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SINGULARITY OF THE GENERATOR SUBALGEBRA IN q-GAUSSIAN ALGEBRAS

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Abstract. Given $-1 < q < 1$ and a separable real Hilbert space $\mathcal{H}_{\mathbb{R}}$ with dimension no less than 2, we prove that the generator subalgebra in the q-Gaussian algebra $\Gamma_q(\mathcal{H}_{\mathbb{R}})$ is singular.

Introduction

For a real number $-1 < q < 1$ and for a given separable real Hilbert space $\mathcal{H}_{\mathbb{R}}$, Bożejko and Speicher [3] introduced the q-deformed Fock space $\mathcal{F}_q(\mathcal{H}_{\mathbb{R}})$, in order to construct the q-commutation relation

$$\ell^*(e)\ell(f) - q\ell(f)\ell^*(e) = \langle e, f\rangle\, Id.$$

They also studied the q-Gaussian algebra $\Gamma_q(\mathcal{H}_{\mathbb{R}})$ as the von Neumann algebra generated by the q-Gaussian variables $\{\ell(e) + \ell^*(e) : e \in \mathcal{H}_{\mathbb{R}}\}$ in [4]. This family of von Neumann algebras has attracted lots of attention since as q varies, we obtain interesting interpolation between different von Neumann algebras: the free group factor when $q = 0$, the hyperfinite II_1 factor when $q = -1$ and $L^\infty(X)$ when $q = 1$. It is known that assuming $\dim \mathcal{H}_{\mathbb{R}} \geq 2$, these $\Gamma_q(\mathcal{H}_{\mathbb{R}})$ are II_1 factors [9], non-injective [7], strongly solid [1] and for small q we actually get back free group factors [5].

One interesting family of subalgebras of $\Gamma_q(\mathcal{H}_{\mathbb{R}})$ is the one generated by a single q-Gaussian variable $\ell(e) + \ell^*(e)$ for some $e \in \mathcal{H}_{\mathbb{R}}$. In Éric Ricard's proof [9] of factoriality for $\Gamma_q(\mathcal{H}_{\mathbb{R}})$, these subalgebras played a fundamental role. Moreover, when $q = 0$ those $\ell(e) + \ell^*(e)$ are exactly Voiculescu's semi-circular elements in the free probability theory and those subalgebras are just the generator subalgebras of free group factors. This motivates us to call such subalgebras *the generator subalgebras of q-Gaussian algebras.*

The (genuine) generator subalgebras are long known to be singular and even maximal amenable inside the free group factors [8]. Thus it is natural to ask whether the same holds for generator subalgebras in q-Gaussian algebras. The problem is that for $q \neq 0$ and $e_1, e_2 \in \mathcal{H}_{\mathbb{R}}$ mutually orthogonal, $\ell(e_1) + \ell^*(e_1)$ and $\ell(e_2) + \ell^*(e_2)$ are no longer freely independent. Another difficulty in dealing with q-Fock spaces is that it is hard to control the operator norms of $W(e^{\otimes n})$ (see Section 1 for the notation). However as we will see, for generator subalgebras

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coming from orthogonal vectors, the situation is not so far from the free case and the computation is manageable. The main result of this paper is

Theorem. *For any $-1 < q < 1$ and for any separable real Hilbert space $\mathcal{H}_{\mathbb{R}}$ with dimension no less than 2, the generator subalgebras in q-Gaussian algebras are singular.*

The author strongly believes that the generator subalgebras should be maximal amenable inside q-Gaussian algebras, but he is not able to prove it. Thus we leave it as a question:

Question. Are the generator subalgebras of $\Gamma_q(\mathcal{H}_{\mathbb{R}})$ maximal amenable? Are they disjoint from other maximal amenable subalgebras, in the sense of [11]?

Upon completion of this paper, the author learned that Bikram and Mukherjee [2] independently proved (among other things) similar results for generator subalgebras in q-deformed Araki-Woods algebras which contain the main theorem in this paper as a special case. Their approach to show the mixing property of the generator subalgebra is by proving that the left-right measure of the generator subalgebra is Lebesgue absolutely continuous. The proof essentially boils down to the tracial case and is very similar to ours. Indeed, both the proof of one key ingredient of their paper, [2, Theorem 5.3] and the proof of Theorem 2 in the current paper, are strongly motivated by Ricard's work [9].

1. Preliminaries

Let $-1 < q < 1$ be a fixed real number and let $\mathcal{H}_{\mathbb{R}}$ be a separable real Hilbert space with dimension no less than 2. Denote by $\mathcal{H} := \mathcal{H}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ the complexification of $\mathcal{H}_{\mathbb{R}}$. Define an inner product on $\bigoplus_{n\geq 0} \mathcal{H}^{\otimes n}$ by

$$\langle e_1 \otimes \cdots \otimes e_n, f_1 \otimes \cdots \otimes f_m \rangle_q = \delta_n(m) \sum_{\sigma \in S_m} q^{|\sigma|} \left\langle e_1 \otimes \cdots \otimes e_n, f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(m)} \right\rangle,$$

where S_m is the group of permutations on $\{1, \cdots, m\}$, $|\sigma|$ stands for the number of inversions of σ, $\mathcal{H}^{\otimes 0} = \mathbb{C}\Omega$ is the space spanned by the vacuum vector Ω and the inner product on the right-hand side is the usual one on the tensor product of Hilbert spaces. The q-deformed Fock space $\mathcal{F}_q(\mathcal{H}_{\mathbb{R}})$ is the completion of $(\bigoplus_{n\geq 0} \mathcal{H}^{\otimes n}, \langle \cdot, \cdot \rangle_q)$. We will simply write $\|\cdot\|$ to be the norm induced by this inner product.

For $e \in \mathcal{H}_{\mathbb{R}}$, we define the *left creation operator* $\ell(e)$ on $\mathcal{F}_q(\mathcal{H}_{\mathbb{H}})$ by $\ell(e)(\Omega) = e$ and

$$\ell(e)(e_1 \otimes \cdots \otimes e_n) = e \otimes e_1 \otimes \cdots \otimes e_n, \tag{1}$$

for $n \geq 1$. $\ell(e)$ is a bounded operator and its adjoint $\ell^*(e)$ is called the *(left) annihilation operator*, which is given by $\ell^*(e)(\Omega) = 0$ and

$$\ell^*(e)(e_1 \otimes \cdots \otimes e_n) = \sum_{1\leq i\leq n} q^{(i-1)} \langle e, e_i \rangle\, e_1 \otimes \cdots \otimes \hat{e}_i \otimes \cdots \otimes e_n, \tag{2}$$

for $n \geq 1$, where $\hat{e}_i$ means a removed letter. One can define similarly the right creation and annihilation operators.

For $e \in \mathcal{H}_{\mathbb{R}}$, let

$$W(e) = \ell(e) + \ell^*(e)$$

and let $\Gamma_q(\mathcal{H}_{\mathbb{R}})$ be the von Neumann algebra generated by $\{W(e) : e \in \mathcal{H}_{\mathbb{R}}\}$. We call it the *$q$-Gaussian algebra associated with $\mathcal{H}_{\mathbb{R}}$*. It is known [4] that $\Gamma_q(\mathcal{H}_{\mathbb{R}})$ is a finite

von Neumann algebra with Ω a separating and cyclic trace vector. Consequently, each element $x \in \Gamma_q(\mathcal{H}_{\mathbb{R}})$ is uniquely determined by $\xi = x \cdot \Omega \in \mathcal{F}_q(\mathcal{H}_{\mathbb{R}})$ and we write $x = W(\xi)$. This notation is consistent with the above definition for $W(e), e \in \mathcal{H}_{\mathbb{R}}$. Moreover, an easy induction shows that $\Gamma_q(\mathcal{H}_{\mathbb{R}})\Omega$ contains all the simple tensors $e_1 \otimes \cdots \otimes e_n \in \mathcal{H}^{\otimes n}$.

Here we record two facts that will be used in this paper.

- Let $e \in \mathcal{H}$ be a unit vector; then

$$\|e^{\otimes n}\|^2 = [n]_q!, \tag{3}$$

where $[k]_q = \dfrac{1-q^k}{1-q}$ and $[n]_q! = [1]_q \cdots [n]_q$.

- (**Wick formula**) Let $e_1 \otimes \cdots \otimes e_n \in \mathcal{H}^{\otimes n}$; then

$$\begin{aligned} W(e_1 \otimes \cdots \otimes e_n) = \sum_{i=0}^{n} \sum_{\sigma \in S_n/(S_{n-i} \times S_i)} & q^{|\sigma|} \ell(e_{\sigma(1)}) \cdots \ell(e_{\sigma(n-i)}) \\ & \times \ell^*(e_{\sigma(n-i+1)}) \cdots \ell^*(e_{\sigma(n)}), \end{aligned} \tag{4}$$

where σ is the representative of the *right* coset of $S_{n-i} \times S_i$ in S_n with a minimal number of inversions. For each coset such a representative is unique thus the above formula is well defined.

From now on we fix a unit vector $e \in \mathcal{H}_{\mathbb{R}}$ and we call the von Neumann subalgebra $\Gamma_q(\mathbb{R}e) \subset \Gamma_q(\mathcal{H}_{\mathbb{R}})$ a *generator subalgebra*. It is shown by Ricard [9] that this gives a maximal abelian subalgebra (masa) of $\Gamma_q(\mathcal{H}_{\mathbb{R}})$, however we will not need this fact in our proof.

Let $T : \mathcal{H}_{\mathbb{R}} \to \mathcal{H}_{\mathbb{R}}$ be an $\mathbb{R}$-linear contraction. We still denote by T its complexification. Then the first quantization $\mathcal{F}_q(T)$, is the bounded operator on $\mathcal{F}_q(\mathcal{H}_{\mathbb{R}})$ given by

$$\mathcal{F}_q(T) = Id_{\mathbb{C}\Omega} \oplus \bigoplus_{n \geq 1} T^{\otimes n}.$$

The *second quantization of* T, is the unique unital completely positive map $\Gamma_q(T)$ on $\Gamma_q(\mathcal{H}_{\mathbb{R}})$ given by

$$\Gamma_q(T)(W(\xi)) = W(\mathcal{F}_q(T)(\xi)).$$

In particular, if $T = E_e : \mathcal{H}_{\mathbb{R}} \to \mathbb{R}e$ is the orthogonal projection, then $\Gamma_q(E_e)$ is the conditional expectation of $\Gamma_q(\mathcal{H}_{\mathbb{R}})$ onto $\Gamma_q(\mathbb{R}e)$.

2. Singularity of the generator subalgebra $\Gamma_q(\mathbb{R}e)$

Recall that a von Neumann subalgebra $A \subset M$ is called *singular*, if the normalizer of A, defined by $\mathcal{N}_M(A) := \{u \in \mathcal{U}(M) : uAu^* = A\}$, is contained in A. As is well known, singularity is closely related to a weakly mixing property: we say that for a finite von Neumann algebra (M, τ), a subalgebra A is *weakly mixing* in M if there exists a sequence of unitaries $\{u_n\}$ in A, such that

$$\lim_{n \to \infty} \|E_A(au_nb) - E_A(a)u_nE_A(b)\|_2 = 0, \forall a, b \in M,$$

where $\|x\|_2^2 = \tau(x^*x)$ for any $x \in M$. If the above limit equals 0 for any sequence of unitaries $\{u_n\}$ in A which converges to 0 weakly, then A is said to be *mixing* in M. Clearly for diffuse subalgebras, mixing implies weakly mixing and weakly mixing implies singularity (see [6], [10]).

We will use this sufficient condition to show the singularity of the generator subalgebra. In order to prove that the generator subalgebra is mixing in the q-Gaussian algebra, we will need the following basis criteria from [10, Theorem 11.4.1]. The statement is slightly stronger than that in [10], but the proof is the same. For the convenience of the reader as well as for completeness, we include the proof here.

Proposition 1. *Let M be a separable finite von Neumann algebra and $A \subset M$ a diffuse subalgebra. Let $Y \subset M$ be a subset whose linear span is $\|\cdot\|_2$-dense in $L^2(M)$ and $\{v_n\} \subset A$ is an orthonormal basis for $L^2(A)$. If*

$$\sum_n \|E_A(av_nb) - E_A(a)v_nE_A(b)\|_2^2 < \infty,$$

for all $a, b \in Y$, then A is mixing in M. In particular, A is singular in M.

Proof. Let $\{u_n\}_{n\geq 1}$ be a sequence of unitaries in A which converges to 0 weakly. Let $a, b \in Y$ be arbitrary elements in Y, $D := \sup\{\|a\|, \|b\|\}$ and let $\epsilon > 0$ be fixed. By the hypothesis we can find a k such that

$$\sum_{i>k} \|E_A(av_ib) - E_A(a)v_iE_A(b)\|_2^2 < \epsilon^2.$$

Since $u_n \to 0$ weakly, we can choose an n_0 such that

$$|\tau(u_n^*v_i)| \leq \frac{\epsilon}{kD^2},$$

for all $1 \leq i \leq k$ and for all $n \geq n_0$.

Write $u_n = \sum_i \tau(u_nv_i^*)v_i$; then we have

(5)
$$\begin{aligned}\|E_A(au_nb) - E_A(a)u_nE_A(b)\|_2 &= \left\|\sum_i \tau(u_nv_i^*)\left(E_A(av_ib) - E_A(a)v_iE_A(b)\right)\right\|_2 \\ &\leq \left\|\sum_{1\leq i\leq k} \tau(u_nv_i^*)\left(E_A(av_ib) - E_A(a)v_iE_A(b)\right)\right\|_2 \\ &\quad + \left\|\sum_{i>k} \tau(u_nv_i^*)\left(E_A(av_ib) - E_A(a)v_iE_A(b)\right)\right\|_2.\end{aligned}$$

For $n \geq n_0$,

(6)
$$\begin{aligned}&\left\|\sum_{1\leq i\leq k} \tau(u_nv_i^*)\left(E_A(av_ib) - E_A(a)v_iE_A(b)\right)\right\|_2 \\ &\leq \sum_{1\leq i\leq k} \frac{\epsilon}{kD^2}\|E_A(av_ib) - E_A(a)v_iE_A(b)\|_2 \leq \sum_{1\leq i\leq k} \frac{\epsilon}{kD^2}\cdot 2\|a\|\|b\|\|v_i\|_2 \leq 2\epsilon.\end{aligned}$$

On the other hand, an easy application of the Cauchy-Schwarz inequality gives that

$$\begin{aligned}
&\left\| \sum_{i>k} \tau(u_n v_i^*) \left(E_A(av_ib) - E_A(a)v_iE_A(b)\right) \right\|_2 \\
&\leq \left(\sum_{i>k} |\tau(u_n v_i^*)|^2 \right)^{1/2} \left(\sum_{i>k} \|E_A(av_nb) - E_A(a)v_nE_A(b)\|_2^2 \right)^{1/2} \\
&\leq ||u_n||_2 \cdot \epsilon \leq \epsilon.
\end{aligned} \tag{7}$$

Therefore,

$$\lim_{n\to\infty} \|E_A(au_nb) - E_A(a)u_nE_A(b)\|_2 = 0, \forall a, b \in Y.$$

Noticing that $\|u_n\| = 1$, a density argument then completes the proof. □

The main result of this paper is

Theorem 2. *Let q be a real number between $(-1, 1)$ and let $\mathcal{H}_{\mathbb{R}}$ be a separable real Hilbert space with dimension greater than or equal to 2. Let $e \in \mathcal{H}_{\mathbb{R}}$ be a unit vector and write $A = \Gamma_q(\mathbb{R}e)$ as the generator subalgebra of the q-Gaussian algebra $M = \Gamma_q(\mathcal{H}_{\mathbb{R}})$. Suppose that $\{e_i : 0 \leq i \leq \dim(\mathcal{H}_{\mathbb{R}}) - 1\}$ is an orthonormal basis for $\mathcal{H}_{\mathbb{R}}$, where $e_0 = e$. We write E_A as the conditional expectation from M onto A, which can be obtained via the second quantization of E_e.*

Define

$$v_0 = W(\Omega) = 1, v_j = \frac{W(e^{\otimes j})}{\|W(e^{\otimes j})\|_2}, \forall j \in \mathbb{N}$$

and let

$$Y = \{W(f_1 \otimes \cdots \otimes f_s) : s \geq 0, f_t \in \{e_i\}, \forall 1 \leq t \leq s\} \subset M.$$

Then $\{v_j\}_{j\geq 0}$ is an orthonormal basis for $L^2(A)$ and span(Y) is dense in $L^2(M)$ in the $\|\cdot\|_2$-norm, such that

$$\sum_j \|E_A(av_jb) - E_A(a)v_jE_A(b)\|_2^2 < \infty, \tag{8}$$

for all $a, b \in Y$. Consequently, A is mixing (thus singular) in M.

Proof. The statements that $\{v_j\}_{j\geq 0}$ is an orthonormal basis for $L^2(A)$ and that *span*(Y) is dense in $L^2(M)$ in the $\|\cdot\|_2$-norm are clear from definitions. We just need to show the estimate (8).

To this end, first note that if either a or b is from A, then (8) is trivially true. Indeed, in this case $E_A(av_jb) - E_A(a)v_jE_A(b) = 0, \forall j \geq 0$.

Therefore we can assume that $a, b \in M \ominus A$ are of the form

$$a = W(f_1 \otimes \cdots \otimes f_s), b = W(g_1 \otimes \cdots \otimes g_t),$$

where $s, t \in \mathbb{N}$, $f_k, g_l \in \{e_i\}_{i\geq 0}, 1 \leq k \leq s, 1 \leq l \leq t$ and there is some $1 \leq k_0 \leq s, 1 \leq l_0 \leq t$ such that $f_{k_0}, g_{l_0} \in \{e_i\}_{i\geq 1}$.

Moreover, by the Wick formula (4), the form of the annihilation operator and the assumption that $\{e_i : 0 \leq i \leq \dim(\mathcal{H}_{\mathbb{R}}) - 1\}$ is an orthonormal set, we can also assume that for each $i \geq 1$, the multiplicity of e_i in the word form of a is the same as in the word form of b.

For simplicity, we will show the conclusion (8) when b is of the form

$$b = W(f_1 \otimes \cdots f_m \otimes e^{\otimes l}),$$

where $f_1, \cdots, f_m \in \{e_i : i \geq 1\}$. Assume also that the multiplicity of e in the word form of a is n (n may differ from l). The other cases are completely similar.

For each $N \geq 0$, let

(9)
$$C_N = \|E_A(av_N b) - E_A(a)v_N E_A(b)\|_2^2 = \frac{\left\|\mathcal{F}_q(E_e)\left(aW(e^{\otimes N})(f_1 \otimes \cdots f_m \otimes e^{\otimes l})\right)\right\|^2}{\|W(e^{\otimes N})\|_2^2}.$$

Our goal is to estimate C_N when N is large.

First we apply the Wick formula to a. Note that there are finitely many terms in the expansion, thus it suffices to estimate each term. Also note that each $f_i, 1 \leq i \leq m$ has to appear in this expansion of a as an annihilation operator, in order to get anything non-zero under $\mathcal{F}_q(E_e)$. Again for simplicity, we consider here only the terms of the form

(10)
$$D_N = \frac{1}{\|W(e^{\otimes N})\|_2^2}\left\|\mathcal{F}_q(E_e)\left(\ell^n(e)\ell^*(f_1)\cdots\ell^*(f_m)W(e^{\otimes N})(f_1 \otimes \cdots f_m \otimes e^{\otimes l})\right)\right\|^2,$$

and the rest of the cases are almost identical.

Next, we apply the Wick formula to $W(e^{\otimes N})$ inside the above expression for D_N,

(11)
$$\begin{aligned}
D_N &= \frac{1}{\|W(e^{\otimes N})\|_2^2}\left\|\mathcal{F}_q(E_e)\left(\ell^n(e)\ell^*(f_1)\cdots\ell^*(f_m)W(e^{\otimes N})(f_1 \otimes \cdots f_m \otimes e^{\otimes l})\right)\right\|^2 \\
&= \frac{1}{\|W(e^{\otimes N})\|_2^2}\left\|\mathcal{F}_q(E_e)\left(\ell^n(e)\ell^*(f_1)\cdots\ell^*(f_m)\sum_{i=0}^{N}\sum_{\sigma} q^{|\sigma|}\ell^{N-i}(e)\ell^{*i}(e)\right.\right. \\
&\qquad\qquad \left.\left.\times (f_1 \otimes \cdots f_m \otimes e^{\otimes l})\right)\right\|^2 \\
&= \frac{1}{\|W(e^{\otimes N})\|_2^2}\left\|\mathcal{F}_q(E_e)\left(\ell^n(e)\ell^*(f_1)\cdots\ell^*(f_m)\sum_{i=0}^{l}\sum_{\sigma} q^{|\sigma|}\ell^{N-i}(e)\ell^{*i}(e)\right.\right. \\
&\qquad\qquad \left.\left.\times (f_1 \otimes \cdots f_m \otimes e^{\otimes l})\right)\right\|^2,
\end{aligned}$$

where the last equality comes from the orthogonality of $\{e_i\}$.

For each $0 \leq i \leq l$, we have

(12)
$$\mathcal{F}_q(E_e)\left(\ell^n(e)\ell^*(f_1)\cdots\ell^*(f_m)\ell^{N-i}(e)\ell^{*i}(e)(f_1 \otimes \cdots f_m \otimes e^{\otimes l})\right) \in \mathbb{C}e^{\otimes(N+n+l-2i)},$$

therefore if we let

(13)
$$D_{N,i} = \frac{1}{\|W(e^{\otimes N})\|_2^2}\left\|\mathcal{F}_q(E_e)\left(\ell^n(e)\ell^*(f_1)\cdots\ell^*(f_m)\sum_{\sigma} q^{|\sigma|}\ell^{N-i}(e)\ell^{*i}(e)\right.\right.$$
$$\left.\left.\times (f_1 \otimes \cdots f_m \otimes e^{\otimes l})\right)\right\|^2,$$

then $D_N = \sum_{i=0}^{l} D_{N,i}$.

Now there is an $M_1 = M_1(l, m, q) > 0$, such that

(14)
$$D_{N,i} \leq M_1 \cdot \frac{|S_N/(S_{N-i} \times S_i)|^2}{\|W(e^{\otimes N})\|_2^2} \Big\| \mathcal{F}_q(E_e) \Big(\ell^n(e)\ell^*(f_1) \cdots \ell^*(f_m) \times (e^{\otimes(N-i)} \otimes f_1 \otimes \cdots f_m \otimes e^{\otimes(l-i)}) \Big) \Big\|^2 .$$

For $\ell^n(e)\ell^*(f_1)\cdots\ell^*(f_m)(e^{\otimes(N-i)} \otimes f_1 \otimes \cdots f_m \otimes e^{\otimes(l-i)})$ to contribute something non-zero in $\mathcal{F}_q(\mathbb{R}e)$, each $l^*(f_j)$ has to first pass $e^{\otimes(N-i)}$ then hit $f_1 \otimes \cdots \otimes f_m$. The definition for annihilation operators (2) then implies that there exists an $M_2 = M_2(m, l) > 0$, such that

$$\begin{aligned} &\Big\| \mathcal{F}_q(E_e) \Big(\ell^n(e)\ell^*(f_1) \cdots \ell^*(f_m)(e^{\otimes(N-i)} \otimes f_1 \otimes \cdots f_m \otimes e^{\otimes(l-i)}) \Big) \Big\|^2 \\ &\quad \leq M_2 \cdot |q|^{2N} \|e^{\otimes(N+n+l-2i)}\|^2. \end{aligned} \tag{15}$$

Combining the previous two inequalities, we obtain

$$\begin{aligned} D_{N,i} &\leq M_1 \cdot M_2 \cdot |q|^{2N} \cdot \frac{|S_N/(S_{N-i} \times S_i)|^2}{\|W(e^{\otimes N})\|_2^2} \cdot \|e^{\otimes(N+n+l-2i)}\|^2 \\ &= M_1 \cdot M_2 \cdot |q|^{2N} \cdot \left| \frac{N!}{(N-i)!i!} \right|^2 \cdot \frac{[N+n+l-2i]_q!}{[N]_q!} \\ &\leq M_1 \cdot M_2 \cdot |q|^{2N} \cdot N^{2i} \cdot \frac{[N+n+l-2i]_q!}{[N]_q!}. \end{aligned} \tag{16}$$

Since $|q| < 1$, it is then clear that $\sum_N D_{N,i} < \infty$ which implies that

$$\sum_N C_N < \infty.$$

The mixing property and the singularity of A in M then follow from Proposition 1. □

Remark 3. The reason we use Proposition 1 to show the mixing property, instead of proving it directly, is that the basis we choose may be unbounded in the operator (uniform) norm hence the density argument may fail.

Note that in the proof of the theorem, we never used the fact that A is maximal abelian. In fact, A being singular and abelian implies that it is maximal abelian. Thus our proof recovers Ricard's results from [9].

Corollary 4. *With the same assumptions as in the previous theorem, the generator subalgebra is a masa in the q-Gaussian algebra.*

Corollary 5. *For any separable real Hilbert space $\mathcal{H}_\mathbb{R}$ with dimension greater than or equal to 2, $\Gamma_q(\mathcal{H}_\mathbb{R})$ is a II_1 factor.*

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