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ON THE NOTION OF RANDOM CHAOS

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Abstract. Deterministic chaos is investigated for random dynamical systems in dimension one. Some well-known as well as new Li-Yorke-type theorems are randomized. Deterministic chaos exhibited by random dynamics is therefore called random chaos for brevity. Chaotic random dynamics are also studied for multivalued maps.

1. Introduction

The notion of random chaos seems to be a logical contradiction, because deterministic chaos used to be traditionally interpreted as an antipode to stochastic randomness. On the other hand, we presented in [2, Examples 1 and 2] simple illustrative examples of randomly perturbed logistic and tent maps which exhibit a very similar behaviour like paradigmatic (for deterministic chaos) unperturbed maps. Further results of this type were given in physical journals (see e.g. [17, 22, 26, 32]). Nice examples of the relationship between deterministic chaos and noise can be found in [19]. Hence, unlike in [29], where stochastic chaos was exclusively studied under the same name "random chaos", we should rather speak here about deterministic chaos in random dynamical systems or, more appropriately w.r.t. our approach, about randomized deterministic chaos.

It is well known (see Theorem 2 below) that continuous interval maps are chaotic in the sense of Devaney if and only if they admit an m-orbit, for some $m \neq 2^k$, where $k \in \mathbb{N} \cup \{0\}$. The same is true (see Theorem 3 below) for continuous circle maps, provided they have fixed points. The first statement for interval maps is equivalent to a theorem of Bowen and Franks [9] (see Proposition 1 below) formulated in terms of a positive topological entropy. This theorem significantly generalizes those reflected by the title "Period $\neq 2^n$ implies chaos" (in the sense of Li–Yorke), obtained independently by Oono [24] and Graw [13], and in particular the celebrated "Period three implies chaos" by Li and Yorke [21]. For continuous circle maps, the related analogies were formulated in terms of a positive topological entropy in [1, Chapter 3]. In particular, these results yield "Periods one, two, three imply chaos on S^1" proved by Siegberg [28].

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Hence, our main aim is to randomize possibly all these deterministic theorems. It will be fully done for continuous interval maps (see Theorem 4 below), while for continuous circle maps we will be able to randomize only one way (sufficiency) implications (see Theorem 5 below). The related obstructions will be described.

Furthermore, we will give necessary and sufficient conditions to random chaos in terms of random orbits for a class of multivalued maps, studied in deterministic case by the author in [3]. Rather curiously, unlike in other theorems presented in this paper, any nontrivial (i.e. different from a fixed point) random orbit implies there random chaos, while random chaos reversely implies the existence of random orbits with all even periods, jointly with random fixed points.

All the results are based on our universal randomization technique (see Proposition 8 below) developed in [2,5], and suitable definitions of random chaos introduced here for the first time.

2. Some preliminaries

By $\mathbb{N}$, $\mathbb{R}$, we denote as usual the set of natural numbers and the set of real numbers, respectively. By $\mathbb{I}$, we denote an arbitrary compact interval in $\mathbb{R}$, i.e., $\mathbb{I} = [a, b]$, where $a, b \in \mathbb{R}$, $a \neq b$. By S^1, we denote an arbitrary circle with a finite radius $r > 0$ in the plane $\mathbb{R}^2$, i.e., $S^1 = \{(x, y) \in \mathbb{R}^2 \colon (x - x_0)^2 + (y - y_0)^2 = r^2$, where $(x_0, y_0) \in \mathbb{R}^2$ is a fixed center$\}$.

The triple (Ω, Σ, μ) denotes a complete measure space with measure μ and the pair (X, d) always denotes a metric space with metric d. For σ-algebras Σ_1 and Σ_2, by $\Sigma_1 \otimes \Sigma_2$, we mean the product σ-algebra. The symbol $\mathcal{B}(X)$ denotes the σ-algebra of Borel sets in X.

We naturally identify a relation $\varphi \subset A \times B$ with a mapping $\varphi \colon A \to \mathcal{P}(B)$. Furthermore, if $\varphi \subset (A \times B) \times C$, i.e., $\varphi \colon A \times B \to \mathcal{P}(C)$, then for any $x \in A$ we define the relation $\varphi_x := \varphi(x, \cdot)$. A multivalued map $\varphi \colon A \multimap B$ is a relation with nonempty values, i.e., $\varphi \colon A \to \mathcal{P}(B) \setminus \{\emptyset\}$. We also identify a single-valued map $f \colon A \to B$ with the multivalued map with one-element values fulfilling the condition $\varphi(x) = \{f(x)\}$, for every $x \in A$.

By a superposition of a map $f \colon B \to C$ with a map $\varphi \subset A \times B$, we mean the relation $f \circ \varphi \subset A \times C$ defined by $f \circ \varphi(a) = f(\varphi(a))$, for $a \in A$. By a product of relations $F_i \subset A \times B$, for $i \in I$, we mean the relation $\prod_{i\in I} F_i$ defined by $(\prod_{i\in I} F_i)(a) = \prod_{i\in I} F_i(a)$, for $a \in A$.

We denote by

$$\varphi_+^{-1}(B) := \{\omega \in \Omega \colon \varphi(\omega) \cap B \neq \emptyset\} \text{ and } \varphi^{-1}(B) := \{\omega \in \Omega \colon \varphi(\omega) \subset B\}$$

the *large* and *small preimages* of the set $B \subset X$, under the relation $\varphi \subset \Omega \times X$, respectively.

Definition 1. A relation $\varphi \subset \Omega \times X$ with closed values is called *measurable* if $\varphi_+^{-1}(F) \in \Sigma$, for every closed $F \subset X$. It is called *weakly measurable* if $\varphi_+^{-1}(G) \in \Sigma$, for every open $G \subset X$.

It will be convenient to recall here several facts from the measure theory (see e.g. [6, Chapter I.3]). Every measurable relation is weakly measurable. For a compact-valued $\varphi \subset \Omega \times X$, the notions of measurability and weak measurability coincide. The product of at most countably many weakly measurable relations is weakly measurable. The superposition of a single-valued continuous map with a weakly measurable relation is weakly measurable (which is clear from the definition). The superposition of a single-valued measurable map with a (single-valued) Carathéodory map is measurable. A measurable mapping $\varphi\colon \Omega \multimap X$, where X is a separable complete space, has according to the Kuratowski–Ryll–Nardzewski theorem (see e.g. [6, Theorem I.3.49]), a (single-valued) measurable selection.

Definition 2. Assume that $\varphi\colon A \multimap A$. A sequence $\{x_i\}_{i=0}^{k-1} \in A^k$ is called a *k-orbit* of the multivalued map φ, if $x_{i+1} \in \varphi(x_i)$, for $i < k-1$, $x_0 \in \varphi(x_{k-1})$, and there is no $m < k$ such that $m|k$ and $x_{sm+i} = x_i$, for $i < m$, and $s < \frac{k}{m}$, where $m|k$ means that m is a divisor of k.

Any sequence $\{x_i\}_{i=0}^{\infty}$ of elements $x_i \in A$ will be called a (deterministic) *orbit* of mapping $\varphi\colon A \multimap A$, provided $x_{i+1} \in \varphi(x_i)$, for all $i = 0, 1, \dots$.

Definition 3. Let $A \subset X$ be a closed subset and $\varphi\colon \Omega \times A \multimap X$ be a multivalued map with closed values. We say that φ is a *random operator* if it is measurable with respect to σ-algebra $\Sigma \otimes \mathcal{B}(X)$.

Definition 4. Let $A \subset X$ be a closed subset and $\varphi\colon \Omega \times A \multimap X$ be a random operator. A sequence of measurable functions $\{\xi_i\}_{i=0}^{k-1}$, where $\xi_i\colon \Omega \to A$, for $i = 0, \dots, k-1$, is called a *random k-orbit* of the operator φ if

(a) $\xi_{i+1}(\omega) \in \varphi(\omega, \xi_i(\omega))$, for $i = 0, \dots, k-2$, and $\xi_0(\omega) \in \varphi(\omega, \xi_{k-1}(\omega))$, for almost all $\omega \in \Omega$,
(b) the sequence $\{\xi_i\}_{i=0}^{k-1}$ is not formed by going p-times around a shorter subsequence of m consecutive elements, where $mp = k$ (for almost all $\omega \in \Omega$).

Equivalently, we can write:

(a) $\mu(\Omega \setminus \{\omega \in \Omega\colon \forall i < k-1\, \xi_{i+1}(\omega) \in \varphi(\omega, \xi_i(\omega))$
$\wedge\, \xi_0(\omega) \in \varphi(\omega, \xi_{k-1}(\omega))\}) = 0$,
(b) there is no $m < k$ such that $m \mid k$ and

$$\mu(\Omega \setminus \{\omega \in \Omega\colon \forall s < \tfrac{k}{m}\, \forall i < m\ \xi_{sm+i}(\omega) = \xi_i(\omega)\}) = 0.$$

Obviously, any sequence $\{\xi_i\}_{i=0}^{\infty}$ of measurable functions $\xi_i\colon \Omega \to A$, where $A \subset X$ is closed, will be called a *random orbit* of the random operator $\varphi\colon \Omega \times A \multimap X$, provided $\xi_{i+1}(\omega) \in \varphi(\omega, \xi_i(\omega))$, for all $i = 0, 1, \dots$, and almost all $\omega \in \Omega$.

Definition 5. A map $\varphi\colon X \multimap Y$ with closed values is said to be *upper semicontinuous* (u.s.c.) if, for every open $B \subset Y$, the set $\varphi^{-1}(B)$ is open in X, or equivalently, if $\varphi_+^{-1}(B)$ is closed in X. It is said to be *lower semicontinuous* (l.s.c.) if, for every open $B \subset Y$, the set $\varphi_+^{-1}(B)$ is open in X, or equivalently, if $\varphi^{-1}(B)$ is closed in X. If it is both u.s.c. and l.s.c., then it is called *continuous*.

Of course, if φ is u.s.c. or l.s.c., then it is measurable. If a single-valued $f\colon X \to Y$ is u.s.c. or l.s.c., then it is *continuous*. An l.s.c. map $\varphi\colon X \multimap Y$ with closed, convex values, where X is a compact space and Y is a Banach space has, according to Michael's selection theorem (cf. [6, Theorem I.3.30]), a single-valued continuous selection $f\colon X \to Y$ (written $f \subset \varphi$). For more details about semicontinuous multivalued maps, see e.g. [6, Chapter I.3].

Finally, let us recall the notions of deterministic chaos under our consideration. We will restrict ourselves to the most popular three definitions.

Definition 6 (cf. [21])**.** Let (X, d) be a metric space and $f\colon X \to X$ be a continuous mapping. We say that f is *chaotic in the sense of Li–Yorke* if there exists an uncountable "scrambled" subset $S \subset X$ such that, for all $x, y \in S$, $x \neq y$, the following two conditions are satisfied:

$(i)_{\mathrm{LY}}$ $\limsup_{n\to\infty} d\big(f^n(x), f^n(y)\big) > 0$,

$(ii)_{\mathrm{LY}}$ $\liminf_{n\to\infty} d\big(f^n(x), f^n(y)\big) = 0$.

Definition 7 (cf. [12])**.** Let (X, d) be a metric space and $f\colon X \to X$ be a continuous mapping. We say that f is *chaotic in the sense of Devaney* if there exists a positively invariant "chaotic" subset $S \subset X$ such that the following three conditions are satisfied:

$(i)_{\mathrm{D}}$ $f|_S\colon S \to S$ is topologically positive, i.e., there exists an orbit of $f|_S$ which is dense in S or, equivalently (according to the Birkhoff transitive theorem) that, for every two open subsets $U, V \subset S$, there exists $n \in \mathbb{N}$ such that
$$f^n(U) \cap V \neq \emptyset,$$

$(ii)_{\mathrm{D}}$ the set $\mathrm{Per}(f|_S)$ of periodic points of $f|_S$ is dense in S, i.e., $\overline{\mathrm{Per}(f|_S)} = S$, where the bar stands for the closure in S,

$(iii)_{\mathrm{D}}$ for every $\varepsilon > 0$ and any $s \in S$, there exist $\delta > 0$, $r \in \{x \in S\colon d(s, x) < \varepsilon\}$, and $n \in \mathbb{N}$ such that
$$d\big(f^n|_S(s), f^n|_S(r)\big) > \delta.$$

Remark 1. If $S \subset X$ is infinite, then $(i)_{\mathrm{D}}, (ii)_{\mathrm{D}} \Rightarrow (iii)_{\mathrm{D}}$. In other words, for an infinite subset S, condition $(iii)_{\mathrm{D}}$ is superfluous (cf. [8]). For interval maps, where $X = \mathbb{I}$, even $(i)_{\mathrm{D}} \Rightarrow (ii)_{\mathrm{D}}, (iii)_{\mathrm{D}}$, i.e., the sole transitivity implies Devaney's chaos (cf. [31]).

In order to define topological entropy, let us again consider a metric space (X, d) and a continuous mapping $f\colon X \to X$. For the metric
$$d_{n,f}(x, y) := \sup_{0\le j\le n} d\big(f^j(x), f^j(y)\big),$$
we say that the set $S \subset X$ is (n, ε)*-separated* for f, if $d_{n,f}(x, y) > \varepsilon$, for every pair of points $x, y \in S$, $x \neq y$.

The *number of different orbits of the length* n is then defined by
$$r(n, \varepsilon, f) := \max\{\#(S)\colon S \subset X \text{ is } (n, \varepsilon)\text{-separated for } f\},$$
where $\#(S) = \mathrm{card}(S)$ is the cardinality of S.

The related *asymptotic growth rate* can be characterized by means of the function
$$h(\varepsilon, f) := \limsup_{n\to\infty} \frac{\log(r(n, \varepsilon, f))}{n},$$
which satisfies $0 \le h(\varepsilon, f) \le \infty$ and, on compact metric spaces X, $0 \le h(\varepsilon, f) < \infty$.

Hence, we are ready to give the third definition of deterministic chaos.

Definition 8. By *topological entropy* $h(f)$ of $f\colon X \to X$, we mean
$$h(f) := \lim_{\varepsilon\to 0+} h(\varepsilon, f).$$
We say that f is chaotic in the sense that it has a *positive topological entropy* if simply $h(f) > 0$.

For more details and properties of topological entropy as well as for its relationship to various sorts of deterministic chaos (including those in Definition 6 and Definition 7) see e.g. [1, Chapter 4]).

3. Review of deterministic results to be randomized

It will be convenient to recall some deterministic results to be randomized. At first, let us recall the relevant statements for interval maps. The following theorem was obtained independently by Oono [24] and Graw [13].

Theorem 1 (cf. [13,24])**.** *Let $f\colon \mathbb{I} \to \mathbb{I}$, $\mathbb{I} \subset \mathbb{R}$, be a continuous map which has an m-orbit, for some natural $m \neq 2^k$, $k \in \mathbb{N} \cup \{0\}$. Then f is chaotic in the sense of Li–Yorke.*

A very important generalization of Theorem 1, which we state here in the form of a proposition, is due to Bowen and Franks [9] (cf. also [1, Theorem 4.4.20]).

Proposition 1 (cf. [9])**.** *Let $f\colon \mathbb{I} \to \mathbb{I}$, $\mathbb{I} \subset \mathbb{R}$, be a continuous map. It has an m-orbit, for some natural $m \neq 2^k$, $k \in \mathbb{N} \cup \{0\}$, if and only if f has a positive topological entropy, i.e., $h(f) > 0$.*

In view of for instance [20, Theorem], Proposition 1 can be equivalently reformulated as follows:

Theorem 2. *Let $f\colon \mathbb{I} \to \mathbb{I}$, $\mathbb{I} \subset \mathbb{R}$, be a continuous map. It has an m-orbit, for some natural $m \neq 2^k$, $k \in \mathbb{N} \cup \{0\}$, if and only if f is chaotic in the sense of Devaney.*

Remark 2. Further possible equivalent reformulations of Proposition 1 can be obtained by means of results in [20].

Lemma 1 (cf. [11,16])**.** *Let $f\colon \mathbb{I} \to \mathbb{I}$, $\mathbb{I} \subset \mathbb{R}$, be a continuous map which is chaotic in the sense of Devaney. Then f admits a 6-orbit.*

The following statements concern circle maps. Although, from our point of view, the most relevant result, Theorem 3 below, is like well-known folklore (see e.g. the arguments in [1, Chapters 3, 4]), we could not see its explicit formulation in the literature. Hence, we will deduce it here from the following series of results due to various authors.

Lemma 2 (cf. [7, Proposition 3.2])**.** *Let $f\colon S^1 \to S^1$ be a continuous self-map. Then, for any $n \in \mathbb{N}$, the map f is chaotic in the sense of Devaney (resp. Li–Yorke) if and only if $f^n\colon S^1 \to S^1$ is so.*

Proposition 2 (cf. e.g. [15, 18], and the references therein)**.** *The space $C(S^1)$ of continuous self-maps on the circle S^1 can be decomposed as follows:*

$$C(S^1) = C_1(S^1) \cup C_2(S^1) \cup C_3(S^1),$$

where

$$\begin{aligned}
C_1(S^1) &:= \{f \in C(S^1)\colon f \text{ has neither periodic orbits nor fixed points}\},\\
C_2(S^1) &:= \{f \in C(S^1)\colon f^n \text{ has, for some } n \in \mathbb{N}, \text{ either a fixed point or}\\
&\qquad\quad m\text{-orbits, where } m = 1, 2, 2^2, \dots\},\\
C_3(S^1) &:= \{f \in C(S^1)\colon f^n \text{ has, for some } n \in \mathbb{N}, m\text{-orbits, for all } m \in \mathbb{N}\}.
\end{aligned}$$

Proposition 3 (cf. [23, Main Theorem, p. 454])**.** *Let $f\colon S^1 \to S^1$ be a continuous self-map on the circle. Then the following two conditions are equivalent:*

(I) *$h(f) > 0$, i.e., f has a positive topological entropy,*
(II) *f is chaotic in the sense of Devaney.*

Proposition 4 (cf. [28, Theorem, p. 353 and Theorem 3.2, p. 367])**.** *Let $f\colon S^1 \to S^1$ be a continuous self-map on the circle S^1. If f has m-orbits, for $m = 1, 2, 3$, then f is chaotic in the sense of Li–Yorke, and $h(f) \geq \frac{\log 2}{3} > 0$, where $h(f)$ stands for the topological entropy of f.*

Now, let us recall that to every continuous self-map $f\colon S^1 \to S^1$, one can associate the integer degree $d = d(f)$ such that $|d - 1| = N(f)$, where $N(f)$ stands for the Nielsen number of f. It means that its lifting F satisfies $F(x + 1) = F(x) + d$. For more details, see e.g. [1, Chapter 3].

Proposition 5 (cf. [1, p. 264])**.** *Let $f\colon S^1 \to S^1$ be a continuous map of degree $d = 0$ or $d = -1$. Then $h(f) > 0$ if and only if f has an m-orbit such that $m \neq 2^k$, $k \in \mathbb{N} \cup \{0\}$.*

Proposition 6 (cf. [1, p. 265])**.** *Let $f\colon S^1 \to S^1$ be a continuous map of degree $d = 1$. Then the following conditions are equivalent:*

(a) *$h(f) > 0$, where $h(f)$ stands for the topological entropy of f,*
(b) *f has n-orbits as well as m-orbits, for some $n, m \in \mathbb{N}$, such that neither $\frac{n}{m}$ nor $\frac{m}{n}$ is an integer,*
(c) *f has n-orbits as well as m-orbits, for some $n, m \in \mathbb{N}$, such that neither $\frac{n}{m}$ nor $\frac{m}{n}$ is of the form 2^k, $k \in \mathbb{N} \cup \{0\}$.*

Proposition 7 (cf. [10])**.** *Let $f\colon S^1 \to S^1$ be a continuous self-map and suppose that f has a fixed point and an n-orbit, for some $n \in \mathbb{N}$. Then one of the following possibilities holds:*

(i) *f possesses an m-orbit, for every $m > n$,*
(ii) *f possesses an m-orbit, for every m satisfying $n \triangleright m$, where the symbol "$\triangleright$" stands for the inequality in Sharkovsky's ordering of natural numbers:*

$$3 \triangleright 5 \triangleright 7 \triangleright \cdots \triangleright 2 \cdot 3 \triangleright 2 \cdot 5 \triangleright 2 \cdot 7 \triangleright \cdots \triangleright 2^2 \cdot 3 \triangleright 2^2 \cdot 5 \triangleright 2^2 \cdot 7 \triangleright \cdots \triangleright 8 \triangleright 4 \triangleright 2 \triangleright 1.$$

We are ready to prove the following theorem for circle maps.

Theorem 3. *Let $f\colon S^1 \to S^1$ be a continuous self-map on the circle S^1 which has a fixed point. Then the mapping f is chaotic in the sense of Devaney if and only if it has an m-orbit, for some natural $m \neq 2^k$, $k \in \mathbb{N} \cup \{0\}$.*

Proof. "if part" If f has an m-orbit, for some $m \neq 2^k$, $k \in \mathbb{N} \cup \{0\}$, then f^m has, according to Proposition 7, n-orbits, for $n = 1, 2, 3, \ldots$, because $m \triangleright m \cdot n > m$, for all $n \in \mathbb{N}$, where $m \neq 2^k$, $k \in \mathbb{N} \cup \{0\}$. Thus, according to Proposition 4, $h(f^m) > 0$ by which f^m is, in view of Proposition 3, chaotic in the sense of Devaney. Consequently, in view of Lemma 2, the same is true for f itself.

"only if part" Let f be chaotic in the sense of Devaney. Then, according to Proposition 3, $h(f) > 0$. For maps of degrees $d = -1$ or $d = 0$ or $d = 1$, in view of Proposition 5 and Proposition 6, f has an m-orbit such that $m \neq 2^k$, $k \in \mathbb{N} \cup \{0\}$, as claimed. Since the maps of degrees d with $d \geq 2$ or $d \leq -3$ possess periodic orbits of all least periods and those with $d = -2$ admit all but possibly except 2-orbits (cf. e.g. [1, Chapter 3]), the claim follows for them trivially. □

Remark 3. For maps having fixed points, Theorem 3 is quite analogous to Theorem 2 for the interval maps.

Remark 4. Observe that, in view of Proposition 5 and Proposition 6, $h(f) = 0$ holds for maps from the classes $C_1(S^1) \cup C_2(S^1)$ in Proposition 2. Thus, in view of Proposition 3, the chaos in the sense of Devaney concerns exclusively the class $C_3(S^1)$ in Proposition 2.

For the chaos in the sense of Li–Yorke, the situation is different (see e.g. [1, 7, 15, 18]. Nevertheless, since the chaos in the sense of Devaney implies the one in the sense of Li–Yorke (but not vice versa; cf. e.g. [7]), only a one way implication takes place as follows:

Corollary 1. *Let $f\colon S^1 \to S^1$ be a continuous self-map on the circle S^1 which has a fixed point. If f has an m-orbit, for some natural $m \neq 2^k$, $k \in \mathbb{N} \cup \{0\}$, then it is chaotic in the sense of Li–Yorke.*

Remark 5. For maps having fixed points, Corollary 1 is quite analogous to Theorem 1 obtained independently by Oono [24] and Graw [13] for the interval maps.

Remark 6. Unlike for the chaos in the sense of Devaney, there exist maps whose topological entropy is zero such that they are chaotic in the sense of Li–Yorke (see e.g. [15, 18, 30]). Since the maps from the class $C_1(S^1)$ are not chaotic in any sense (see e.g. [15, 18]), the maps which are chaotic in the sense of Li–Yorke, but not in the sense of Devaney, belong to the class $C_2(S^1)$ in Proposition 2 (see e.g. [15]).

4. Randomization principle and definitions of random chaos

Our randomization principle is based on the application of Proposition 8 below. Hence, it will be useful to make a partition of Ω as follows:

$$\Omega = \Omega_0 \cup \bigcup_{j=0}^{l-1} \Omega_{i_j}, \tag{1}$$

where

- $\Omega_m \in \Sigma$, for $m = 0, i_0, i_1, \dots, i_{l-1}$,
- $\mu(\Omega_0) = 0$ and $\mu(\Omega_{i_j}) > 0$, for $j = 0, \dots, l-1$,
- $\mathrm{LCM}\{i_j : j = 0, \dots, l-1\} = k$; the abbreviation LCM means the least common multiple of a given set of integers.

The following proposition which was proved in [2], and in a more abstract setting in [5], is crucial for our investigations in this paper.

Proposition 8 (cf. [2, 5])**.** *Let X be a Polish (i.e. complete and separable) space, $A \subset X$ be a closed subset and $\varphi\colon \Omega \times A \multimap X$ be a random operator. Then φ admits a random k-orbit $\{\xi_i\}_{i=0}^{k-1}$ if and only if there exists a partition of Ω as in* (1)*, jointly with an i_j-orbit $\{\xi_i(\omega)\}_{i=0}^{i_j-1}$ of $\varphi_\omega := \varphi(\omega, \cdot)$, for each $\omega \in \Omega_{i_j}$; $j < l$.*

It will also be convenient to recall the randomized versions of the Sharkovsky theorem [27], for interval maps, and a particular case of the Block theorem [10], for circle maps, obtained by means of Proposition 8 in [3] and [4].

Proposition 9 (cf. [3])**.** *Let $f\colon \Omega \times \mathbb{I} \to \mathbb{I}$ be such that $f(\cdot, x)\colon \Omega \to \mathbb{I}$ is measurable, for every $x \in \mathbb{I}$, on a complete measure space Ω, and $f(\omega, \cdot)\colon \mathbb{I} \to \mathbb{I}$ is continuous, for almost every $\omega \in \Omega$. If f has a random n-orbit, for some $n \in \mathbb{N}$, then it also possesses a random m-orbit, for every m satisfying $n \rhd m$, where the symbol "$\rhd$" stands for the inequality in Sharkovsky's ordering of natural numbers (see Proposition* 7*).*

Proposition 10 (cf. [4]). *Let $f\colon \Omega \times S^1 \to S^1$ be such that $f(\cdot, x)\colon \Omega \to S^1$ is measurable, for every $x \in S^1$, on a complete measure space Ω, and $f(\omega, \cdot)\colon S^1 \to S^1$ is continuous, for almost every $\omega \in \Omega$. Assume, furthermore, that f has a random fixed point. If f has a random n-orbit, for some $n \in \mathbb{N}$, where $n = p \cdot 2^t$ with an odd $p > 1$, then it also possesses, for each $m \in \mathbb{N}$, where $n \rhd m > n$, a random m-orbit.*

In view of Proposition 8, it is natural to give the following definitions of random chaos for single-valued random operators.

Definition 9. Let $X = (X, d)$ be a compact metric space and $\Omega = (\Omega, \mu)$ be a complete measure space. Assume that $f\colon \Omega \times X \to X$ is a random operator such that $f_\omega := f(\omega, \cdot)\colon X \to X$ is, for almost all $\omega \in \Omega$, a continuous (single-valued) mapping. We say that f is *weakly randomly chaotic* in a given sense if there exists a measurable subset $\Omega^* \subset \Omega$ with a positive measure, i.e., $\mu(\Omega^*) > 0$, such that f_ω is chaotic in a respective sense, for all $\omega \in \Omega^*$.

Definition 10. Under the assumptions of Definition 9, we say that f is *strongly randomly chaotic* in a given sense if f_ω is chaotic in a respective sense, for almost all $\omega \in \Omega$.

These definitions are really natural, because if, for instance, $X = \mathbb{I} \subset \mathbb{R}$ is a compact interval in $\mathbb{R}$ and f is randomly chaotic in the sense of Devaney, then at least a 6-orbit always exists, for every $\omega \in \Omega^* \subset \Omega$ (see Lemma 1). Moreover, $f_\omega\colon \mathbb{I} \to \mathbb{I}$ admits, according to the well-known Brouwer fixed point theorem, a fixed point, for almost all $\omega \in \Omega$. Thus, by means of Proposition 8 and the randomized Sharkovsky theorem, Proposition 9, infinitely many random m-orbits exist, for all m with $6 \rhd m$, jointly with a random 6-orbit. In other words, *all the given chaotic properties concern the consequences of random periodic orbits.*

Furthermore, since every chaotic (deterministic) map in the sense of Devaney is also chaotic in the sense of Li–Yorke, we have guaranteed, besides other things, that: for every random periodic point p (as an element of a random periodic orbit) and for every point $q(\omega)$, $\omega \in \Omega^*$, from the scrambled set $S_\omega \subset \mathbb{I}$, $\omega \in \Omega^*$, it holds (cf. [21]):

$$\limsup_{n\to\infty} |f^n(\omega, p(\omega)) - f^n(\omega, q(\omega))| > 0,$$

for almost all $\omega \in \Omega^*$.

This property obviously involves all random aperiodic points q (i.e. measurable functions $q\colon \Omega^* \to \bigcup_{\omega\in\Omega^*} S_\omega$ $(\subset \mathbb{I})$). Moreover, since every Carathéodory function $f\colon \Omega \times X \to X$ is superpositionally measurable (i.e. if q is measurable, then so is its image $f(\cdot, q(\cdot))$ on Ω^*), we have verified that the set of random aperiodic orbits is nonempty. At least all constants $q \equiv q(\omega)$, $\omega \in \Omega^*$, can be taken as an example.

The same is true for two required conditions of the Li–Yorke chaos (cf. [21]), when dealing with arbitrary random (measurable) orbits, resp. their elements $q_1, q_2 \in \bigcup_{\omega\in\Omega^*} S_\omega$ $(\subset \mathbb{I})$, $q_1 \neq q_2$:

$$\limsup_{n\to\infty} |f^n(\omega, q_1(\omega)) - f^n(\omega, q_2(\omega))| > 0,$$
$$\liminf_{n\to\infty} |f^n(\omega, q_1(\omega)) - f^n(\omega, q_2(\omega))| = 0,$$

for almost all $\omega \in \Omega^*$.

On the other hand, not every function $r\colon \Omega \to \mathbb{I}$ must be measurable, and so it need not be an element of a random orbit or, in particular, a random periodic orbit. Thus, to verify the appropriate density requirements in the definition of Devaney's chaos, for instance, the one of random periodic points, as elements of random periodic orbits, on the "chaotic" subset of all random orbits, could be rather complicated. This justifies the introduction of Definition 9 and Definition 10 as above.

5. Main theorems

Our first main theorem can be regarded as a randomized Theorem 2.

Theorem 4. *The random operator $f\colon \Omega \times \mathbb{I} \to \mathbb{I}$, where $f_\omega := f(\omega, \cdot)\colon \mathbb{I} \to \mathbb{I}$ is continuous, for almost all $\omega \in \Omega$, is weakly randomly chaotic in the sense of Devaney (see Definition 9) if and only if f admits a random m-orbit, for some natural $m \neq 2^k$, where $k \in \mathbb{N} \cup \{0\}$.*

Proof. "if part" Let f admit a random m-orbit, for some natural $m \neq 2^k$, where $k \in \mathbb{N} \cup \{0\}$. Then, according to Proposition 8, there exist a measurable subset $\Omega_n \subset \Omega$ such that $\mu(\Omega_n) > 0$, jointly with deterministic n-orbits of $f_\omega := f(\omega, \cdot)$, for all $\omega \in \Omega_n$, where $n \mid m$ and $n \neq 2^k$, $k \in \mathbb{N} \cup \{0\}$.

Applying Theorem 2, f_ω is on Ω_n chaotic in the sense of Devaney, which already means (see Definition 9) a weak random Devaney's chaos.

"only if part" If f is weakly randomly chaotic in the sense of Devaney, then, in view of Definition 9, there exists a measurable subset $\Omega_6 \subset \Omega$ such that $\mu(\Omega_6) > 0$, on which f_ω is chaotic in Devaney's sense. According to Lemma 1, f_ω has deterministic 6-orbits, for all $\omega \in \Omega_6$.

Since f_ω has, by virtue of the Brouwer fixed point theorem, a fixed point for almost all $\omega \in \Omega$, there is a partition $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_6$ of Ω such that $\mu(\Omega_0) = 0$, $\mu(\Omega_1) \geq 0$ and $\mu(\Omega_6) > 0$. Applying Proposition 8, f has a random 6-orbit which is the concrete case of a random m-orbit, where $m \neq 2^k$, $k \in \mathbb{N} \cup \{0\}$. □

Remark 7. The "only if part" of Theorem 4 can be improved, by means of Proposition 9, into the form that the weak random Devaney's chaos implies the existence of random m-orbits, with $m = 6$ or $6 \rhd m$, where the symbol "$\rhd$" stands for Sharkovsky's inequality (see Proposition 7), i.e., random m-orbits, for all even natural numbers, and a random fixed point.

As a direct consequence of Theorem 4, we can state in the form of a corollary the following randomization of Theorem 1.

Corollary 2. *Let $f\colon \Omega \times \mathbb{I} \to \mathbb{I}$ be a random operator such that $f_\omega := f(\omega, \cdot)\colon \mathbb{I} \to \mathbb{I}$ is continuous, for almost all $\omega \in \Omega$. If f has a random m-orbit, for some natural $m \neq 2^k$, $k \in \mathbb{N} \cup \{0\}$, then it is weakly randomly chaotic in the sense of Li–Yorke.*

We would also like to randomize Theorem 3 in a similar simple way but, because of the absence (as far as we know)[1] of an analogous result for circle maps to "transitivity implies period six" (see Lemma 1), i.e., of a concrete implied period, we can only randomize Theorem 3 in one way (sufficiency).

[1]Confirmed in a private communication with Professor Louis Block (University of Florida, Gainesville) and Professor Francisco Balibrea Gallego (University of Murcia, Spain) who also informed the author about the related paper [14] of M. C. Hidalgo. Their information was greatly appreciated.

Theorem 5. *Let* $f\colon \Omega\times S^1 \to S^1$ *be a random operator such that* $f_\omega := f(\omega,\cdot)\colon S^1 \to S^1$ *is continuous, for almost all* $\omega \in \Omega$*. Assume, furthermore, that* f *has a random fixed point. If* f *admits a random* m*-orbit, for some natural* $m \neq 2^k$*, where* $k \in \mathbb{N}\cup\{0\}$*, then it is weakly randomly chaotic in the sense of Devaney (see Definition 9).*

Proof. Since f admits a random m-orbit, for some natural $m \neq 2^k$, where $k \in \mathbb{N}\cup\{0\}$, according to Proposition 8, there exists a measurable subset $\Omega_n \subset \Omega$ such that $\mu(\Omega_n) > 0$, jointly with deterministic n-orbits of $f_\omega := f(\omega,\cdot)$, for all $\omega \in \Omega_n$, where $n \mid m$ and $n \neq 2^k$, $k \in \mathbb{N}\cup\{0\}$. Furthermore, according to Proposition 8, f_ω has deterministic fixed points, for almost all $\omega \in \Omega$ and, in particular, for all $\omega \in \Omega_n$.

Applying Theorem 3, f_ω is on Ω_n chaotic in the sense of Devaney which already means (see Definition 9) a weak random Devaney's chaos. □

Remark 8. As we have already pointed out, the "only if part" of Theorem 5 relies on the existence of an implied (by Devaney's chaos) random m-orbit, for some natural $m \neq 2^k$, where $k \in \mathbb{N}\cup\{0\}$. On one side, we have guaranteed that, for every $\omega \in \Omega^* \subset \Omega$, where $\mu(\Omega^*) > 0$, f_ω is chaotic in Devaney's sense, and so deterministic m_ω-orbits exist, $\omega \in \Omega^*$, where $m_\omega \neq 2^k$, $k \in \mathbb{N}\cup\{0\}$. On the other hand, since the periods m_ω depend on $\omega \in \Omega^*$, it is not enough for the knowledge of a concrete, common, natural $m \neq 2^k$, $k \in \mathbb{N}\cup\{0\}$, because in view of Proposition 7, nothing concrete is guaranteed without a further (more detailed) specification of m_ω's, $\omega \in \Omega^*$. Moreover,[2] there need not exist a unique $m \neq 2^k$, where $k \in \mathbb{N}\cup\{0\}$, and a measurable subset $\Omega_m \subset \Omega^*$ with $\mu(\Omega_m) > 0$ such that $m = m_\omega \neq 2^k$, $k \in \mathbb{N}\cup\{0\}$, for all $\omega \in \Omega_m$, $\mu(\Omega_m) > 0$, because it can happen that Ω^* contains a (nonmeasurable) Bernstein set $\Omega_B \subset \Omega^*$ such that $\Omega_B = \{\omega \in \Omega^*\colon f_\omega,\ \omega \in \Omega^*,$ admits an m-orbit$\}$ and, at the same time, $\Omega^* \setminus \Omega_B = \{\omega \in \Omega^*\colon f_\omega,\ \omega \in \Omega^*,$ admits an l-orbit, $l \neq m \wedge l \neq 2^k,\ k \in \mathbb{N}\cup\{0\}\}$. However, every measurable subset of Ω^* with a positive measure (like Ω_m above) intersects, by the basic property of a Bernstein set, both Ω_B and its complement $\Omega^* \setminus \Omega_B$. Thus, it contains elements associated with possible periods m and l, which excludes the existence of $\Omega_m \subset \Omega^*$ with $\mu(\Omega_m) > 0$ as above.

Remark 9. Of course, if such a common concrete period m can be detected (like $m = 6$, for interval maps), then the reverse implication of Theorem 5 can be given in a quite analogous way as in the proof of Theorem 4, and Proposition 10 could then be applied to improving the necessary conditions of Theorem 5, similarly as Proposition 9 in Remark 7. This can happen if all the sets $\Omega_{m_\omega} = \{\omega \in \Omega^*\colon f_\omega,\ \omega \in \Omega^*,$ admits an m_ω-orbit$\}$ are measurable, because at least one of them must have a positive measure; otherwise, there would be a contradiction with a positive measure of Ω^*.

As a direct consequence of Theorem 5, we can state in the form of a corollary the following randomization of Corollary 1.

Corollary 3. *Let* $f\colon \Omega\times S^1 \to S^1$ *be a random operator such that* $f_\omega := f(\omega,\cdot)\colon S^1 \to S^1$ *is continuous, for almost all* $\omega \in \Omega$*. Assume, furthermore, that* f *has a random*

[2]The author is indebted to Professor Robert B. Israel (University of British Columbia, Vancouver, BC, Canada) who kindly suggested these arguments.

fixed point. If f admits a random m-orbit, for some natural $m \neq 2^k$, where $k \in \mathbb{N} \cup \{0\}$, then it is weakly randomly chaotic in the sense of Li–Yorke (see Definition 9).

Remark 10. For a strong random chaos, we still need to know whether the assumed random m-orbit consists (after fixing $\omega \in \Omega$) of deterministic m_ω-orbits such that $m_\omega \,|\, m$ and $m_\omega \neq 2^k$, where $k \in \mathbb{N} \cup \{0\}$, for almost all $\omega \in \Omega$. In particular, it can be satisfied when $m_\omega = m$, for almost all $\omega \in \Omega$.

6. Concluding remarks

Finally, we will mention a possibility of randomizing the Li–Yorke-type results for deterministic multivalued maps. Since the easiest way seems to be via single-valued selections, let us introduce the following two definitions.

Definition 11. Let X be a compact metric space and $\varphi\colon X \multimap X$ be a multivalued mapping with nonempty, closed values. We say that φ is *selectionally chaotic* in a given sense if there exists a compact, connected subset $S \subset X$ which is invariant under $\varphi|_S$, i.e., $\varphi|_S\colon S \multimap S$, such that $\varphi|_S$ has a single-valued continuous selection $f \subset \varphi|_S$ which is chaotic in a respective sense.

Definition 12. Let X be a compact metric space and Ω be a complete measure space. We say that the random operator $\eta\colon \Omega \times X \multimap X$ is *weakly randomly chaotic* in a given sense if there exists a measurable subset $\Omega^* \subset \Omega$ with a positive measure, i.e., $\mu(\Omega^*) > 0$, such that $\eta_\omega := \eta(\omega, \cdot)\colon X \multimap X$ is, for all $\omega \in \Omega^*$, selectionally chaotic in the sense of Definition 11.

Remark 11. For $\Omega = \Omega^*$, we can speak again (like in the single-valued case) about a *strong random chaos* for multivalued random operators.

The following theorem randomizes our statement for deterministic interval maps in [3].

Theorem 6. *Let $\eta\colon \Omega \times \mathbb{I} \multimap \mathbb{I}$ be a random operator such that $\eta_\omega := \eta(\omega, \cdot)\colon \mathbb{I} \multimap \mathbb{I}$ is u.s.c. or l.s.c. with compact, connected values, for almost all $\omega \in \Omega$. Assume, furthermore, that the margins $\sup\{y\colon y \in \eta_\omega(x)\}$ and $\inf\{y\colon y \in \eta_\omega(x)\}$ of η_ω are, for almost all $\omega \in \Omega$, nondecreasing. Then η is weakly randomly chaotic in the sense of Devaney if and only if it admits a random m-orbit, for some natural $m > 1$.*

Proof. "if part" If η has, for some $m > 1$, a random m-orbit, then, according to Proposition 8, there exist some $n > 1$, $n \,|\, m$, and $\Omega_n \subset \Omega$ with $\mu(\Omega_n) > 0$ such that η_ω admits deterministic n-orbits, for all $\omega \in \Omega_n$. Thus, in view of [3, Theorem 6], η_ω is selectionally chaotic in Devaney's sense (see Definition 11), for all $\omega \in \Omega_n$, and subsequently η is weakly randomly chaotic in the same sense (see Definition 12).

"only if part" Let η be weakly randomly chaotic in the sense of Devaney. Then, by Definition 12, there exists a measurable subset $\Omega_6 \subset \Omega$ with $\mu(\Omega_6) > 0$ such that η_ω is, for all $\omega \in \Omega_6$, selectionally chaotic in Devaney's sense. Applying Lemma 1, each single-valued continuous selection $f_\omega\colon S_\omega \to S_\omega$ of $\eta_\omega|_{S_\omega}\colon S_\omega \multimap S_\omega$, $f_\omega \subset \eta_\omega$, $S_\omega \subset \mathbb{I}$, as well as $\eta_\omega\colon \mathbb{I} \multimap \mathbb{I}$ possesses a deterministic 6-orbit, for all $\omega \in \Omega_6$.

If η_ω, $\omega \in \Omega$, is u.s.c., then it has a fixed point, by virtue of the Kakutani–Ky Fan theorem (see e.g. [6, Chapter I.6]). If η_ω, $\omega \in \Omega$, is l.s.c., then it has, by means of the Kuratowski–Ryll–Nardzewski theorem (see e.g. [6, Theorem I.3.49]), a single-valued continuous selection which admits a fixed point by the well-known Brouwer fixed point theorem.

Applying Proposition 8 for the partition $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_6$, when $\mu(\Omega_0) = 0$, $\mu(\Omega_1) > 0$, $\mu(\Omega_6) > 0$, η possesses a random 6-orbit, i.e., $m = 6$, as claimed. $\square$

Remark 12. Let us note that for the "if part" of Theorem 6, η_ω, $\omega \in \Omega$, need not be u.s.c. or l.s.c. In other words, for the sufficiency, these regularity assumptions can be omitted (see [3]).

Remark 13. Observe that, unlike in Theorem 4 or Theorem 5, $m > 1$ can be arbitrary. In particular, m can be equal to 2, by which one implication can be paraphrased as "period two implies (random) chaos", whence the title of [3] in the deterministic case.

Remark 14. In [25], where even "the existence of a fixed point implies deterministic chaos", a different nontraditional approach was employed. The class of maps under consideration was generated there, by means of the inverse limit technique, by solutions of differential inclusions whose right-hand sides are unions of single-valued continuous maps. Their result can be randomized in the way that, roughly speaking, "random fixed points imply strong random chaos", provided randomized differential inclusions determine randomized maps under consideration.

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