.3}$$

Let $v=0$ in Corollary 3.2; for $x>0$ we get

$$K_0(x)<\sqrt{\frac{\pi}{2x}}e^{-x}. \tag{3.4}$$

Combining inequalities (3.3) and (3.4) gives

$$\frac{1}{\sqrt{x+1}}<\sqrt{\frac{2}{\pi}}e^{x}K_0(x)<\frac{1}{\sqrt{x}}$$

for $x>0$. Some stronger results can be found in [36], [37, Theorem 3.3, Corollary 3.4]. Motivated by these considerations, we here conjecture that the double inequality

$$\frac{1}{\sqrt{x+a}}<\sqrt{\frac{2}{\pi}}e^{x}K_0(x)<\frac{1}{\sqrt{x+b}}$$

holds for all $x>0$ with the best constants $a=1/4$ and $b=0$.

Taking $v=n$ in Proposition 3.1 we conclude the following.

Corollary 3.6. *Let $n \in \mathbb{N}$. Then the function*

$$x \mapsto \frac{K_n(x)}{\sqrt{K_{n-1/2}(x) K_{n+1/2}(x)}}$$

is strictly increasing on $(0, \infty)$. Consequently, the double inequality

$$\beta_n \sqrt{K_{n-1/2}(x) K_{n+1/2}(x)} < K_n(x) < \sqrt{K_{n-1/2}(x) K_{n+1/2}(x)}$$

holds for $x > 0$ with the best coefficients

$$\beta_n = \frac{\sqrt{n-1/2}\,\Gamma(n)}{\Gamma(n+1/2)} \quad \textit{and} \quad 1.$$

Remark 3.7. It is easy to check that

$$\beta_n = \frac{1}{\sqrt{\pi(n-1/2)}W_{n-1}},$$

where W_n denotes the Wallis ratio. By the Wallis inequality

$$\frac{1}{\sqrt{\pi(n+1/2)}} < W_n < \frac{1}{\sqrt{\pi(n+1/4)}}, \tag{3.5}$$

proved in [38] by Kazarinoff, we have

$$\sqrt{\frac{n-3/4}{n-1/2}} < \beta_n < 1,$$

which implies that

$$\lim_{n\to\infty} \frac{K_n(x)}{\sqrt{K_{n-1/2}(x) K_{n+1/2}(x)}} = 1$$

or, equivalently,

$$K_n(x) \sim \sqrt{K_{n-1/2}(x) K_{n+1/2}(x)} \quad \text{as } n \to \infty.$$

For $x \in \mathbb{R}$, the Airy function of the first kind can be defined by the improper Riemann integral (see [1, p. 447, (10.4.32)]), namely,

$$\mathrm{Ai}(x) = \frac{1}{\pi} \int_0^\infty \cos\left(\frac{t^3}{3} + xt\right) dt.$$

For $x > 0$, the Airy function and its first derivative are related to the modified Bessel functions of the second kind. More precisely, we have [1, p. 447, (10.4.14), (10.4.16)]

$$\mathrm{Ai}(x) = \frac{\sqrt{x}}{\sqrt{3}\pi} K_{1/3}\left(\frac{2}{3}x^{3/2}\right) \quad \text{and} \quad \mathrm{Ai}'(x) = -\frac{x}{\sqrt{3}\pi} K_{2/3}\left(\frac{2}{3}x^{3/2}\right). \tag{3.6}$$

Now letting $v = 1/3$ in Corollary 3.4 yields

$$\sqrt{\frac{\pi}{2}} \frac{(1+1/x)^{1/3}}{\sqrt{x+1}} e^{-x} < K_{1/3}(x) < 2^{-2/3}\Gamma\left(\frac{1}{3}\right) \frac{(1+1/x)^{1/3}}{\sqrt{x+1}} e^{-x},$$

which, by replacing x with $2x^{3/2}/3$ and simplifying, is equivalent to

$$\frac{\sqrt[6]{2}}{2\sqrt{\pi}} \frac{\exp\left(-2x^{3/2}/3\right)}{\sqrt[6]{2x^{3/2}+3}} < \mathrm{Ai}(x) < \frac{1}{\sqrt{3}\Gamma(2/3)} \frac{\exp\left(-2x^{3/2}/3\right)}{\sqrt[6]{2x^{3/2}+3}} \tag{3.7}$$

for $x>0$, where both the coefficients $\sqrt[6]{2}/(2\sqrt{\pi})\approx 0.316\,64$ and $\left[\sqrt{3}\Gamma(2/3)\right]^{-1}\approx 0.426\,37$ are the best possible.

On the other hand, by the formulas (3.6) we have

$$-\mathrm{Ai}(x)\,\mathrm{Ai}'(x)=\frac{1}{2\pi^2}\left(\frac{2}{3}x^{3/2}\right)K_{1/3}\left(\frac{2}{3}x^{3/2}\right)K_{2/3}\left(\frac{2}{3}x^{3/2}\right), \tag{3.8}$$

which together with Theorem 2.5 gives another sharp estimate for the Airy function by the incomplete gamma function. More precisely, we have

Proposition 3.8. *For $x>0$, the double inequality*

$$\sqrt{\frac{1}{\alpha_1\pi}\Gamma\left(\frac{2}{3},\frac{4}{3}x^{3/2}\right)}<\mathrm{Ai}(x)<\sqrt{\frac{1}{\alpha_2\pi}\Gamma\left(\frac{2}{3},\frac{4}{3}x^{3/2}\right)} \tag{3.9}$$

holds with the best constants $\alpha_1=2\sqrt[3]{6}\approx 3.6342$ and $\alpha_2=\sqrt{3}\sqrt[3]{6}\approx 3.1473$, where

$$\Gamma(a,x)=\int_x^\infty t^{a-1}e^{-t}dt \quad \text{for } a>0 \text{ and } x>0$$

is the incomplete gamma function.

Proof. Utilizing Theorem 2.5, we see that the function $t\to K_v(t)\,K_{1-v}(t)/K_{1/2}(t)^2$ is strictly decreasing on $(0,\infty)$. Therefore, for $v\in(0,1)$ with $v\neq 1/2$, it holds that

$$1<\frac{K_v(t)\,K_{1-v}(t)}{K_{1/2}(t)^2}<\frac{\Gamma(v)\,\Gamma(1-v)}{\Gamma(1/2)^2}=\frac{1}{\sin(v\pi)}$$

with the best constants 1 and $1/\sin(v\pi)$, which is equivalent to

$$\frac{\pi}{2}e^{-2t}<tK_v(t)\,K_{1-v}(t)<\frac{\pi}{2\sin(v\pi)}e^{-2t}.$$

In particular, for $v=1/3$ we have

$$\frac{\pi}{2}e^{-2t}<tK_{1/3}(t)\,K_{2/3}(t)<\frac{\pi}{\sqrt{3}}e^{-2t}.$$

Application of the identity (3.8) yields

$$\frac{1}{4\pi}e^{-2t}<-\mathrm{Ai}(x)\,\mathrm{Ai}'(x)<\frac{1}{2\sqrt{3}\pi}e^{-2t}$$

with $t=2x^{3/2}/3$. Integrating both sides over $[z,\infty)$ with $z>0$ leads to

$$\frac{1}{4\pi}\int_z^\infty e^{-2t}dx<-\int_z^\infty \mathrm{Ai}(x)\,\mathrm{Ai}'(x)\,dx<\frac{1}{2\sqrt{3}\pi}\int_z^\infty e^{-2t}dx.$$

On the other hand, a direct computation leads to

$$-\int_z^\infty \mathrm{Ai}(x)\,\mathrm{Ai}'(x)\,dx=\frac{1}{2}\mathrm{Ai}(z)^2,$$

and making a change of variable $y=2t=4x^{3/2}/3$ gives

$$\int_z^\infty e^{-t}dx=\frac{1}{\sqrt[3]{6}}\int_{4z^{3/2}/3}^\infty y^{2/3-1}e^{-y}dx=\frac{1}{\sqrt[3]{6}}\Gamma\left(\frac{2}{3},\frac{4}{3}z^{3/2}\right).$$

Hence, we conclude that

$$\frac{1}{4\sqrt[3]{6}\pi}\Gamma\left(\frac{2}{3},\frac{4}{3}z^{3/2}\right)<\frac{1}{2}\mathrm{Ai}(z)^2<\frac{1}{2\sqrt{3}\sqrt[3]{6}\pi}\Gamma\left(\frac{2}{3},\frac{4}{3}z^{3/2}\right),$$

which proves the desired inequalities (3.9). □

4. Sharp bounds for certain ratios

A relation between Turán ratio $K_{v-1}(x) K_{v+1}(x) / K_v(x)^2$ and another ratio $x K_{v-1}(x) / K_v(x)$ appeared in physics [17, (4.25)], which can be described by the following formula:

$$\frac{1}{x}\left(\frac{xK_{v-1}(x)}{K_v(x)}\right)' = \frac{K_{v-1}(x) K_{v+1}(x)}{K_v(x)^2} - 1.$$

On the other hand, from the recurrence formulas (2.4) and (2.9) it follows that

$$\frac{K_{v-1}(x) K_{v+1}(x)}{K_v(x)^2} = \left(\frac{K_v'(x)}{K_v(x)}\right)^2 - \frac{v^2}{x^2}. \tag{4.1}$$

Also, the relations among $K_v'(x)/K_v(x)$, $K_{v-1}(x)/K_v(x)$ and $K_{v+1}(x)/K_v(x)$ are given in (2.10).

In this section, we use theorems presented in Section 2 to re-prove certain known inequalities and to give some new inequalities for those ratios mentioned above.

Proposition 4.1. *For $v \in \mathbb{R}$ and $x > 0$, the following inequalities are true:*

$$\frac{K_v'(x)}{K_v(x)} < -\sqrt{1 + \frac{v^2}{x^2}}, \tag{4.2}$$

$$\frac{xK_v(x)}{K_{v+1}(x)} < \sqrt{x^2 + v^2} - v < \frac{xK_{v-1}(x)}{K_v(x)}. \tag{4.3}$$

Furthermore, if $v > 1$, then we have

$$-\sqrt{\frac{v}{v-1} + \frac{v^2}{x^2}} < \frac{K_v'(x)}{K_v(x)} < -\sqrt{1 + \frac{v^2}{x^2}}, \tag{4.4}$$

$$\begin{aligned}\sqrt{\tfrac{v-1}{v}x^2 + (v-1)^2} - (v-1) < \tfrac{xK_v(x)}{K_{v+1}(x)} < \sqrt{x^2+v^2} - v < \tfrac{xK_{v-1}(x)}{K_v(x)} \\ < \sqrt{\tfrac{v}{v-1}x^2 + v^2} - v.\end{aligned} \tag{4.5}$$

Proof. Noting that $K_v(x) > 0$ and $K_v'(x) < 0$ for $v \in \mathbb{R}$, the inequality (4.2) follows easily from inequalities (1.9) and the relation formula (4.1).

Using the first relation (2.10) and (4.2) gives the right-hand inequality of (4.3), while the left-hand inequality of (4.3) follows from the relation connecting the first with the third one in (2.10) and inequality (4.2).

Now applying the double inequality (1.9) to the relation (4.1) yields inequalities (4.4). By means of the relations (2.10), the inequalities (4.4) can be equivalently changed into the inequalities (4.5). □

Remark 4.2. We would like to mention that the inequality (4.2) was proved in [7] and [9], and an equivalent inequality of (4.4) with positive integers $v > 1$ was given by Phillips and Malin in [39]. It was pointed out in [7] that inequalities (4.4) are equivalent to the left-hand inequality of (1.3), and another proof was presented by Segura in [22].

Proposition 4.3. *Let $v \in \mathbb{R}$. Then for $v \in (1/2, 3/2)$ the double inequality*

$$\frac{K_v'(x)}{K_v(x)} > -\frac{2x^2 + 3x + 2v}{2x(x+1)}$$

holds for $x \in (0,\infty)$, while for $v \in (-\infty,1/2) \cup (3/2,\infty)$, the inequality above is reversed.

Proof. Theorem 2.6 tells us that the function $v \mapsto K_v'(x)/K_v(x)$ is concave on $\mathbb{R}$, which implies that for $u, v \in \mathbb{R}$, the inequality

$$\frac{K_{pu+qv}'(x)}{K_{pu+qv}(x)} > p\frac{K_u'(x)}{K_u(x)} + q\frac{K_v'(x)}{K_v(x)}$$

holds for $x \in (0,\infty)$, where $pq > 0$ with $p+q=1$. If $pq < 0$, the inequality is reversed.

Putting $(u,v) = (1/2, 3/2)$, $(p,q) = (3/2 - v, v - 1/2)$ and using formulas (3.1) yield that for $v \in (1/2, 3/2)$,

$$\begin{aligned}\frac{K_v'(x)}{K_v(x)} &> \left(\frac{3}{2} - v\right)\frac{K_{1/2}'(x)}{K_{1/2}(x)} + \left(v - \frac{1}{2}\right)\frac{K_{3/2}'(x)}{K_{3/2}(x)} \\ &= \left(\frac{3}{2} - v\right)\left(-\frac{2x+1}{2x}\right) + \left(v - \frac{1}{2}\right)\left(-\frac{2x^2+3x+3}{2x(x+1)}\right) \\ &= -\frac{2x^2+3x+2v}{2x(x+1)},\end{aligned}$$

which proves the desired inequality. □

Proposition 4.4. *Let $u, v \in \mathbb{R}$ with $u > v$. Then the double inequality*

$$\frac{K_u(x)K_{u-1}(x)}{K_v(x)K_{v-1}(x)} < \frac{K_u(x)^2 - K_{u-1}(x)K_{u+1}(x)}{K_v(x)^2 - K_{v-1}(x)K_{v+1}(x)} < \frac{K_u(x)K_{u+1}(x)}{K_v(x)K_{v+1}(x)} \tag{4.6}$$

holds for $x \in (0,\infty)$. In particular, for $u > 1/2$, we have

$$\frac{1}{x}\frac{K_{u-1}(x)}{K_u(x)} < \frac{K_{u-1}(x)K_{u+1}(x)}{K_u(x)^2} - 1 < \frac{1}{x+1}\frac{K_{u+1}(x)}{K_u(x)}, \tag{4.7}$$

and the inequalities are reversed for $0 < u < 1/2$.

Proof. From Theorem 2.6, we see that for $a > 0$, the function

$$v \mapsto K_{v+a-1}(x)/K_{v+a}(x) - K_{v-1}(x)/K_v(x)$$

is strictly increasing on $\mathbb{R}$. Then, for $a = 1$ and $u, v \in \mathbb{R}$ with $u > v$, we have

$$\frac{K_u(x)}{K_{u+1}(x)} - \frac{K_{u-1}(x)}{K_u(x)} > \frac{K_v(x)}{K_{v+1}(x)} - \frac{K_{v-1}(x)}{K_v(x)},$$

which is equivalent to the right-hand inequality in (4.6), while the left-hand inequality of (4.6) follows from the increasing property of the function $v \mapsto K_{v+a+1}(x)/K_{v+a}(x) - K_{v+1}(x)/K_v(x)$. In fact, for $a = 1$ and $u, v \in \mathbb{R}$ with $u > v$, we obtain

$$\frac{K_{u+2}(x)}{K_{u+1}(x)} - \frac{K_{u+1}(x)}{K_u(x)} > \frac{K_{v+2}(x)}{K_{v+1}(x)} - \frac{K_{v+1}(x)}{K_v(x)}.$$

Then replacing (u,v) by $(u-1, v-1)$ gives the left-hand inequality in (4.6).

Finally, letting $v = 1/2$ and substituting the formulas (3.1) into (4.6), then simplifying give (4.7). □

Taking $u = 1/2$ and $a = 1$ in Theorem 2.9, we immediately conclude the following.

Proposition 4.5. *The function* $x \mapsto xK_{v+1}(x) / [(x+1) K_v(x)]$ *is strictly decreasing on* $(0, \infty)$ *for* $v > 1/2$ *and increasing on* $(0, \infty)$ *for* $0 < v < 1/2$. *Moreover, for* $v > 1/2$, *the double inequality*

$$\frac{x+1}{x} < \frac{K_{v+1}(x)}{K_v(x)} < 2v \frac{x+1}{x}$$

holds for $x > 0$, *where both the lower and upper bounds are sharp. It is reversed for* $0 < v < 1/2$.

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