FOURIER–MUKAI PARTNERS OF ELLIPTIC RULED SURFACES

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ABSTRACT. We study Fourier–Mukai partners of elliptic ruled surfaces. We also describe the autoequivalence group of the derived categories of ruled surfaces with an elliptic fibration, by using another work of the author.

1. Introduction

1.1. Motivations and results. Let $X$ be a smooth projective variety over $\mathbb{C}$ and $D(X)$ the bounded derived category of coherent sheaves on $X$. If $X$ and $Y$ are smooth projective varieties with equivalent derived categories, then we call $X$ and $Y$ Fourier–Mukai partners. We denote by $\text{FM}(X)$ the set of isomorphism classes of the Fourier–Mukai partner of $X$:

$$\text{FM}(X) := \{Y \text{ smooth projective variety } | D(X) \cong D(Y)\} / \cong.$$ 

It is an interesting problem to determine the set $\text{FM}(X)$ for a given $X$. There are several known results in this direction. For example, Bondal and Orlov show that if $K_X$ or $-K_X$ is ample, then $X$ can be entirely reconstructed from $D(X)$, namely $\text{FM}(X) = \{X\}$ ([BO95]). To the contrary, there are examples of non-isomorphic varieties $X$ and $Y$ having equivalent derived categories: There are examples of a K3 surface, an abelian surface or a relatively minimal elliptic surface with non-zero Kodaira dimension in which $\text{FM}(X) \neq \{X\}$ holds.

By the classification of surfaces, relatively minimal elliptic surfaces with negative Kodaira dimension are either rational elliptic surfaces or elliptic ruled surfaces. In [Ue04,Ue11], the author studies the set $\text{FM}(S)$ of rational elliptic surfaces $S$. In this paper, we describe the set $\text{FM}(S)$ of $\mathbb{P}^1$-bundles $S$ over an elliptic curve:

**Theorem 1.1.** Let $\mathcal{E}$ be a normalized (in the sense in [Ha77, Ch. V, §2]) locally free sheaf of rank $2$ on an elliptic curve $E$, and let $f: S = \mathbb{P}(\mathcal{E}) \rightarrow E$ be the $\mathbb{P}^1$-bundle associated with $\mathcal{E}$. If $|\text{FM}(S)| \neq 1$, there is a degree $0$ line bundle $\mathcal{L} \in \hat{E} := \text{Pic}^0 E$ of order $m > 4$ such that $\mathcal{E} = \mathcal{O}_E \oplus \mathcal{L}$. Furthermore in this case, we have

$$\text{FM}(S) = \{\mathbb{P}(\mathcal{O}_E \oplus \mathcal{L}^i) \mid i \in (\mathbb{Z}/m\mathbb{Z})^*\} / \cong.$$ 

This set consists of $\varphi(m)/|H^*_{\hat{E}}|$ elements. Here, $\varphi$ is the Euler function, and $H^*_{\hat{E}}$ is a group defined in §2 with $|H^*_{\hat{E}}| = 2, 4$ or $6$, depending on the choice of $E$ and $\mathcal{L}$.

As an application, in §4 we describe the autoequivalence group of the derived categories of a certain $\mathbb{P}^1$-bundle over an elliptic curve by using the result in [Ue16].

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1.2. Notation and conventions. All varieties will be defined over $\mathbb{C}$, unless specified otherwise. A point on a variety will always mean a closed point. By an elliptic surface, we will always mean a smooth projective surface $S$ together with a smooth projective curve $C$ and a relatively minimal morphism $\pi: S \to C$ whose general fiber is an elliptic curve. Here a relatively minimal morphism means a morphism whose fibers contain no $(-1)$-curves. Such a morphism $\pi$ is called an elliptic fibration.

For an elliptic curve $E$ and some positive integer $m$, we denote the set of points of order $m$ by $mE$. Furthermore, we denote the dual elliptic curve, namely the group scheme $\text{Pic}^0E$ of line bundles on $E$ of degree 0, by $\hat{E}$, and the group of automorphisms of $E$ fixing the origin by $\text{Aut}_0E$.

Let us denote by $D(X)$ the bounded derived category of coherent sheaves on an algebraic variety $X$, and $\text{Auteq} D(X)$ denotes the group of isomorphism classes of $\mathbb{C}$-linear exact autoequivalences of a $\mathbb{C}$-linear triangulated category $D(X)$.

Let $X$ and $Y$ be smooth projective varieties. For an object $\mathcal{P} \in D(X \times Y)$, we define an exact functor $\Phi^\mathcal{P}$, called an integral functor, to be

$$\Phi^\mathcal{P} := \mathbb{R}p_Y^*(\mathcal{P} \otimes p_X^*(-)) : D(X) \to D(Y),$$

where we denote the projections by $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$. By the result in Orlov ([Or97]), for a fully faithful functor $\Phi : D(X) \to D(Y)$, there is an object $\mathcal{P} \in D(X \times Y)$, unique up to isomorphism, such that $\Phi \cong \Phi^\mathcal{P}$. If an integral functor $\Phi^\mathcal{P}$ is an equivalence, it is called a Fourier–Mukai transform.

2. Preliminaries

2.1. Fourier–Mukai transforms on elliptic surfaces. Bridgeland, Maciocia and Kawamata show in [BM01,Ka02] that if a smooth projective surface $S$ has a non-trivial Fourier–Mukai partner $T$, that is, $|\text{FM}(S)| \neq 1$, then both of $S$ and $T$ are abelian varieties, K3 surfaces or elliptic surfaces with non-zero Kodaira dimension. We consider the last case in more detail. Many results in this subsection are shown in [Br98]. We recommend that the readers refer to the original paper [Br98].

Let $\pi : S \to C$ be an elliptic surface. For an object $E$ of $D(S)$, we define the fiber degree of $E$ as

$$d(E) = c_1(E) \cdot F,$$

where $F$ is a general fiber of $\pi$. Let us denote by $r(E)$ the rank of $E$ and by $\lambda_S$ the highest common factor of the fiber degrees of objects of $D(S)$. Equivalently, $\lambda_S$ is the smallest number $d$ such that there is a $d$-section of $\pi$. Consider integers $a$ and $b$ with $a > 0$ and $b$ coprime to $a\lambda_S$. Then, there exists a smooth, 2-dimensional component $J_S(a,b)$ of the moduli space of pure dimension one stable sheaves on $S$, the general point of which represents a rank $a$, degree $b$ stable vector bundle supported on a smooth fiber of $\pi$. There is a natural morphism $J_S(a,b) \to C$, taking a point representing a sheaf supported on the fiber $\pi^{-1}(x)$ of $S$ to the point $x$. This morphism is a relatively minimal elliptic fibration. Furthermore, there is a universal sheaf $\mathcal{U}$ on $J_S(a,b) \times S$ such that the integral functor $\Phi^\mathcal{U}$ is a Fourier–Mukai transform.

Put $J_S(b) := J_S(1,b)$. Obviously, we have $J_S(1) \cong S$. As is shown in [BM01 Lemma 4.2], there is also an isomorphism

$$J_S(a,b) \cong J_S(b).$$
Theorem 2.1 (Proposition 4.4 in [BM01]). Let $\pi : S \to C$ be an elliptic surface and $T$ a smooth projective variety. Assume that the Kodaira dimension $\kappa(S)$ is non-zero. Then the following are equivalent:

1. $T$ is a Fourier–Mukai partner of $S$.
2. $T$ is isomorphic to $J_S(b)$ for some integer $b$ with $(b, \lambda_S) = 1$.

There are natural isomorphisms

$J_S(b) \cong J_S(b + \lambda_S) \cong J_S(-b)$

(see [BM01, Remark 4.5]). Therefore, we can define the subset

$H_S := \{ b \in (\mathbb{Z}/\lambda_S \mathbb{Z})^* \mid J_S(b) \cong S \}$

of the multiplicative group $(\mathbb{Z}/\lambda_S \mathbb{Z})^*$. We can see that $H_S$ is a subgroup of $(\mathbb{Z}/\lambda_S \mathbb{Z})^*$, and there is a natural one-to-one correspondence between the set $\text{FM}(S)$ and the quotient group $(\mathbb{Z}/\lambda_S \mathbb{Z})^*/H_S$ (see [Ue16, §2.6]).

Claim 2.2. When $\lambda_S \leq 4$, we have $|\text{FM}(S)| = 1$.

Proof. When $\lambda_S \leq 2$, $(\mathbb{Z}/\lambda_S \mathbb{Z})^*$ is trivial and hence, $\text{FM}(S) = \{ S \}$. For $\lambda_S > 2$ and $b \in (\mathbb{Z}/\lambda_S \mathbb{Z})^*$, we have $b \neq \lambda_S - b$ in $(\mathbb{Z}/\lambda_S \mathbb{Z})^*$, and hence, the isomorphisms (2.1) yield $|\text{FM}(S)| \leq \phi(\lambda_S)/2$, where $\phi$ is the Euler function. This inequality implies $|\text{FM}(S)| = 1$ for $\lambda_S \leq 4$. □

In general, it is not easy to describe the group $H_S$, equivalently to describe the set $\text{FM}(S)$, concretely. However, even if $\lambda_S \geq 5$, there are several examples in which we can compute the cardinality of the set $\text{FM}(S)$ (see [Ue11, Example 2.6]).

2.2. Some technical lemmas on elliptic curves. Let $F$ be an elliptic curve. For points $x_1, x_2 \in F$, to distinguish the summations as divisors and as elements in the group scheme $F$, we denote by $x_1 \oplus x_2$ the sum of them by the operation of $F$, and

$i \cdot x_1 := x_1 \oplus \cdots \oplus x_1$ (i times).

We also denote by

$ix_1 := x_1 + \cdots + x_1$ (i times)

the divisor on $F$ of degree $i$. It is well known that there is a group scheme isomorphism

(2.2) $F \to \hat{F} \quad x \mapsto \mathcal{O}_F(x - O),$

where $O$ is the origin of $F$. If we identify $\hat{F}$ and $F$ by (2.2), the so-called normalized Poincaré bundle $\mathcal{P}_0$ on $F \times F$ is defined by

$\mathcal{P}_0 := \mathcal{O}_{F \times F}(\Delta_F - F \times O - O \times F),$

where $\Delta_F$ is the diagonal of $F$ in $F \times F$. It satisfies that

$\mathcal{P}_0|_{F \times x} \cong \mathcal{P}_0|_{x \times F} \cong \mathcal{O}_F(x - O)$

for a point $x \in F$.

Let us fix an element $a \in mF$ with a positive integer $m$. Let us denote by $E$ the quotient variety $F/\langle a \rangle$, by

$q : F \to E$
the dual isogeny of \( q \). Define a subgroup of \((\mathbb{Z}/m\mathbb{Z})^*\) as

\[
H_F^a := \{ k \in (\mathbb{Z}/m\mathbb{Z})^* \mid \exists \phi \in \text{Aut}_0 F \text{ such that } \phi(a) = k \cdot a \}.
\]

Recall that

- \( F \cong \mathbb{C}/(\mathbb{Z} + \sqrt{-1}\mathbb{Z}) \), \( \text{Aut}_0 F = \{ \pm 1, \pm \sqrt{-1} \} \) when \( j(F) = 1728 \),
- \( F \cong \mathbb{C}/(\mathbb{Z} + \omega\mathbb{Z}) \), \( \text{Aut}_0 F = \{ \pm 1, \pm \omega, \pm \omega^2 \} \) when \( j(F) = 0 \), and
- \( \text{Aut}_0 F = \{ \pm 1 \} \) when \( j(F) \neq 0, 1728 \).

(See [Ha77, Ch. IV, §4].) Here \( j(F) \) is the \( j \)-invariant of \( F \), and we put \( \omega = \frac{-1 + \sqrt{-3}}{2} \). We use the following technical lemmas in the proof of Theorem 1.1.

**Lemma 2.3.** Suppose that \( m > 3 \). Then exactly one of the following three cases for \( F \) and \( a \in mF \) occurs:

1. The equality \( H_F^a = \{ \pm 1 \} \) holds.
2. We have \( j(F) = 1728 \), and there is an integer \( n \) such that \( m \) divides \( n^2 + 1 \).
   (Note that this condition implies that \( \pm n \in (\mathbb{Z}/m\mathbb{Z})^* \).) Moreover, the point \( a \in F \) is an element in the subgroup

\[
\langle \frac{n}{m} + \frac{1}{m} \sqrt{-1} \rangle
\]

of \( F \cong \mathbb{C}/(\mathbb{Z} + \sqrt{-1}\mathbb{Z}) \), and the equality \( H_F^a = \{ \pm 1, \pm n \} \) holds.
3. We have \( j(F) = 0 \), and there is an integer \( n \) such that \( m \) divides \( n^2 + n + 1 \).
   (Note that this condition implies that \( \pm n \in (\mathbb{Z}/m\mathbb{Z})^* \).) Moreover, the point \( a \in F \) is an element in the subgroup

\[
\langle \frac{n + 1}{m} + \frac{1}{m} \omega \rangle
\]

of \( F \cong \mathbb{C}/(\mathbb{Z} + \omega\mathbb{Z}) \), and the equality \( H_F^a = \{ \pm 1, \pm n, \pm n^2 \} \) holds.

**Proof.** When \( j(F) \neq 0, 1728 \), obviously the case (1) occurs.

Next, let us consider the case \( j(F) = 1728 \). Put \( a = \frac{x}{m} + \frac{y}{m} \sqrt{-1} \) for some \( x, y \in \mathbb{Z} \), and suppose first that an equality

\[
\sqrt{-1}a = n \cdot a
\]

holds for some \( n \in \mathbb{Z} \). Then we have

\[
(2.4) \quad nx \equiv -y, \quad ny \equiv x \pmod{m}.
\]

Hence, we know that \( a = \frac{ny}{m} + \frac{x}{m} \sqrt{-1} \), and since the order of \( a \) in \( F \) is \( m \), \( m \) and \( y \) are coprime. The coprimality and the equations (2.4) yield that \( m \) divides \( n^2 + 1 \). The coprimality also implies that the subgroups \( \langle a \rangle \) and \( \langle \frac{n}{m} + \frac{1}{m} \sqrt{-1} \rangle \) coincide. We know from \( \text{Aut}_0 F = \{ \pm 1, \pm \sqrt{-1} \} \) that \( H_F^a = \{ \pm 1, \pm n \} \) holds.

In the case (3), the proof is similar.

It follows from the conditions on \( m \) and \( n \) that \( |H_F^a| = 2, 4 \) and 6 in the case (1), (2) and (3) respectively, hence any of the two cases do not occur at the same time.
Define
\[ H^E_m := \{ k \in (\mathbb{Z}/m\mathbb{Z})^* \mid \exists \hat{\psi} \in \text{Aut}_0 \hat{E} \text{ such that } \hat{\psi}(\mathcal{L}) = \mathcal{L}^k \} \]
\[ = \{ k \in (\mathbb{Z}/m\mathbb{Z})^* \mid \exists \psi \in \text{Aut}_0 E \text{ such that } \psi^* \mathcal{L} = \mathcal{L}^k \} \]
for a line bundle \( \mathcal{L} \in m\hat{E} \).

**Lemma 2.4.** In each case of Lemma 2.3, the equality \( H^E_F = H^E_m \) holds for any \( \mathcal{L} \in m\hat{E} \) with \( \ker \hat{q} = \langle \mathcal{L} \rangle \). (In particular, there is an isomorphism \( F \cong E \) in the cases (2) and (3), since their \( j \)-invariants coincide.)

*Proof.* Let us consider the case (2) first. Let \( L \) be the lattice generated by 1 and \( \sqrt{-1} \) in \( \mathbb{C} \) so that \( F \) with \( j(F) = 1728 \) is isomorphic to \( \mathbb{C}/L \). Moreover, the elliptic curve \( E = F/\langle a \rangle \) is isomorphic to \( \mathbb{C}/(L + \langle a \rangle) \). We can see that the lattice \( L + \langle a \rangle \) is preserved by the complex multiplication by \( \sqrt{-1} \). (Hence, \( j(E) = 1728 \), which implies \( F \cong E \)). It turns out that the quotient morphism
\[ q: F \cong \mathbb{C}/L \to \mathbb{C}/(L + \langle a \rangle) \]
induced by the inclusion \( L \hookrightarrow L + \langle a \rangle \) is compatible with the complex multiplication by \( \sqrt{-1} \).

Take an element \( \frac{1}{m} \in \mathbb{C}/L(\cong F) \), and put
\[ a := \frac{ny}{m} + \frac{y}{m} \sqrt{-1} \]
for the integer \( n \) in (2) and some \( y \in (\mathbb{Z}/m\mathbb{Z})^* \). We define \( \mathcal{L}' \) to be the element in \( \hat{E} \) corresponding to \( q(\frac{1}{m}) \in E \) via \( E \cong \hat{E} \). Then we have
\[ \sqrt{-1}q(\frac{1}{m}) = q(\frac{1}{m} \sqrt{-1}) = q(y^{-1}a - \frac{n}{m}) = -nq(\frac{1}{m}) \]
and this implies the equality \( H^E_F = \{ \pm 1, \pm n \} = H^E_{\mathcal{L}'} \). We can also see that
\[ \langle a, \frac{1}{m} \rangle = \langle \frac{\sqrt{-1}}{m}, \frac{1}{m} \rangle = \ker[m] \]
where \([m]\) is the multiplication map by \( m \). Recall that \([m] = \hat{q} \circ q \) and \( \ker q = \langle a \rangle \). Consequently, we have \( \ker \hat{q} = \langle \mathcal{L}' \rangle \). For any \( \mathcal{L} \in m\hat{E} \) with \( \ker \hat{q} = \langle \mathcal{L} \rangle \), the equality \( H^E_m = H^E_{\mathcal{L}'} \) holds, which gives the assertion.

The proof of the case (3) is similar.

Next let us take an element \( \mathcal{L} \in \ker \hat{q} \), and suppose that \( |H^E_m| = 4 \) or 6, namely the case (2) or (3) occurs for \( \hat{E} \) and \( \mathcal{L} \in m\hat{E} \). Then we have already shown above that \( H^E_F = H^E_{\mathcal{L}'} \) (just by replacing the roles of \( \hat{E} \) and \( F \)). Consequently, in the case (1), we again obtain the assertion. \( \square \)

### 2.3. \( \mathbb{P}^1 \)-bundles on an elliptic curve over a field of arbitrary characteristic.

In this subsection, we refer to a result which is needed in the proof of Theorem 1.1. The results and notation here over a positive characteristic field are not logically needed in this paper, but we leave them to explain a background of Problem 4.1.

Let \( k \) be an algebraically closed field of characteristic \( p \geq 0 \). Suppose that \( E \) is an elliptic curve defined over \( k \), \( E \) is a normalized, in the sense of [Ha77, Ch. V, §2], locally free sheaf of rank 2 on \( E \), and \( f: S = \mathbb{P}(\mathcal{E}) \to E \) is a \( \mathbb{P}^1 \)-bundle on \( E \). Set \( e := -\text{deg} \mathcal{E} \). Then we can see that \( e = 0 \) or \(-1 \) if \(-K_S \) is nef, and in particular, if \( S \) has an elliptic fibration \( \pi: S \to \mathbb{P}^1 \). Furthermore, when the locally free sheaf \( \mathcal{E} \)
is decomposable and \( e = 0 \), it turns out that \( \mathcal{E} = \mathcal{O}_E \oplus \mathcal{L} \) for some \( \mathcal{L} \in \hat{E} \). When \( e = -1 \), it is indecomposable (see [Ha77, Theorem V.2.12]).

Maruyama considers the conditions that \( \mathbb{P}^1 \)-bundles over an elliptic curve have an elliptic fibration in terms of elementary transformations of ruled surfaces. See [Ma71, Theorem 4]. Togashi also describes the conditions in terms of vector bundles on elliptic curves as follows. His proof is independent of both the proof and the result of Maruyama.

**Theorem 2.5** ([To11]). We use the above notation.

(1) For \( e = 0 \), \( S \) has an elliptic fibration in the cases (1-1), (1-2) and (1-5). Moreover, we have the following:

<table>
<thead>
<tr>
<th>( \mathcal{E} )</th>
<th>singular fibers</th>
<th>( p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathcal{O}_E \oplus \mathcal{O}_E )</td>
<td>no singular fibers</td>
<td>( p \geq 0 )</td>
</tr>
<tr>
<td>( \mathcal{O}_E \oplus \mathcal{L} ), ord ( \mathcal{L} = m &gt; 1 )</td>
<td>( 2 \times m \mathcal{I}_0 )</td>
<td>( p \geq 0 )</td>
</tr>
<tr>
<td>( \mathcal{O}_E \oplus \mathcal{L} ), ord ( \mathcal{L} = \infty )</td>
<td>( p \geq 0 )</td>
<td></td>
</tr>
<tr>
<td>indecomposable</td>
<td>( m \mathcal{I}_0 ) (a wild fiber)</td>
<td>( p &gt; 0 )</td>
</tr>
</tbody>
</table>

(2) Suppose that \( e = -1 \) and \( p \neq 2 \). Then, \( S \) has an elliptic fibration with three singular fibers of type \( 2 \mathcal{I}_0 \) (see the notation in [BHPV, Ch. V, §7]).

We use Theorem 2.5 to describe the set \( \text{FM}(S) \) for \( \mathbb{P}^1 \)-bundles \( S \) over an elliptic curve in Theorem 1.1.

**Remark 2.6.** Let \( C_0 \) be a section of \( f \) satisfying \( \mathcal{O}_S(C_0) \cong \mathcal{O}_{\mathbb{P}^1}(1) \) (see [Ha77, p. 373]), \( F \) be a general fiber of \( \pi \), and \( F_f \) a fiber of \( f \). Then [Ha77, Corollary V.2.11] tells us that

\[
K_S \equiv -2C_0 - eF_f,
\]

and by the canonical bundle formula of elliptic fibrations, we have

\[
K_S \equiv -\frac{2}{m}F
\]

in the case (1-2), and

\[
K_S \equiv -\frac{1}{2}F
\]

in the case (2). Then, we can see that \( F \cdot F_f = m \) (resp. \( F \cdot C_0 = 2 \)), and hence, we have \( \lambda_S = m \) (resp. \( \lambda_S = 2 \)) in (1-2) (resp. in (2)).

3. Proof of Theorem 1.1

We give the proof of Theorem 1.1 in this section. Before giving the proof, we need several claims.

Let us take a cyclic group \( G = \mathbb{Z}/m\mathbb{Z} \) for an integer \( m > 1 \) and a generator \( g \) of \( G \). For integers \( i \in (\mathbb{Z}/m\mathbb{Z})^* \), define representations

\[
\rho_{\mathbb{P}^1}: G \to \text{Aut}(\mathbb{P}^1) \quad \text{as} \quad \rho_{\mathbb{P}^1}(g)(y) = \zeta y,
\]

and

\[
\rho_{F,i}: G \to \text{Aut}(F) \quad \text{as} \quad \rho_{F,i}(g)(x) = T^{-a}x,
\]

where \( F \) is an elliptic curve, \( a \) is an element of \( _mF \), \( T_a \) is the translation by \( a \) and \( \zeta \) is a primitive \( m \)-th root of unity in \( \mathbb{C} \). Let us consider the diagonal action

\[
\rho_i := \rho_{F,i} \times \rho_{\mathbb{P}^1}: G \to \text{Aut}(F \times \mathbb{P}^1)
\]
induced by \( \rho_{\mathbb{P}^1} \) and \( \rho_{F,i} \). Set 

\[ S_i := (F \times \mathbb{P}^1)/G, \]

the quotient of \( F \times \mathbb{P}^1 \) by the action \( \rho_i \). Then we have the following commutative diagram:

\[
\begin{array}{ccc}
F & \xleftarrow{p_1} & F \times \mathbb{P}^1 \\
q & & q_1 \\
E := F/\langle a \rangle & \xleftarrow{f_i} & S_i \\
\pi_i & & \xrightarrow{\pi} \mathbb{P}^1/G \cong \mathbb{P}^1
\end{array}
\]

(3.2)

Here, every vertical arrow is the quotient morphism of the action of \( G \). We can readily see that \( f_i \) is a \( \mathbb{P}^1 \)-bundle and \( \pi_i \) is an elliptic fibration. Note that the quotient morphism \( q \) does not depend on the choice of \( i \), and that the left square in (3.2) is a fiber product. We can also see that \( \pi_i \) has exactly two multiple fibers of type \( m_i 0 \) over the branch points \( q_{p_1}(0), q_{p_1}(\infty) \) of \( q_{p_1} \), and it fits into the cases (1-2) in Theorem 2.5 Consequently, there is a line bundle \( \mathcal{L}_i \in \hat{\mathcal{E}} \) such that 

\[ S_i \cong \mathbb{P}(\mathcal{O}_E \oplus \mathcal{L}_i) \]

for each \( i \). Furthermore, because the left square in (3.2) is a fiber product, we have 

\[ q^* \mathcal{L}_i = \mathcal{O}_F, \]

which implies that \( \langle \mathcal{L}_i \rangle = \text{Ker} \hat{q} \) for the dual isogeny \( \hat{q} : \hat{E} \to \hat{F} \) of \( q \). Therefore, the subgroup \( \langle \mathcal{L}_i \rangle \) of \( \hat{E} \) does not depend on the choice of \( i \). In particular, we have an inclusion

\[
\{ S_i \mid i \in (\mathbb{Z}/m\mathbb{Z})^* \}/ \cong \to \{ \mathbb{P}(\mathcal{O}_E \oplus \mathcal{L}_i^1) \mid i \in (\mathbb{Z}/m\mathbb{Z})^* \}/ \cong .
\]

(3.3)

We will see below that these sets actually coincide by checking their cardinality. Let us start the following claim.

Claim 3.1. If there is a group automorphism \( \psi_1 \in \text{Aut}_0 E \) such that \( \psi_1^* \mathcal{L} \cong \mathcal{L}^{\pm i^{-1}j} \) holds for \( i, j \in (\mathbb{Z}/m\mathbb{Z})^* \), then we have 

\[ \mathbb{P}(\mathcal{O}_E \oplus \mathcal{L}^*) \cong \mathbb{P}(\mathcal{O}_E \oplus \mathcal{L}^j). \]

Consequently, the cardinality of the right hand side of (3.3) is at most \( \varphi(m)/|H_{\hat{E},1}| \).

Proof. The result follows from [Ha77, Ex. II.7.9(b)]. \( \square \)

We also have the following.

Claim 3.2. For \( i, j \in (\mathbb{Z}/m\mathbb{Z})^* \), \( S_i \cong S_j \) if and only if there is a group automorphism \( \phi_1 \in \text{Aut}_0 F \) such that \( \phi_1(a) = (\pm i^{-1}j) \cdot a \) holds. Consequently, the cardinality of the left hand side of (3.3) is \( \varphi(m)/|H_{\hat{F},1}| \).

Proof. Suppose that there is an isomorphism \( \psi : S_i \to S_j \). As in the proof of Claim 3.1, \( \psi \) induces an automorphism \( \psi_1 \) of \( E \) which is compatible with \( \psi \). It is also satisfied that the dual isogeny \( \psi_1^* \) preserves the subgroup \( \langle \mathcal{L}_1 \rangle \) of \( \hat{E} \), and hence \( \psi_1 \) lifts an automorphism \( \phi_1 \) of \( F \cong \hat{F} \cong E/\langle \mathcal{L}_1 \rangle \). Since the left square in (3.2) is a fiber product, \( \psi \) lifts to an automorphism \( \phi \) of \( F \times \mathbb{P}^1 \). We can see that \( \phi \) is of the form \( \phi_1 \times \phi_2 \) for some \( \phi_2 \in \text{Aut} \mathbb{P}^1 \). Since any translation on \( F \) descends to a translation on \( E \), replacing \( \phi_1 \) if necessary, we may assume that \( \phi_1 \in \text{Aut}_0 F \). Since \( \phi \) descends to \( \psi \), it should satisfy

\[ \phi \circ \rho_i(g) = \rho_j(g^k) \circ \phi \]
for any \( g \in G \) and some \( k \in \mathbb{Z} \). By observing the action on \( \mathbb{P}^1 \), we know that \( k = 1 \) or \( m - 1 \), and moreover

\[
\phi_2(y) = \begin{cases} 
\lambda y & \text{(in the case } k = 1 \text{)}, \\
\lambda / y & \text{(in the case } k = m - 1 \text{)},
\end{cases}
\]

for \( y \in \mathbb{P}^1 \) and some \( \lambda \in \mathbb{C}^* \). In the former case, we obtain that \( \phi_1(a) = (i^{-1}j) \cdot a \) holds, and in the latter case, \( \phi_1(a) = (-i^{-1}j) \cdot a \) holds. \( \square \)

For \( m \leq 3 \), we can easily see from Claims 3.1 and 3.2 that both sides of (3.3) coincide. And hence, suppose that \( m > 3 \). Then, it follows from Lemma 2.4, Claims 3.1 and 3.2 that both sides of (3.3) coincide:

\[
\{ S_i \mid i \in (\mathbb{Z}/m\mathbb{Z})^* \} / \cong = \{ \mathbb{P}(\mathcal{O}_E \oplus \mathcal{L}_i^1) \mid i \in (\mathbb{Z}/m\mathbb{Z})^* \} / \cong.
\]

The cardinality of this set is \( \varphi(m)/|H_\mathcal{E}_1| \).

Claim 3.3. In the above notation, \( S_i \cong J_{S_i}(i) \) for all \( i \) with \( i \in (\mathbb{Z}/m\mathbb{Z})^* \).

Proof. Take an element \( j \in (\mathbb{Z}/m\mathbb{Z})^* \) such that \( ij = 1 \). Henceforth, we identify \( F \) and \( \hat{F} \) as group schemes by (2.2). For the normalized Poincaré bundle \( P_0 \) given in §2.2, we define

\[ P := P_0 \otimes p_1^* \mathcal{O}_F(iO) \otimes p_2^* \mathcal{O}_F(jO). \]

Here, we regard elements \( i, j \in (\mathbb{Z}/m\mathbb{Z})^* \) as integers satisfying \( 1 \leq i, j \leq m - 1 \). Then the line bundle \( P \) satisfies

\[
P|_{x \times F} \cong \mathcal{O}_F(x + (j-1)O) \quad \text{and} \quad P|_{F \times y} \cong \mathcal{O}_F(y + (i-1)O)
\]

for any \( x, y \in F \). Let us consider the commutative diagram:

\[
\begin{array}{ccc}
x \times F & \xrightarrow{T_a} & F \times F \\
\downarrow T_{i,a} & & \downarrow T_{a} \times T_{i,a} \\
(x \oplus a) \times F & \xrightarrow{T_a} & F \times (y \oplus i \cdot a)
\end{array}
\]

Here, the left vertical morphism is defined by the composition of morphisms

\[ x \times F \cong F \xrightarrow{T_{i,a}} F \cong (x \oplus a) \times F \]

and similarly, the right vertical arrow is also defined by \( T_a \). Now we have

\[
((T_a \times T_{i,a})^* P)|_{F \times y} \cong T_a^*(P|_{F \times (y \oplus i \cdot a)})
\]

\[
\cong P|_{F \times (y \oplus i \cdot a)} \otimes \mathcal{O}_F(a - O)^{-i} \\
\cong \mathcal{O}_F(y + ia - O) \otimes \mathcal{O}_F(a - O)^{-i} \\
\cong \mathcal{O}_F(y + (i - 1)O) \\
\cong P|_{F \times y}.
\]
Using ord \( a = m \), we also have
\[
((T_a \times T_{i-a})^*P)|_{x \times F} \cong T_{i-a}(P|_{(x \oplus a) \times F}) \\
\cong P|_{(x \oplus a) \times F} \otimes \mathcal{O}_F(i \sigma - i \theta)^{-j} \\
\cong \mathcal{O}_F(x + a + (j - 2)\theta) \otimes \mathcal{O}_F(i \sigma - i \theta)^{-j} \\
\cong \mathcal{O}_F(x + (j - 1)\theta) \\
\cong P|_{x \times F}.
\]
Hence, we obtain \((T_a \times T_{i-a})^*P \cong \mathcal{P}\) by [Ha77, Ex. III.12.4]. Let us define \(\Delta_{\mathbb{P}^1} (\cong \mathbb{P}^1)\) to be the diagonal in \(\mathbb{P}^1 \times \mathbb{P}^1\). For the projection \(p_1 : (F \times F) \times \Delta_{\mathbb{P}^1} \rightarrow F \times F\), define a sheaf
\[
\mathcal{U} := p_1^*\mathcal{P}.
\]
We regard \(\mathcal{U}\) as a sheaf on \((F \times \mathbb{P}^1) \times (F \times \mathbb{P}^1)\). Then for any \(g \in G\), we have
\[
((\rho_1(g) \times \rho_i(g))^*\mathcal{U})|_{x \times \mathbb{P}^1 \times y \times \mathbb{P}^1} \cong (\rho_{\mathbb{P}^1}(g) \times \rho_{\mathbb{P}^1}(g))^*(\mathcal{U}|_{(x \oplus a) \times \mathbb{P}^1 \times (y \oplus i-a) \times \mathbb{P}^1}) \\
\cong \rho_{\mathbb{P}^1}(g)^*\mathcal{O}_{\Delta_{\mathbb{P}^1}} \\
\cong \mathcal{O}_{\Delta_{\mathbb{P}^1}} \\
\cong \mathcal{U}|_{x \times \mathbb{P}^1 \times y \times \mathbb{P}^1}
\]
for \(x, y \in F\). In the second isomorphism, we identify \(\mathbb{P}^1\) with \(\Delta_{\mathbb{P}^1}\). We also have
\[
((\rho_1(g) \times \rho_i(g))^*\mathcal{U})|_{F \times x \times F \times z} \cong (T_a \times T_{i-a})^*(\mathcal{U}|_{F \times \zeta \times F \times \zeta}) \\
\cong (T_a \times T_{i-a})^*\mathcal{P} \\
\cong \mathcal{P}
\]
for any \(z \in \mathbb{P}^1\), and note that both \((\rho_1(g) \times \rho_i(g))^*\mathcal{U})|_{F \times z_1 \times F \times z_2}\) and \(\mathcal{U}|_{F \times z_1 \times F \times z_2}\) are zero for \(z_1 \neq z_2 \in \mathbb{P}^1\), since \(\text{Supp}\mathcal{U} = (F \times F) \times \Delta_{\mathbb{P}^1}\). Then, by [Ha77, Ex. III.12.4] we can check \((\rho_1(g) \times \rho_i(g))^*\mathcal{U} \cong \mathcal{U}\), equivalently
\[
(\rho_1(g) \times \text{id}_{F \times \mathbb{P}^1})^*\mathcal{U} \cong (\text{id}_{F \times \mathbb{P}^1} \times \rho_i(g^{-1}))^*\mathcal{U}.
\]
Push forward both sides by the morphism \(q_1 \times \text{id}_{F \times \mathbb{P}^1}\). Then we obtain
\[
(q_1 \times \text{id}_{F \times \mathbb{P}^1})_*\mathcal{U} \cong (\text{id}_{S_1} \times \rho_i(g^{-1}))^*(q_1 \times \text{id}_{F \times \mathbb{P}^1})_*\mathcal{U},
\]
that is, the sheaf \((q_1 \times \text{id}_{F \times \mathbb{P}^1})_*\mathcal{U}\) is \(G\)-invariant with respect to the diagonal action of \(G\) on \(S_1 \times (F \times \mathbb{P}^1)\), induced by the trivial action on \(S_1\) and \(p_1\) on \(F \times \mathbb{P}^1\). Since \(G\) is cyclic, we can conclude that \((q_1 \times \text{id}_{F \times \mathbb{P}^1})_*\mathcal{U}\) is \(G\)-equivariant, and hence there is a coherent sheaf \(\mathcal{U}'\) on \(S_1 \times S_1\) such that
\[
(3.6) \quad (q_1 \times \text{id}_{F \times \mathbb{P}^1})_*\mathcal{U} \cong (\text{id}_{S_1} \times q_1)^*\mathcal{U}'.
\]
For \(x \times z \in F \times \mathbb{P}^1\), we have \(\mathcal{U}|_{F \times x \times x \times z} \cong \mathcal{P}|_{F \times x}\), which is a line bundle of degree \(i\) on \(F\) by (3.5). The isomorphism (3.6) yields
\[
\mathcal{U}'|_{S_1 \times q_i(x \times z)} \cong ((q_1 \times \text{id}_{F \times \mathbb{P}^1})_*\mathcal{U})|_{S_1 \times (x \times z)} \cong (q_1 \times \text{id}_{F \times \mathbb{P}^1})_*\mathcal{U}|_{F \times x \times z} = (q_1 \times \text{id}_{F \times \mathbb{P}^1})_*\mathcal{U}|_{F \times x \times z}.
\]
Here, the second isomorphism follows from [BO95, Lemma 1.3] and the smoothness of \(q_1\). Since \(\mathcal{U}|_{(F \times \mathbb{P}^1) \times (x \times z)}\) is actually a sheaf on \(F \times x \times z \times x \times z\) and the restriction \(q_1|_{F \times z}\) is an isomorphism for \(z \in \mathbb{P}^1 \setminus \{0, \infty\}\), \(\mathcal{U}'|_{S_1 \times q_i(x \times z)}\) is also a line bundle of degree \(i\) on \(F_z \times q_i(x \times z)\) for such \(z\). Here, \(F_z(\cong F)\) is a fiber of \(\pi_1\) over the point
fits into the case (1-2) with $\lambda$. Since $S_i \neq y \in S_i$, this morphism is injective, and hence $S_i$ and $J_{S_i}(i)$ are birational over $\mathbb{P}^1$. Then, [BHPV, Proposition III.8.4] implies the result. \hfill \Box

Now, we obtain the following.

**Proposition 3.4.** Let $E$ be an elliptic curve, and define $S$ to be a $\mathbb{P}^1$-bundle $\mathbb{P} (\mathcal{O}_E \oplus \mathcal{L})$ over $E$ for a line bundle $\mathcal{L} \in m \hat{E}$ for $m > 0$. Then we have

$$\text{FM}(S) = \{ \mathbb{P} (\mathcal{O}_E \oplus \mathcal{L}') \mid i \in (\mathbb{Z}/m\mathbb{Z})^* \} \cong \mathbb{Z}/2.4,$$

This set consists of $\varphi(m)/|H^E_{\mathbb{C}}|$ elements. In the case $m > 3$, the cardinality $|H^E_{\mathbb{C}}|$ is 2, 4 or 6, depending on the choice of $E$ and $\mathcal{L}$.

**Proof.** The first statement is a direct consequence of Theorem 2.1 and the equation (3.4) and Claim 3.3. The second is a direct consequence of Claim 3.1. We can compute the cardinality of $H^E_{\mathbb{C}}$ by Lemmas 2.3 and 2.4. \hfill \Box

We are in a position to show Theorem 1.1.

**Proof of Theorem 1.1** The condition $|\text{FM}(S)| \neq 1$ implies that $S$ has an elliptic fibration $\pi: S \to \mathbb{P}^1$ (see [BM01]). Hence, either of the cases (1-1), (1-2) or (2) in Theorem 2.6 occurs (recall that we work over $\mathbb{C}$). In each case, we see from Remark 2.6 that $\lambda_S = 1, m$ and 2 respectively. It follow from Claim 2.2 that $S$ actually fits into the case (1-2) with $m > 4$. Now set $S = \mathbb{P} (\mathcal{O}_E \oplus \mathcal{L})$ for some $\mathcal{L} \in m \hat{E}$ for some $m > 4$. Then the assertion follows from Proposition 3.4. \hfill \Box

4. **Further questions**

4.1. **Autoequivalences.** Let $S := \mathbb{P} (\mathcal{O}_E \oplus \mathcal{L})$ be a $\mathbb{P}^1$-bundle over an elliptic curve $E$ with non-trivial Fourier–Mukai partners, where $\mathcal{L} \in m E$ for some $m > 4$, as in Theorem 1.1. Then, the group $H_S$ defined in [2.1] coincides with the group $H^E_{\mathbb{C}}$, by the results in [2.3]. Note that there are no $(-2)$-curves on $S$, and hence no twist functors associated with $(-2)$-curves appear in $\text{Aut} \, D(S)$. Moreover, we can see that the $\mathbb{P}^1$-bundle $f: S \to E$ has two sections $C_0$ and $C_1$, and $mC_0$ and $mC_1$ are the multiple fibers of $\pi$. We can also check that

$$\langle \mathcal{O}_S(D) \mid D \cdot F = 0 \rangle = \langle \mathcal{O}_S(C_0), \mathcal{O}_S(C_1) \rangle$$

in $\text{Pic}(S)$, where $F$ is a smooth fiber of $\pi$. Therefore, by the main theorem of [Ue16], we have the following short exact sequence:

$$1 \to \langle \otimes \mathcal{O}_S(C_0), \otimes \mathcal{O}_S(C_1) \rangle \times \text{Aut} \, S \times \mathbb{Z}[2] \to \text{Aut} \, D(S) \to \langle \{ \begin{pmatrix} c & a \\ d & b \end{pmatrix} \in \Gamma_0(m) \mid b \in H^E_{\mathbb{C}} \} \rangle \to 1.$$

Here for an integer $b$, coprime with $m$, we again denote by $b$ the corresponding element in $H^E_{\mathbb{C}}(\mathbb{Z}/m\mathbb{Z})^*$, and $\Gamma_0(m)$ is the congruence subgroup of $\text{SL}(2, \mathbb{Z})$, defined in [Ue16].

Since other ruled surfaces with an elliptic fibration have no non-trivial Fourier–Mukai partners, the description of their autoequivalence groups is directly given by [Ue16].
For a $\mathbb{P}^1$-bundle over an elliptic curve without elliptic fibrations, a description of the autoequivalence group will be given in the forthcoming paper [Ue].

### 4.2. Positive characteristic

The proof of Theorem 1.1 does not work over positive characteristic fields. We finish this section to raise the following:

**Problem 4.1.**

1. In the notation of §2.3 consider the case $e = -1$ and $p = 2$. Study when $S$ has an elliptic fibration, and if it has, study the singular fibers of the fibration.

2. Describe the set $FM(S)$ for a $\mathbb{P}^1$-bundle $S$ on an elliptic curve over a positive characteristic field.

In [Ma71 Theorem 4], Maruyama states that in the case $e = -1$ and $p = 2$, $S$ (in his notation) has an elliptic fibration. But it seems to the author that he gave no proof of this statement. See also [Ma71 Remark 7].

Furthermore, in the case (1-5) in Theorem 2.5, if $p \geq 5$, $S$ may have non-trivial Fourier–Mukai partners, since $\lambda_S = p$ (we omit the proof of this fact here). It is also an interesting question to describe $FM(S)$ in this case. [KU85 Examples 4.7, 4.8] fit into the case (1-5). To show Theorem 1.1 the equality (3.4) was a key. The author believes that the description in [KU85 Examples 4.7, 4.8] should be useful to describe $FM(S)$ for $S$ in the case (1-5).

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### References


H. Uehara, A trichotomy for the autoequivalence groups on smooth projective surfaces (in preparation).

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