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CRITERIA FOR THE EXISTENCE OF PRINCIPAL EIGENVALUES OF TIME PERIODIC COOPERATIVE LINEAR SYSTEMS WITH NONLOCAL DISPERSAL

XIONGXIONG BAO AND WENXIAN SHEN

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ABSTRACT. The current paper establishes criteria for the existence of principal eigenvalues of time periodic cooperative linear nonlocal dispersal systems with Dirichlet type, Neumann type or periodic type boundary conditions. It is shown that such a nonlocal dispersal system has a principal eigenvalue in the following cases: the nonlocal dispersal distance is sufficiently small; the spatial inhomogeneity satisfies a so-called vanishing condition; or the spatial inhomogeneity is nearly globally homogeneous. Moreover, it is shown that the principal eigenvalue of a time periodic cooperative linear nonlocal dispersal system (if it exists) is algebraically simple. A linear nonlocal dispersal system may not have a principal eigenvalue. The results established in the current paper extend those in literature for time independent or periodic nonlocal dispersal equations to time periodic cooperative nonlocal dispersal systems and will serve as a basic tool for the study of cooperative nonlinear systems with nonlocal dispersal.

1. INTRODUCTION

Internal dispersal (diffusion), which describes the movements or interactions of the organisms in the underlying systems, occurs in many evolution systems arising in applied sciences. There are several disparate approaches to model internal dispersal. In the case that the movements or interactions of the organisms occur randomly between adjacent spatial locations, differential operators such as Δ have been widely used to model the dispersal; see, for example, [1], [2], [5], [20], [32], [42], etc. They are frequently referred to as *random dispersal operators*. In contrast, integral operators, which are referred to as *nonlocal dispersal operators*, are often used to describe the movements or interactions of the organisms in an underlying system which occur between adjacent as well as nonadjacent spatial locations; see, for example, [15], [17], [21], [22], etc. For the study of various dynamical aspects of nonlocal dispersal evolution equations, the reader is referred to [3], [6]–[8], [10], [17], [18], [21]-[28], [31], [33]–[36], [38], [39].

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It is of great importance for the study of nonlinear dispersal evolution equations/systems to establish a spectral theory for linear dispersal evolution equations/systems. The objective of the current paper is to develop the principal eigenvalue theory for the following three linear cooperative systems with nonlocal dispersal and time periodic dependence:

$$\mathbf{u}_t = \int_D \kappa(y-x)\mathbf{u}(t,y)dy - \mathbf{u}(t,x) + A_1(t,x)\mathbf{u}(t,x), \quad x \in \bar{D}, \tag{1.1}$$

where $D \subset \mathbb{R}^N$ is a smooth bounded domain, $\mathbf{u}(t,x) \in \mathbb{R}^K$ $(K > 1)$ and $A_1(t,x) = (a_{kj}(t,x))_{K\times K}$ with $a_{kj}(t,x)$ being continuous in $t \in \mathbb{R}$ and $x \in \bar{D}$ and $a_{kj}(t+T,x) = a_{kj}(t,x)$ for $k,j = 1,\dots,K$;

$$\mathbf{u}_t = \int_D \kappa(y-x)[\mathbf{u}(t,y) - \mathbf{u}(t,x)]dy + A_2(t,x)\mathbf{u}(t,x), \quad x \in \bar{D}, \tag{1.2}$$

where $D \in \mathbb{R}^N$ and $A_2(t,x)$ has the same properties as $A_1(t,x)$; and

$$\mathbf{u}_t = \int_{\mathbb{R}^N} \kappa(y-x)[\mathbf{u}(t,y) - \mathbf{u}(t,x)]dy + A_3(t,x)\mathbf{u}(t,x), \quad x \in \mathbb{R}^N, \tag{1.3}$$

where $A_3(t,x) = (a_{kj}(t,x))_{K\times K}$ with $a_{kj}(t,x)$ being continuous in $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$ and $a_{kj}(t+T,x) = a_{kj}(t,x+p_l\mathbf{e}_l) = a_{kj}(t,x)$ for $p_l > 0$, $\mathbf{e}_l = (\delta_{l1},\dots,\delta_{lN})$ with $\delta_{ln} = 1$ if $l = n$ and $\delta_{ln} = 0$ if $l \neq n$, $l,n = 1,\dots,N$. In (1.1)-(1.3) $\kappa(\cdot)$ is a nonnegative C^1 function with compact support, $\kappa(0) > 0$, $\int_{\mathbb{R}^N}\kappa(z)dz = 1$.

In order to state the standing assumptions, let

$$D_1 = D_2 = \bar{D}, \quad D_3 = \mathbb{R}^N. \tag{1.4}$$

Throughout this paper, we assume that:

- **(H1):** For each given $1 \le i \le 3$, $A_i(\cdot,\cdot)$ is cooperative in the sense that for any $t \in \mathbb{R}$, $x \in D_i$, and $1 \le k \neq j \le K$, $a_{kj}(t,x) \ge 0$.
- **(H2):** For each given $1 \le i \le 3$, $A_i(\cdot,\cdot)$ is strongly irreducible in the sense that for any two nonempty subsets S, S' of $\{1,2,\dots,K\}$ which form a partition of $\{1,2,\dots,K\}$ and any $t \in \mathbb{R}$ and $x \in D_i$ there exist $k \in S$, $j \in S'$ satisfying that $|a_{kj}(t,x)| > 0$.

The following eigenvalue problems are associated with the coupled nonlocal dispersal systems (1.1), (1.2), and (1.3), respectively:

$$\begin{cases} -\mathbf{u}_t + \int_D \kappa(y-x)\mathbf{u}(t,y)dy - \mathbf{u}(t,x) + A_1(t,x)\mathbf{u}(t,x) = \lambda\mathbf{u}(t,x), \quad x \in \bar{D}, \\ \mathbf{u}(t+T,x) = \mathbf{u}(t,x), \end{cases} \tag{1.5}$$

$$\begin{cases} -\mathbf{u}_t + \int_D \kappa(y-x)[\mathbf{u}(t,y) - \mathbf{u}(t,x)]dy + A_2(t,x)\mathbf{u}(t,x) = \lambda\mathbf{u}(t,x), \quad x \in \bar{D}, \\ \mathbf{u}(t+T,x) = \mathbf{u}(t,x), \end{cases} \tag{1.6}$$

and

$$\begin{cases} -\mathbf{u}_t + \int_{\mathbb{R}^N} \kappa(y-x)[\mathbf{u}(t,y) - \mathbf{u}(t,x)]dy + A_3(t,x)\mathbf{u}(t,x) = \lambda\mathbf{u}(t,x), \quad x \in \mathbb{R}^N, \\ \mathbf{u}(t+T,x) = \mathbf{u}(t,x+p_l\mathbf{e}_l) = \mathbf{u}(t,x), \ \ l = 1,2,\dots,N. \end{cases} \tag{1.7}$$

The so-called principal spectrum point of (1.5) (resp. (1.6), (1.7)) plays a special role in the study of the related nonlinear systems. Roughly, let $\sigma(A_1)$ (resp. $\sigma(A_2)$, $\sigma(A_3)$) be the spectrum of the eigenvalue problem (1.5) (resp. (1.6), (1.7)).

Let $\lambda_i(A_i) = \sup\{\text{Re}\lambda \mid \lambda \in \sigma(A_i)\}$. $\lambda_1(A_1)$ (resp. $\lambda_2(A_2)$, $\lambda_3(A_3)$) is called the *principal spectrum point* of the eigenvalue problem (1.5) (resp. (1.6), (1.7)) (see Definition 2.2 for details). $\lambda_1(A_1)$ (resp. $\lambda_2(A_2)$, $\lambda_3(A_3)$) is called the *principal eigenvalue* of (1.5) (resp. (1.6), (1.7)) if it is an isolated eigenvalue with a positive eigenfunction (see Definition 2.2 for details).

Observe that (1.5), (1.6), and (1.7) can be viewed as the nonlocal dispersal counterparts of the following eigenvalue problems with random dispersal and Dirichlet, Neumann, and periodic boundary conditions, respectively:

$$(1.8) \qquad \begin{cases} -\mathbf{u}_t + \Delta\mathbf{u}(t,x) + A_1(t,x)\mathbf{u}(t,x) = \lambda\mathbf{u}(t,x), \quad x \in D, \\ \mathbf{u}(t,x) = 0, \quad x \in \partial D, \\ \mathbf{u}(t+T,x) = \mathbf{u}(t,x), \end{cases}$$

$$(1.9) \qquad \begin{cases} -\mathbf{u}_t + \Delta\mathbf{u}(t,x) + A_2(t,x)\mathbf{u}(t,x) = \lambda\mathbf{u}(t,x), \quad x \in D, \\ \frac{\partial \mathbf{u}}{\partial n} = 0, \quad x \in \partial D, \\ \mathbf{u}(t+T,x) = \mathbf{u}(t,x), \end{cases}$$

and

$$(1.10) \qquad \begin{cases} -\mathbf{u}_t + \Delta\mathbf{u}(t,x) + A_3(t,x)\mathbf{u}(t,x) = \lambda\mathbf{u}(t,x), \quad x \in \mathbb{R}^N, \\ \mathbf{u}(t+T,x) = \mathbf{u}(t,x+p_l\mathbf{e_l}) = \mathbf{u}(t,x), \;\; l = 1,2,\dots,N. \end{cases}$$

In fact, it is proved for $K = 1$ that the principal eigenvalue of (1.8), (1.9), and (1.10) can be approximated by the principal spectrum points of (1.5), (1.6), and (1.7), respectively, by properly rescaling the kernels and that the initial value problems of random dispersal evolution equations can also be approximated by the initial value problems of the related nonlocal dispersal evolution equations with properly rescaled kernels (see [11], [12] and [40]). Hence we may say that (1.1), (1.2), and (1.3) correspond to problems having Dirichlet type boundary conditions, Neumann type boundary conditions, and periodic boundary conditions, respectively.

When $K = 1$, there are many results on the principal eigenvalues of time independent or periodic random dispersal eigenvalue problems (1.8), (1.9) and (1.10) (see [5], [13], [20], and references therein) and their nonlocal dispersal counterparts (1.5), (1.6) and (1.7) (see [9], [10], [16], [19], [24], [30], [37], [40], [41] and references therein). It is known that a random dispersal operator always has a principal eigenvalue, but a nonlocal dispersal operator may not have a principal eigenvalue (see [9] and [37] for some examples), which reveals some essential differences between nonlocal dispersal and random dispersal operators. We point out that, very recently, Ding and Liang [14] studied the existence of the principal eigenvalues of generalized convolution or integral operators on the circle in periodic media. They showed that such a generalized convolution operator has a principal eigenvalue provided that its kernel satisfies the so-called uniformly irreducible condition (see [14, Theorem 2.4]).

When $K > 1$, principal eigenvalue theory for (1.8), (1.9) and (1.10) has also been well established. For example, it is known that the principal eigenvalue of (1.8), (resp. (1.9), (1.10)) always exists. However, to the best of our knowledge, the principal eigenvalue problems (1.5), (1.6) and (1.7) associated to coupled nonlocal dispersal systems are hardly studied in the literature. In this paper, we will establish some criteria for the existence of principal eigenvalues of (1.5), (1.6) and (1.7). The results obtained in this paper extend many existing ones for principal eigenvalues of time independent and time periodic nonlocal dispersal equations to time periodic coupled nonlocal dispersal systems. They provide basic tools for the study of nonlinear nonlocal dispersal evolution systems. We will discuss some applications of the results established in this paper elsewhere.

The rest of the paper is organized as follows. In Section 2 we introduce standing notation and state the main results of this paper. In Section 3, we present some basic properties for coupled nonlocal dispersal systems. In Section 4, we prove the main results.

2. Notation and main results

In this section, we first introduce the standing notation, present a comparison principle, and give the definition of principal spectrum points and principal eigenvalues of (1.1), (1.2) and (1.3), which will be used throughout this paper. We then state the main results of the current paper.

For any $\alpha = (\alpha_1, \ldots, \alpha_K)^\top$, we put $|\alpha| = \sqrt{\sum_{i=1}^K \alpha_i^2}$. Let

$$(\mathbb{R}^K)^+ = \{\alpha = (\alpha_1, \ldots, \alpha_K)^\top \,|\, \alpha_i \in \mathbb{R}, \alpha_i \geq 0, i = 1, 2, \ldots, K\}$$

and

$$(\mathbb{R}^K)^{++} = \{\alpha = (\alpha_1, \ldots, \alpha_K)^\top \,|\, \alpha_i \in \mathbb{R}, \alpha_i > 0, i = 1, 2, \ldots, K\}.$$

Let

$$X_1 = X_2 = C(\bar{D}, \mathbb{R}^K)$$

with norm $\|\mathbf{u}\|_{X_i} = \sup_{x \in \bar{D}} |\mathbf{u}(x)|$ $(i = 1, 2)$,

$$X_3 = \{\mathbf{u} \in C^b_{\rm unif}(\mathbb{R}^N, \mathbb{R}^K) | \mathbf{u}(x + p_l \mathbf{e}_l) = \mathbf{u}(x),\ l = 1, \ldots, N\}$$

with norm $\|\mathbf{u}\|_{X_3} = \sup_{x \in \mathbb{R}^N} |\mathbf{u}(x)|$, and

$$X_i^+ = \{\mathbf{u} \in X_i \,|\, \mathbf{u}(x) \in (\mathbb{R}^K)^+\}, \quad X_i^{++} = \{\mathbf{u} \in X_i \,|\, \mathbf{u}(x) \in (\mathbb{R}^K)^{++}\},\ i = 1, 2, 3.$$

For given $\mathbf{u}, \mathbf{v} \in X_i$, we define

$$\mathbf{u} \leq \mathbf{v}\ \ (\mathbf{v} \geq \mathbf{u}) \quad \text{if} \quad \mathbf{v} - \mathbf{u} \in X_i^+ \quad \text{and} \quad \mathbf{u} \ll \mathbf{v}\ \ (\mathbf{v} \gg \mathbf{u}) \quad \text{if} \quad \mathbf{v} - \mathbf{u} \in X_i^{++}.$$

Let

$$\mathcal{X}_1 = \mathcal{X}_2 = \{\mathbf{u} \in C(\mathbb{R} \times \bar{D}, \mathbb{R}^K) \,|\, \mathbf{u}(t + T, x) = \mathbf{u}(t, x)\}$$

with norm $\|\mathbf{u}\|_{\mathcal{X}_i} = \sup_{t \in \mathbb{R}, x \in \bar{D}} |\mathbf{u}(t, x)|$ $(i = 1, 2)$,

$$\mathcal{X}_3 = \{\mathbf{u} \in C(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^K) \,|\, \mathbf{u}(t + T, x) = \mathbf{u}(t, x + p_l \mathbf{e}_l) = \mathbf{u}(t, x),\ l = 1, \ldots, N\}$$

with norm $\|\mathbf{u}\|_{\mathcal{X}_3} = \sup_{t \in \mathbb{R}, x \in \mathbb{R}^N} |\mathbf{u}(t, x)|$, and

$$\mathcal{X}_i^+ = \{\mathbf{u} \in \mathcal{X}_i |\ \mathbf{u}(t, x) \in (\mathbb{R}^K)^+\}, \quad \mathcal{X}_i^{++} = \{\mathbf{u} \in \mathcal{X}_i |\ \mathbf{u}(t, x) \in (\mathbb{R}^K)^{++}\},\ i = 1, 2, 3.$$

For any $\mathbf{u}, \mathbf{v} \in \mathcal{X}_i$, we define

$$\mathbf{u} \leq \mathbf{v}\ (\mathbf{v} \geq \mathbf{u}) \quad \text{if} \quad \mathbf{v} - \mathbf{u} \in \mathcal{X}_i^+ \ \text{and} \ \mathbf{u} \ll \mathbf{v}\ (\mathbf{v} \gg \mathbf{u}) \quad \text{if} \quad \mathbf{v} - \mathbf{u} \in \mathcal{X}_i^{++}.$$

General semigroup theory (see [29]) guarantees that for every $\mathbf{u}_0 \in X_i$, (1.1) (resp. (1.2), (1.3)) has a unique solution $\mathbf{u}_1(t, x; s, \mathbf{u}_0)$ (resp. $\mathbf{u}_2(t, x; s, \mathbf{u}_0)$, $\mathbf{u}_3(t, x; s, \mathbf{u}_0)$) with $\mathbf{u}_1(s, \cdot; s, \mathbf{u}_0) = \mathbf{u}_0 \in X_1$ (resp. $\mathbf{u}_2(s, \cdot; s, \mathbf{u}_0) = \mathbf{u}_0 \in X_2$, $\mathbf{u}_3(s, \cdot; s, \mathbf{u}_0) = \mathbf{u}_0 \in X_3$). Put

$$(\mathbf{\Phi}_i(t, s; A_i)\mathbf{u}_0)(\cdot) := \mathbf{u}_i(t, \cdot; s, \mathbf{u}_0), \quad i = 1, 2, 3.$$

Definition 2.1. A continuous function $\mathbf{u}(t,x)$ on $[0,\tau)\times\bar{D}$ is called a supersolution (or subsolution) of (1.1) if for any $x\in\bar{D}$, $\mathbf{u}(t,x)$ is continuously differentiable on $[0,\tau)$ and satisfies that for each $x\in\bar{D}$,

$$\mathbf{u}_t \geq \text{ (or } \leq) \int_D \kappa(y-x)\mathbf{u}(t,y)dy - \mathbf{u}(t,x) + A_1(t,x)\mathbf{u}(t,x), \quad x\in\bar{D},$$

for $t\in[0,\tau)$. Supersolutions and subsolutions of (1.2) and (1.3) are defined in an analogous way.

Proposition 2.1 (Comparison principle)**.** *Assume that* (H1) *and* (H2) *hold.*

(1) *If $\mathbf{u}^-(t,x)$ and $\mathbf{u}^+(t,x)$ are bounded subsolution and supersolution of* (1.1) *(resp.* (1.2), (1.3)*) on $[0,\tau)$, respectively, and $\mathbf{u}^-(0,x)\leq\mathbf{u}^+(0,x)$ for any $x\in D_1$ (resp. $x\in D_2$, $x\in D_3$), then $\mathbf{u}^-(t,x)\leq\mathbf{u}^+(t,x)$ for any $t\in[0,\tau)$ and $x\in D_1$ (resp. $x\in D_2$, $x\in D_3$).*
(2) *For given $1\leq i\leq 3$ and $\mathbf{u}_0\in X_i^+$, $\mathbf{u}_i(t,\cdot;s,\mathbf{u}_0)$ exists for all $t\geq s$.*
(3) *For given $1\leq i\leq 3$ and $\mathbf{u}_0^1(\cdot),\mathbf{u}_0^2(\cdot)\in X_i$, if $\mathbf{u}_0^1\leq\mathbf{u}_0^2$ and $\mathbf{u}_0^1\neq\mathbf{u}_0^2$, then $\mathbf{\Phi}_i(t,s;A_i)\mathbf{u}_0^1\ll\mathbf{\Phi}_i(t,s;A_i)\mathbf{u}_0^2$ for all $t>s$.*

Proof. Note that $A_i(t,x)$ is cooperative and strongly irreducible for any $t\in\mathbb{R}$ and $x\in D_i$. Then the results of (1)-(3) follow from the similar argument in [37, Propositions 2.1 and 2.2] and [18, Propositions 3.1 and 3.2]. $\square$

Define

$$(\mathcal{K}_i\mathbf{u})(t,x) := \int_D \kappa(y-x)\mathbf{u}(t,y)dy \quad \text{for} \quad \mathbf{u}\in\mathcal{X}_i, \quad i=1,2,$$
$$(\mathcal{K}_3\mathbf{u})(t,x) := \int_{\mathbb{R}^N} \kappa(y-x)\mathbf{u}(t,y)dy \quad \text{for} \quad \mathbf{u}\in\mathcal{X}_3$$

and

$$(\mathcal{A}_i\mathbf{u})(t,x) := -\mathbf{u}(t,x) + A_i(t,x)\mathbf{u}(t,x) \quad \text{for } \mathbf{u}\in\mathcal{X}_i,\ i=1,3,$$
$$(\mathcal{A}_2\mathbf{u})(t,x) := -\int_D \kappa(y-x)dy\cdot\mathbf{u}(t,x) + A_2(t,x)\mathbf{u}(t,x) \quad \text{for } \mathbf{u}\in\mathcal{X}_2.$$

Definition 2.2. For given $1\leq i\leq 3$, let $\sigma(-\partial_t+\mathcal{K}_i+\mathcal{A}_i)$ be the spectrum of $-\partial_t+\mathcal{K}_i+\mathcal{A}_i$ on $\mathcal{X}_i$. Let

$$\lambda_i(A_i) = \sup\{\mathrm{Re}\lambda \,|\, \lambda\in\sigma(-\partial_t+\mathcal{K}_i+\mathcal{A}_i)\}.$$

We call $\lambda_i(A_i)$ the principal spectrum point of $-\partial_t+\mathcal{K}_i+\mathcal{A}_i$. If $\lambda_i(A_i)$ is an isolated eigenvalue of $-\partial_t+\mathcal{K}_i+\mathcal{A}_i$ with a positive eigenfunction $\mathbf{v}$ (i.e. $\mathbf{v}\in\mathcal{X}_i^+$), then $\lambda_i(A_i)$ is called the principal eigenvalue of $-\partial_t+\mathcal{K}_i+\mathcal{A}_i$ or it is said that $-\partial_t+\mathcal{K}_i+\mathcal{A}_i$ has a principal eigenvalue.

Lemma 2.1. *For given $1\leq i\leq 3$, assume that $A_i(t,x)$ satisfies* (H1) *and* (H2)*. For any given $x\in D_i$, the eigenvalue problem*

$$\begin{cases} -\frac{d\boldsymbol{\phi}(t)}{dt} + A_i(t,x)\boldsymbol{\phi}(t) = \lambda\boldsymbol{\phi}(t), \quad \boldsymbol{\phi}(t)\in\mathbb{R}^K, \\ \boldsymbol{\phi}(t+T) = \boldsymbol{\phi}(t) \end{cases} \tag{2.1}$$

has a unique real eigenvalue, denoted by $\lambda_i(x)$, which has a unique corresponding positive eigenfunction, denoted by $\boldsymbol{\phi}_i(t,x)$, with $|\boldsymbol{\phi}_i(0,x)|=1$.

Proof. $A_i(t,x)$ is a cooperative, strongly irreducible, and T-periodic matrix for any given $x \in D_i$ by (H1) and (H2). Let $\mathbf{I}$ be the $K \times K$ identity matrix and $\mathbf{U}(t;x)$ be the fundamental matrix solution of $\frac{d}{dt}\mathbf{U}(t;x) = A_i(t,x)\mathbf{U}(t;x)$ with $\mathbf{U}(0;x) = \mathbf{I}$ for any given $x \in D_i$. By [42, Theorem 4.1.1], $\mathbf{U}(T;x) : \mathbb{R}^K \to \mathbb{R}^K$ is a compact and strongly positive linear operator. Then the Krein-Rutmann theorem (or the Perron-Frobenius theorem in our present finite-dimensional case (see [42, Theorem 4.3.1]), implies for any given $x \in D_i$ that the spectral radius $r(\mathbf{U}(T;x))$ is an algebraic simple isolated eigenvalue of $\mathbf{U}(T;x)$ with a unique positive eigenvector $\boldsymbol{\phi}^i(x) \in (\mathbb{R}^K)^+$ with $|\boldsymbol{\phi}^i(x)| = 1$. The lemma follows with $\lambda_i(x) = \frac{1}{T}\ln r(\mathbf{U}(T;x))$ and $\boldsymbol{\phi}_i(t,x) = e^{-\lambda_i(x)t}\mathbf{U}(t;x)\boldsymbol{\phi}^i(x)$. $\square$

We remark that $\lambda_i(x)$ and $\boldsymbol{\phi}_i(t,x)$ are as smooth in x as $A_i(t,x)$ in x. We also remark that when $A_i(t,x) \equiv A_i(x)$, $\lambda_i(x)$ is the largest real part of the eigenvalues of the matrix $A_i(x)$. Let

$$h_i(x) = \begin{cases} -1 + \lambda_i(x), & \text{for } i = 1,3, \\ -\int_D k(y-x)dy + \lambda_2(x), & \text{for } i = 2. \end{cases} \tag{2.2}$$

Our main results on the principal spectral points and principal eigenvalues of coupled nonlocal dispersal operators can then be stated as follows.

Theorem 2.1. *Let $1 \le i \le 3$ be given. $\lambda_i(A_i)$ is the principal eigenvalue of $-\partial_t + \mathcal{K}_i + \mathcal{A}_i$ if and only if $\lambda_i(A_i) > \max_{x\in D_i} h_i(x)$.*

Theorem 2.2. *Let $1 \le i \le 3$ be given.*

(1) *Let $\alpha_i = \max_{x\in D_i} h_i(x)$. If there is a bounded domain $D_0 \subset D_i$ such that $1/(\alpha_i - h_i(\cdot)) \notin L^1(D_0)$, then the principal eigenvalue of $-\partial_t + \mathcal{K}_i + \mathcal{A}_i$ exists.*

(2) *Let $\eta_i = \min_{t\in\mathbb{R},x\in D_i}\{\phi_{i1}(t,x),\dots,\phi_{iK}(t,x)\}$ and*

$$\tilde{\eta}_i = \max_{t\in\mathbb{R},x\in D_i}\{\phi_{i1}(t,x),\dots,\phi_{iK}(t,x)\}$$

for $i = 1,3$, where $\boldsymbol{\phi}_i(t,x) = (\phi_{i1}(t,x),\dots,\phi_{iK}(t,x))^\top$ is as in Lemma 2.1. If $\max_{x\in D_i}\lambda_i(x) - \min_{x\in D_i}\lambda_i(x) < \frac{\eta_i}{\tilde{\eta}_i}\inf_{x\in D_i}\int_{D_i}\kappa(y-x)dy$ in the case $i = 1$ and $\max_{x\in D_i}\lambda_i(x) - \min_{x\in D_i}\lambda_i(x) < \frac{\eta_i}{\tilde{\eta}_i}$ in the case $i = 3$, then the principal eigenvalue of $-\partial_t + \mathcal{K}_i + \mathcal{A}_i$ exists for $i = 1,3$.

(3) *Suppose that $\kappa(z) = \frac{1}{\delta^N}\widetilde{\kappa}(\frac{z}{\delta})$ for some $\delta > 0$ and $\widetilde{\kappa}(\cdot)$ with $\widetilde{\kappa}(z) \ge 0$, $\mathrm{supp}(\widetilde{\kappa}) = B(0,1) := \{z \in \mathbb{R}^N \mid \|z\| < 1\}$, $\int_{\mathbb{R}^N}\widetilde{\kappa}(z)dz = 1$, and $\widetilde{k}(\cdot)$ being symmetric with respect to 0. Then the principal eigenvalue of $-\partial_t + \mathcal{K}_i + \mathcal{A}_i$ exists for $0 < \delta \ll 1$.*

Theorem 2.3. *Let $1 \le i \le 3$ be given. If $\lambda_i(A_i)$ is the principal eigenvalue of $-\partial_t + \mathcal{K}_i + \mathcal{A}_i$, then it is an algebraically simple eigenvalue of $-\partial_t + \mathcal{K}_i + \mathcal{A}_i$ with a positive eigenfunction.*

Corollary 2.1. *Let $1 \le i \le 3$ be given. If $h_i(x)$ is C^N and there is some $x_0 \in D_i$ such that $h_i(x_0) = \max_{x\in D_i} h_i(x)$ and the partial derivatives of $h_i(x)$ up to order $N-1$ at x_0 are zero, then the principal eigenvalue of $-\partial_t + \mathcal{K}_i + \mathcal{A}_i$ exists.*

Proof of Corollary 2.1. Assume that $h_i(x)$ is C^N and there is some $x_0 \in D_i$ such that $\alpha_i = h_i(x_0) = \max_{x\in D_i} h_i(x)$, and the partial derivatives of $h_i(x)$ up to order

$N-1$ at x_0 are zero. Then there are bounded domain $D_0 \subset D_i$ and $M>0$ such that

$$0 \le \alpha_i - h_i(x) \le M|x-x_0|^N, \quad x \in D_0.$$

Therefore

$$\frac{1}{\alpha_i - h_i(x)} \ge \frac{1}{M|x-x_0|^N} \quad \text{for} \quad x \in D_0.$$

This implies that $1/(\alpha_0 - h_i(\cdot)) \notin L^1(D_0)$. The corollary then follows from Theorem 2.2 (1). $\square$

Remark 2.1.

(1) The condition in Corollary 2.1 is called the *vanishing condition.*
(2) It remains open whether the analogous result holds for the case $i=2$ in Theorem 2.2 (2). When $K=1$, Theorem 2.2 (2) holds for $i=1,2,3$ under some weaker conditions (see [30, Theorem B (2)]).
(3) If $N=1$ or 2, $h_i(\cdot)$ is C^N, and $\max_{x\in D_i} h_i(x) = h_i(x_0)$ for some $x_i \in \mathrm{Int}(D_i)$, then the conditions in Theorem 2.2 (1) are always satisfied and hence the principal eigenvalue of $-\partial_t + \mathcal{K}_i + \mathcal{A}_i$ exists. The principal eigenvalue may not exist if $N \ge 3$ (see [30], [37]).

3. Basic properties

In this section, we present some basic properties for principal spectrum points of coupled nonlocal dispersal systems (1.5), (1.6) and (1.7). Throughout this section, we assume that (H1) and (H2) hold.

Let $\mathcal{K}_i : \mathcal{X}_i \to \mathcal{X}_i$ and $\mathcal{A}_i : \mathcal{X}_i \to \mathcal{X}_i$ be defined as in Section 2 for $i=1,2,3$. Consider the eigenvalue problem

$$-\partial_t \mathbf{u} + \mathcal{K}_i \mathbf{u} + \mathcal{A}_i \mathbf{u} = \lambda \mathbf{u}, \quad i=1,2,3. \tag{3.1}$$

Recall that $\mathbf{I}$ denotes the $K \times K$ identity matrix. Without loss of generality, we assume that $-A_i(t,x)$ is positive definite for any $(t,x) \in \mathbb{R} \times D_i$, that is, for any $\mathbf{u} \in \mathbb{R}^K$,

$$-\mathbf{u}^T A_i(t,x)\mathbf{u} \ge 0 \quad \forall (t,x) \in \mathbb{R} \times D. \tag{3.2}$$

In fact, there exists $M>0$ such that for any $\mathbf{u} \in \mathbb{R}^K$,

$$-\mathbf{u}^T (A_i(t,x) - M\mathbf{I})\mathbf{u} \ge 0 \quad \forall (t,x) \in \mathbb{R} \times D.$$

Hence $-\widetilde{A}_i(t,x)$ is positive definite for any $(t,x) \in \mathbb{R} \times D_i$, where $\widetilde{A}_i(t,x) = A_i(t,x) - M\mathbf{I}$. If $-A_i(t,x)$ is not positive definite for some $(t,x) \in \mathbb{R} \times D_i$, we may consider the eigenvalue problem

$$-\partial_t \mathbf{u} + \mathcal{K}_i \mathbf{u} + \widetilde{\mathcal{A}}_i \mathbf{u} = \widetilde{\lambda} \mathbf{u},$$

where $\widetilde{\mathcal{A}}_i \mathbf{u} = \mathcal{A}_i \mathbf{u} - M\mathbf{u}$.

Let $\mathcal{X}_i \oplus i\mathcal{X}_i = \{\mathbf{u} + i\mathbf{v} | \mathbf{u}, \mathbf{v} \in \mathcal{X}_i\}$. Observe that if $\alpha \in \mathbb{C}$ is such that $\alpha \mathbf{I} + \partial_t - \mathcal{A}_i$ is invertible, then (3.1) with $\lambda = \alpha$ has nontrivial solutions in $\mathcal{X}_i \oplus i\mathcal{X}_i$ is equivalent to

$$\mathcal{K}_i(\alpha \mathbf{I} + \partial_t - \mathcal{A}_i)^{-1}\mathbf{v} = \mathbf{v}$$

has nontrivial solutions in $\mathcal{X}_i \oplus i\mathcal{X}_i$. Recall that $h_i(x)$ is given by (2.2).

Proposition 3.1. *Let $1 \le i \le 3$ be given. Then*

$$\big[\min_{x\in D_i} h_i(x), \max_{x\in D_i} h_i(x)\big] \subset \sigma(-\partial_t + \mathcal{A}_i).$$

Proof. Assume that $h_i(x_0) \in \rho(-\partial_t + \mathcal{A}_i)$ for some $x_0 \in D_i$. Put $h_i = h_i(x_0)$. Then

$$-\partial_t \mathbf{u} + \mathcal{A}_i \mathbf{u} - h_i \mathbf{u} = \mathbf{v}(t)$$

has a unique solution $\mathbf{u}(\cdot, \cdot; x_0, \mathbf{v}) \in \mathcal{X}_i$ for any $\mathbf{v} \in \mathcal{X}_i$ with $\mathbf{v}(t,x) \equiv \mathbf{v}(t)$. This implies that $\mathbf{u}(t; x_0, \mathbf{v}) := \mathbf{u}(t, x_0; x_0, \mathbf{v})$ is a solution of

$$-\partial_t \mathbf{u} + A_i(t, x_0)\mathbf{u} - \lambda_i(x_0)\mathbf{u} = \mathbf{v}(t).$$

Therefore the Fredholm alternative implies that

$$-\frac{d\mathbf{u}}{dt} + A_i(t, x_0)\mathbf{u} - \lambda_i(x_0)\mathbf{u} = 0$$

has no nontrivial solution $\mathbf{u}(t)$ with $\mathbf{u}(t+T) = \mathbf{u}(t)$. Recall that $\boldsymbol{\phi}_i(t, x_0)$ is an eigenfunction of (2.1) with $x = x_0$ corresponding to $\lambda_i(x_0)$. Then $\mathbf{u}(t; x_0, \boldsymbol{\phi}_i) := \boldsymbol{\phi}_i(t, x_0)$ satisfies

$$-\partial_t \mathbf{u}(t; x_0, \boldsymbol{\phi}_i) + A_i(t, x_0)\mathbf{u}(t; x_0, \boldsymbol{\phi}_i) - \lambda_i(x_0)\mathbf{u}(t; x_0, \boldsymbol{\phi}_i) = 0,$$

which is a contradiction. Hence $h_i(x_0) \in \sigma(-\partial_t + \mathcal{A}_i)$ for any $x_0 \in D_i$ and the proposition follows. $\square$

Let

$$X_0 = \begin{cases} C(D_i, \mathbb{R}), & i = 1, 2, \\ \{u \in C(D_i, \mathbb{R}) \,|\, u(\cdot + p_l \mathbf{e}_l) = u(\cdot)\}, & i = 3. \end{cases}$$

Proposition 3.2. *Let $1 \le i \le 3$ be given; then $(\alpha \mathbf{I} + \partial_t - \mathcal{A}_i)^{-1}$ exists for every $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha > \max_{x \in D_i} h_i(x)$. Moreover, one has for any $\alpha \in \mathbf{C}$ with $\operatorname{Re}\alpha > \max_{x \in D_i} h_i(x)$ and any $v_i(\cdot) \in X_0$ that*

$$\left((\alpha \mathbf{I} + \partial_t - \mathcal{A}_i)^{-1} \boldsymbol{\phi}_i v_i\right)(t, x) = \frac{1}{\alpha - h_i(x)} \boldsymbol{\phi}_i(t, x) v_i(x) \quad \forall\ (t, x) \in \mathbb{R} \times D_i,$$

where $\boldsymbol{\phi}_i(t, x) \in \mathcal{X}_i^+$ is as in Lemma 2.1.

Proof. Let $\widetilde{X} = \{\mathbf{u}(t) \in C(\mathbb{R}, \mathbb{R}^K) |\ \mathbf{u}(t+T) = \mathbf{u}(t)\}$. First of all, Floquent theory for periodic ordinary differential equations implies that for any $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha > \max_{x \in D_i} h_i(x)$ and any given $x \in D_i$, $(\alpha \mathbf{I} + \partial_t - \mathcal{A}_i(x))^{-1}$ exists in $\widetilde{X}$, where $(\mathcal{A}_i(x)\mathbf{u})(t) = (\mathcal{A}\mathbf{u})(t, x)$ for $\mathbf{u} \in \widetilde{X}$. Hence $(\alpha \mathbf{I} + \partial_t - \mathcal{A}_i)^{-1}$ exists in $\mathcal{X}_i$.

By Lemma 2.1, $\lambda_i(x)$ is the principal eigenvalue of (2.1) with the positive eigenvector $\boldsymbol{\phi}_i(t, x)$ for any $1 \le i \le 3$ and any given $x \in D_i$. By the definition of $\mathcal{A}_i$ and $h_i(x)$, we have

$$-\partial_t \boldsymbol{\phi}_i(t, x) + (\mathcal{A}_i \boldsymbol{\phi}_i)(t, x) = h_i(x) \boldsymbol{\phi}_i(t, x)$$

for any $x \in D_i$, which implies that

$$-\partial_t \boldsymbol{\phi}_i(t, x) v_i(x) + (\mathcal{A}_i \boldsymbol{\phi}_i v_i)(t, x) = h_i(x) \boldsymbol{\phi}_i(t, x) v_i(x)$$

for every $v_i \in X_0$ and given $x \in D_i$. Hence

$$\Big((\alpha \mathbf{I} + \partial_t - \mathcal{A}_i) \boldsymbol{\phi}_i v_i\Big)(t, x) = (\alpha - h_i(x)) \boldsymbol{\phi}_i(t, x) v_i(x)$$

for every $\alpha \in \mathbf{C}$, each $x \in D_i$ and each $v_i \in X_0$. It then follows that for any $\alpha \in \mathbf{C}$ with $\operatorname{Re}\alpha > \max_{x \in D_i} h_i(x)$ and any $v_i \in X_0$,

$$\Big((\alpha \mathbf{I} + \partial_t - \mathcal{A}_i)^{-1} \boldsymbol{\phi}_i v_i\Big)(t, x) = \frac{1}{\alpha - h_i(x)} \boldsymbol{\phi}_i(t, x) v_i(x) \quad \forall\ (t, x) \in \mathbb{R} \times D_i. \tag{3.3}$$

This completes the proof. $\square$

Proposition 3.3. *Let $1 \le i \le 3$ be given.*

(1) *$-\partial_t + \mathcal{A}_i$ generates a positive semigroup of contractions on $\mathcal{X}_i$.*
(2) *$\mathcal{K}_i(\alpha\mathbf{I} + \partial_t - \mathcal{A}_i)^{-1}$ is compact for every $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha > \max_{x\in D_i} h_i(x)$.*

Proof. (1) Define $T(s)\mathbf{u}$ by

$$(T(s)\mathbf{u})(t,x) = \mathbf{u}(t-s,x)$$

for $s \ge 0$ and $\mathbf{u} \in \mathcal{X}_i$. Then $T(s)$ is a continuous semigroup generated by $-\partial_t$ on $\mathcal{X}_i$. It is clear that $\|T(s)\mathbf{u}\| \le \|\mathbf{u}\|$ for all $s \ge 0$ and $\mathbf{u} \in \mathcal{X}_i$ and that for any $\mathbf{u} \in \mathcal{X}_i$ with $\mathbf{u}(\cdot,\cdot) \ge \mathbf{0}$, $T(s)\mathbf{u} \ge \mathbf{0}$ for any $s \ge 0$. Therefore, $\{T(s)\}_{s\in\mathbb{R}^+}$ is a positive continuous semigroup of contraction on $\mathcal{X}_i$ with generator $-\partial_t$.

On the other hand, given $\mu > 0$ and $\mathbf{u} \in \mathcal{X}_i$, (3.2) yields

$$\begin{aligned}\|\mathbf{u}\|\|(\mu\mathbf{I}-\mathcal{A}_i)\mathbf{u}\| &\ge |\mathbf{u}^T(t,x)((\mu\mathbf{I}-\mathcal{A}_i)\mathbf{u})(t,x)| = \mu|\mathbf{u}(t,x)|^2 - \mathbf{u}^T(t,x)\mathcal{A}_i(t,x)\mathbf{u}(t,x)\\ &\ge \mu|\mathbf{u}(t,x)|^2\end{aligned}$$

for all $(t,x) \in \mathbb{R}\times D_i$, which implies that $\|(\mu\mathbf{I} - \mathcal{A}_i)\mathbf{u}\| \ge \mu\|\mathbf{u}\|$ for any $\mathbf{u} \in \mathcal{X}_i$ and $\mu > 0$. By [29, Theorem 1.4.2], we have that $\mathcal{A}_i$ is dissipative on $\mathcal{X}_i$. It then follows from [29, Corollary 3.3.3] that $-\partial_t + \mathcal{A}_i$ is the infinitesimal generator of continuous semigroup of contraction on $\mathcal{X}_i$.

(2) First, Proposition 3.2 guarantees that $(\alpha\mathbf{I} + \partial_t - \mathcal{A}_i)^{-1}$ exists for every α with $\operatorname{Re}\alpha > \max_{x\in D_i} h_i(x)$. Next for any bounded sequence $\{\mathbf{u}_n\} \in \mathcal{X}_i \oplus \mathcal{X}_i$, let

$$\mathbf{w}_n = \left(\mathcal{K}_i(\alpha\mathbf{I} + \partial_t - \mathcal{A}_i)^{-1}\right)\mathbf{u}_n$$

and

$$\mathbf{v}_n = (\alpha\mathbf{I} + \partial_t - \mathcal{A}_i)^{-1}\,\mathbf{u}_n.$$

Then

$$\mathbf{w}_n = \mathcal{K}_i\mathbf{v}_n.$$

By the boundedness of $\mathcal{A}_i$, both $\{\mathbf{v}_n\}$ and $\{\partial_t\mathbf{v}_n\}$ are bounded sequences in $\mathcal{X}_i \oplus \mathcal{X}_i$. Then $\{\mathbf{w}_n\} = \{\mathcal{K}_i\mathbf{v}_n\}$, $\{\partial_t\mathbf{w}_n\}$, and $\{\partial_{x_i}\mathbf{w}_n\}$ $(i = 1, 2, \ldots, N)$ are bounded sequences in $\mathcal{X}_i \oplus \mathcal{X}_i$. We can then show that $\{\mathbf{w}_n\}$ has a convergent subsequence. This completes the proof. □

Proposition 3.4. *For given $1 \le i \le 3$ and $\alpha > \max_{x\in D_i} h_i(x)$, let*

$$r(\mathcal{K}_i(\alpha\mathbf{I} + \partial_t - \mathcal{A}_i)^{-1})$$

be the spectral radius of $\mathcal{K}_i(\alpha\mathbf{I} + \partial_t - \mathcal{A}_i)^{-1}$. Let $x_0 \in D_i$ be such that $h_i(x_0) = \max_{x\in D_i} h_i(x)$. If there is a bounded domain $D_0 \subset D_i$ such that $1/(h_i(x_0) - h_i(\cdot)) \notin L^1(D_0)$, then there is $\alpha > \max_{x\in D_i} h_i(x)$ such that

$$r(\mathcal{K}_i(\alpha\mathbf{I} + \partial_t - \mathcal{A}_i)^{-1}) > 1.$$

Proof. By Proposition 3.2, $(\alpha\mathbf{I} + \partial_t - \mathcal{A}_i)^{-1}$ exists for all $\alpha \in \mathbb{R}$ with $\alpha > h_i(x_0) = \max_{x\in D_i} h_i(x)$, and

$$\left((\alpha\mathbf{I} + \partial_t - \mathcal{A}_i)^{-1}\boldsymbol{\phi}_i v_i\right)(t,x) = \frac{1}{\alpha - h_i(x)}\boldsymbol{\phi}_i(t,x)v_i(x) \tag{3.4}$$

for all $v_i \in X_0$. Let $\delta > 0$ be such that $B(0,\delta) \subset \operatorname{supp}(\kappa(\cdot))$. By (3.4) we have

$$\Big(\mathcal{K}_i(\alpha\mathbf{I} + \partial_t - \mathcal{A}_i)^{-1}\boldsymbol{\phi}_i v_i\Big)(t,x) = \int_{D_i} \frac{\kappa(y-x)\boldsymbol{\phi}_i(t,y)v_i(y)}{\alpha - h_i(y)}dy.$$

Note that $\boldsymbol{\phi}_i(t,x) \geq \eta_i \mathbf{e}$ for any $t \in \mathbb{R}$ and $x \in D_i$, where

$$\eta_i = \min_{t\in\mathbb{R}, x\in D_i} \{\phi_{i1}(t,x), \dots, \phi_{iK}(t,x)\}$$

and $\mathbf{e} = (1, 1, \dots, 1)^\top$. Then, given $v_i \in X_0$ with $v_i(x) \geq 0$,

$$\Big(\mathcal{K}_i(\alpha\mathbf{I} + \partial_t - \mathcal{A}_i)^{-1}\boldsymbol{\phi}_i v_i\Big)(t,x) \geq \int_{D_i} \frac{\eta_i \kappa(y-x)v_i(y)}{\alpha - h_i(y)} dy\, \mathbf{e}. \tag{3.5}$$

Let $\sigma > 0$ and $x_1 \in D_i$ be such that $2\sigma < \delta$, $B(x_1, \sigma) \subset D_0$, $B(x_1, 2\sigma) \subset D_i$, and

$$\lim_{\alpha \to h_i(x_0)} \int_{B(\sigma,x_1)} \frac{1}{\alpha - h_i(y)} dy = \infty$$

$(B(x_1, r) = \{x \in \mathbb{R}^N \,|\, |x| < r\})$. Let $v_i(x) \in X_0$ be such that $v_i(x) = 1$ if $x \in B(\sigma, x_1)$ and $v_i(x) = 0$ if $x \in D_i \setminus B(2\sigma, x_1)$.

Clearly, for every $x \in D_i \setminus B(2\sigma, x_1)$ and $\gamma > 1$,

$$\Big(\mathcal{K}_i(\alpha\mathbf{I} + \partial_t - \mathcal{A}_i)^{-1}\boldsymbol{\phi}_i v_i\Big)(t,x) > \gamma \boldsymbol{\phi}_i(t,x)v_i(x) = 0.$$

For $x \in B(2\sigma, x_1)$, there is $\widetilde{M} > 0$ such that $\kappa(y-x) \geq \widetilde{M}$ for $y \in B(\sigma, x_1)$. It then follows from (3.5) that for $x \in B(2\sigma, x_1)$,

$$\begin{aligned}\Big(\mathcal{K}_i(\alpha\mathbf{I} + \partial_t - \mathcal{A}_i)^{-1}\boldsymbol{\phi}_i v_i\Big)(t,x) &\geq \int_{B(\sigma,x_1)} \frac{\eta_i \kappa(y-x)v_i(y)}{\alpha - h_i(y)} dy\, \mathbf{e} \\ &\geq \int_{B(\sigma,x_1)} \frac{1}{\alpha - h_i(y)} dy\, \eta_i \widetilde{M}\mathbf{e}.\end{aligned}$$

Then there is $\gamma > 1$ such that, for $\alpha > h_i(x_0)$ and $\alpha - h_i(x_0) \ll 1$,

$$\Big(\mathcal{K}_i(\alpha\mathbf{I} + \partial_t - \mathcal{A}_i)^{-1}\boldsymbol{\phi}_i v_i\Big)(t,x) \geq \gamma \phi_i(t,x)v_i(x).$$

This implies that $r(\mathcal{K}_i(\alpha\mathbf{I} + \partial_t - \mathcal{A}_i)^{-1}) \geq \gamma > 1$ for $\alpha > h_i(x_0)$ and $\alpha - h_i(x_0) \ll 1$. $\square$

4. Principal eigenvalues of coupled nonlocal dispersal

In this section, we investigate criteria for the existence of principal eigenvalues of coupled nonlocal dispersal systems with time periodic dependence and prove Theorems 2.1-2.3. Throughout this section, we assume that (H1) and (H2) hold. Recall that $\lambda_i(A_i) = \sup\{\mathrm{Re}\,\lambda \,|\, \lambda \in \sigma(-\partial_t + \mathcal{K}_i + \mathcal{A}_i)\}$ is the principal spectrum point of $-\partial_t + \mathcal{K}_i + \mathcal{A}_i$ $(i = 1, 2, 3)$. We first prove a lemma.

Lemma 4.1.

(1) *For given $1 \leq i \leq 3$, if there is $\alpha_0 > \max_{x\in D_i} h_i(x)$ such that $r(\mathcal{K}_i(\alpha_0\mathbf{I} + \partial_t - \mathcal{A}_i)^{-1}) > 1$, then $\lambda_i(A_i) > \max_{x\in D_i} h_i(x)$, $r(\mathcal{K}_i(\lambda_i(A_i)\mathbf{I} + \partial_t - \mathcal{A}_i)^{-1}) = 1$, and $\lambda_i(A_i)$ is an isolated eigenvalue of $-\partial_t + \mathcal{K}_i + \mathcal{A}_i$ of finite multiplicity with a positive eigenfunction.*

(2) *If $\lambda_i(A_i)$ is an eigenvalue of $-\partial_t + \mathcal{K}_i + \mathcal{A}_i$ with a positive eigenfunction, then it is geometrically simple.*

Proof. (1) Suppose that there is $\alpha_0 > \max_{x\in D_i} h_i(x)$ such that

$$r(\mathcal{K}_i(\alpha\mathbf{I} + \partial_t - \mathcal{A}_i)^{-1}) > 1.$$

By Proposition 3.3, $\mathcal{K}_i(\alpha\mathbf{I} + \partial_t - \mathcal{A}_i)^{-1}$ is a compact operator for any $\alpha \in \mathbb{C}$ with $\mathrm{Re}\,\alpha > \max_{x\in D_i} h_i(x)$. It then follows from [4, Theorem 2.2] that $\lambda_i(A_i) > \max_{x\in D_i} h_i(x)$, $r(\mathcal{K}_i(\lambda_i(A_i)\mathbf{I}+\partial_t-\mathcal{A}_i)^{-1}) = 1$, and $\lambda_i(A_i)$ is an isolated eigenvalue of $-\partial_t + \mathcal{K}_i + \mathcal{A}_i$ with finite multiplicity and a positive eigenfunction.

(2) We show that $\lambda_i(A_i)$ is geometrically simple. Suppose that $\mathbf{v}(t,x)$ is a positive eigenfunction of $-\partial_t+\mathcal{K}_i+\mathcal{A}_i$ associated to $\lambda_i(A_i)$. By Proposition 2.1, $\mathbf{v}(t,x) > \mathbf{0}$ for $t \in \mathbb{R}$ and $x \in D_i$. Assume that $\mathbf{v}'(t,x)$ is also an eigenfunction of $-\partial_t+\mathcal{K}_i+\mathcal{A}_i$ associated to $\lambda_i(A_i)$. Then there is $\eta \in \mathbb{R}$ such that

$$\mathbf{w}(t,x) \geq 0 \quad \forall\, t \in \mathbb{R},\ x \in D_i \quad \text{and} \quad \mathbf{w}(t_0,x_0) = 0,$$

where $\mathbf{w}(t,x) = \mathbf{v}(t,x) - \eta\mathbf{v}'(t,x)$. Then by Proposition 2.1 again, $\mathbf{w}(t,x) \equiv 0$ and hence $\mathbf{v}(t,x) = \eta\mathbf{v}'(t,x)$. This implies that $\lambda_i(A_i)$ is geometrically simple. This completes the proof. □

Proof of Theorem 2.1. We prove the case $i = 1$. Other cases can be proved similarly.

First, assume that $\lambda = \lambda_1(A)$ is the principal eigenvalue of $-\partial_t + \mathcal{K}_1 + \mathcal{A}_1$ with a positive eigenfunction $\boldsymbol{\psi}_1(t,x) \in \mathcal{X}_1^+$. Then we have

$$-\partial_t\boldsymbol{\psi}_1(t,x) + (\mathcal{K}_1\boldsymbol{\psi}_1)(t,x) + (\mathcal{A}_1\boldsymbol{\psi}_1)(t,x) = \lambda_1(A_1)\boldsymbol{\psi}_1(t,x), \quad x \in D_1. \tag{4.1}$$

Recall that $\mathbf{\Phi}_1(t,s;A_i)$ is the solution operator of (1.1). Note that $\mathbf{u}(t,x) = e^{\lambda t}\boldsymbol{\psi}_1(t,x)$ is a solution of (1.1), which implies that

$$\left(\mathbf{\Phi}_1(t,0;A_1)\boldsymbol{\psi}_1(0,\cdot)\right)(t,x) = e^{\lambda t}\boldsymbol{\psi}_1(t,x).$$

Then by the comparison principle, we have $\boldsymbol{\psi}_1 \gg \mathbf{0}$.

For any given $x_0 \in D_1$, we know that $\boldsymbol{\phi}_1(t,x_0)$ is an eigenfunction of (2.1) corresponding to the eigenvalue $\lambda_1(x_0)$, that is,

$$-\frac{d}{dt}\boldsymbol{\phi}_1(t,x_0) + A_1(t,x_0)\boldsymbol{\phi}_1(t,x_0) = \lambda_1(x_0)\boldsymbol{\phi}_1(t,x_0). \tag{4.2}$$

By (4.1),

$$-\partial_t\boldsymbol{\psi}_1(t,x_0) - \boldsymbol{\psi}_1(t,x_0) + (\mathcal{A}_1\boldsymbol{\psi}_1)(t,x_0) < \lambda_1(A_1)\boldsymbol{\psi}_1(t,x_0) \tag{4.3}$$

for every $x_0 \in D_1$. (4.2) and (4.3) yield

$$\lambda_1(A_1) > -1 + \lambda_1(x_0) \qquad \text{for } x_0 \in D_1.$$

Hence $\lambda_1(A_1) > \max_{x\in D_1}\{-1 + \lambda_1(x)\}$.

Conversely, assume that $\lambda_1(A_1) > \max_{x\in D_1} h_1(x)$. By Lemma 4.1, $\lambda_1(A_1)$ is the principal eigenvalue of $-\partial_t + \mathcal{K}_1 + \mathcal{A}_1$. This completes the proof. □

Proof of Theorem 2.2. (1) By Proposition 3.4, $r(\mathcal{K}_i(\alpha\mathbf{I} + \partial_t - \mathcal{A}_i)^{-1}) > 1$ for $0 < \alpha - \max_{x\in D_i} h_i(x) \ll 1$. Lemma 4.1 implies that $\lambda_i(A_i)$ is the principal eigenvalue of $-\partial_t + \mathcal{K}_i - \mathcal{A}_i$.

(2) We prove the case $i = 3$. Case $i = 1$ can be proved similarly. By Proposition 3.2, we have

$$\left(\mathcal{K}_3(\alpha\mathbf{I} + \partial_t - \mathcal{A}_3)^{-1}\boldsymbol{\phi}_3 v_3\right)(t,x) = \int_{\mathbb{R}^N} \frac{\kappa(y-x)}{\alpha - h_3(y)}\boldsymbol{\phi}_3(t,y)v_3(y)dy$$

for every $\alpha > \max_{x\mathbb{R}^N} h_3(x)$ and any $v_3(x) \in X_0$ with $v_3(x) \geq 0$.

Note that $\boldsymbol{\phi}_3(t,x) \geq \eta_3\mathbf{e}$ for any $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$. Let $v_3(x) \equiv 1$ for all $x \in \mathbb{R}^N$. Then

$$\left(\mathcal{K}_3(\alpha\mathbf{I} + \partial_t - \mathcal{A}_3)^{-1}\boldsymbol{\phi}_3\right)(t,x) \geq \int_{\mathbb{R}^N} \frac{\kappa(y-x)}{\alpha - h_3(y)} dy\ \eta_3\mathbf{e}.$$

Take $\alpha = -1 + \max_{x\in\mathbb{R}^N} \lambda_3(x) + \epsilon$ for $\epsilon \in (0,1)$. Then for any $0 < \epsilon < 1$,

$$\begin{aligned}
&\left(\mathcal{K}_3(\alpha\mathbf{I} + \partial_t - \mathcal{A}_3)^{-1}\boldsymbol{\phi}_3\right)(t,x)\\
&\geq \frac{1}{\max_{x\in\mathbb{R}^N}\lambda_3(x) - \min_{x\in\mathbb{R}^N}\lambda_3(x) + \epsilon} \int_{\mathbb{R}^N} \kappa(y-x)dy\ \eta_3\mathbf{e}\\
&= \frac{\eta_3\mathbf{e}}{\max_{x\in\mathbb{R}^N}\lambda_3(x) - \min_{x\in\mathbb{R}^N}\lambda_3(x) + \epsilon}.
\end{aligned} \tag{4.4}$$

By $\max_{x\in\mathbb{R}^N}\lambda_3(x) - \min_{x\in\mathbb{R}^N}\lambda_3(x) < \frac{\eta_3}{\tilde{\eta}_3}$ and (4.4), for $0 < \epsilon \ll 1$, there is $\gamma > 1$ such that $\left(\mathcal{K}_3(\alpha\mathbf{I} + \partial_t - \mathcal{A}_3)^{-1}\boldsymbol{\phi}_3\right)(t,x) \geq \gamma\boldsymbol{\phi}_3(t,x)$ and hence

$$r(\mathcal{K}_3(\alpha\mathbf{I} + \partial_t - \mathcal{A}_3)^{-1}) > 1.$$

It then follows from Lemma 4.1 that the principal eigenvalue of $-\partial_t + \mathcal{K}_3 - \mathcal{A}_3$ exists.

(3) It can be proved by similar arguments as in [24, Theorem 2.6]. □

Proof of Theorem 2.3. Note that for $\alpha > \max_{x\in D_i} h_i(x)$, $(\alpha\mathbf{I} + \partial_t - \mathcal{A}_i)^{-1}$ exists. For given $\alpha > \max_{x\in D_i} h_i(x)$, let

$$(\mathbf{U}_\alpha\mathbf{u})(t,x) = \left(\mathcal{K}_i(\alpha\mathbf{I} + \partial_t - \mathcal{A}_i)^{-1}\mathbf{u}\right)(t,x) \quad \text{and} \quad r(\alpha) = r(\mathbf{U}_\alpha).$$

By Proposition 3.3, $\mathbf{U}_\alpha : \mathcal{X}_i \to \mathcal{X}_i$ is a positive and compact operator. Let $\lambda_i(A_i)$ be the principal eigenvalue of $-\partial_t + \mathcal{K}_i + \mathcal{A}_i$. By Lemma 4.1, $\lambda_i(A_i)$ is an isolated geometrically simple eigenvalue of $-\partial_t + \mathcal{K}_i + \mathcal{A}_i$. Let $\alpha_0 = \lambda_i(A_i)$. Then $r(\alpha_0) = 1$ and $r(\alpha_0)$ is an isolated geometrically simple eigenvalue of $\mathbf{U}_{\alpha_0}$ with $\boldsymbol{\phi}_i(\cdot,\cdot)$ being a positive eigenfunction.

The rest of the proof is similar to that in [31, Theorem 3.1]. □

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School of Science, Chang'an University, Xi'an, Shaanxi 710064, People's Republic of China

E-mail address: `baoxx2016@chd.edu.cn`

Department of Mathematics and Statistics, Auburn University, Auburn University, Alabama 36849

E-mail address: `wenxish@auburn.edu`