

## SCHUR-WEYL DUALITY AND THE FREE LIE ALGEBRA

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ABSTRACT. We prove an analogue of Schur-Weyl duality for the space of homogeneous Lie polynomials of degree  $r$  in  $n$  variables.

### 1. INTRODUCTION

Let  $k$  be a commutative ring and  $V$  a given  $k$ -module. Put  $E = \text{End}_k(V)$ . The centralizer of a set  $X \subseteq E$  of  $k$ -linear endomorphisms of  $V$  is the set

$$Z_E(X) = \{f \in E \mid fx = xf, \text{ for all } x \in X\}.$$

Suppose further that  $V$  has a given  $(A, B)$ -bimodule structure, where  $A$  and  $B$  are  $k$ -algebras. Let  $\bar{A}, \bar{B} \subseteq \text{End}_k(V)$  be the sets of  $k$ -linear endomorphisms of  $V$  induced by the actions of  $A$  and  $B$ , respectively. Since the actions of  $A$  and  $B$  commute, we have inclusions

$$(1) \quad \bar{A} \subseteq \text{End}_B(V) \quad \text{and} \quad \bar{B} \subseteq \text{End}_A(V),$$

where  $\text{End}_A(V) = Z_E(\bar{A})$  and  $\text{End}_B(V) = Z_E(\bar{B})$ . When the inclusions in (1) are equalities then we say that the triple  $(A, V, B)$  satisfies Schur-Weyl duality. This implies that  $\bar{A}$  and  $\bar{B}$  both have the double centralizer property, that is,

$$(2) \quad Z_E(Z_E(\bar{A})) = \bar{A} \quad \text{and} \quad Z_E(Z_E(\bar{B})) = \bar{B}.$$

If  $\bar{A}$  has the double centralizer property and  $\bar{B} = Z_E(\bar{A}) = \text{End}_A(V)$ , then  $\bar{A} = Z_E(\bar{B}) = \text{End}_B(V)$  as well, and  $(A, V, B)$  satisfies Schur-Weyl duality. But the two equalities in (2) do not by themselves imply that  $(A, V, B)$  satisfies Schur-Weyl duality.

Assume henceforth that  $k$  is a field. An important example of the duality above is given by the  $r^{\text{th}}$  tensor power  $V = T^r(k^n) = (k^n)^{\otimes r}$  of the space  $k^n$  of  $n$ -dimensional column vectors, regarded as an  $(A, B)$ -bimodule, where  $A = k \text{GL}_n(k)$  and  $B = k \Sigma_r$  are respectively the group algebras of the general linear group  $\text{GL}_n(k)$  and symmetric group  $\Sigma_r$ , with  $\text{GL}_n(k)$  acting diagonally on the left and  $\Sigma_r$  acting on the right by place permutation. To be precise, the commuting actions are given by

$$g \cdot (v_1 \otimes \cdots \otimes v_r) = gv_1 \otimes \cdots \otimes gv_r \quad \text{and} \quad (v_1 \otimes \cdots \otimes v_r) \cdot \sigma = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(r)}$$

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for all  $v_1, \dots, v_r \in k^n$ ,  $g \in \text{GL}_n(k)$ , and  $\sigma \in \Sigma_r$ . In this setting the assertion that the triple  $(k \text{GL}_n(k), T^r(k^n), k\Sigma_r)$  satisfies Schur-Weyl duality is the classical Schur-Weyl duality between representations of  $\text{GL}_n(k)$  and  $\Sigma_r$ , which is known to hold whenever  $|k| > r$ . (See [2] for a proof.)

In this note, we investigate the analogue of classical Schur-Weyl duality when tensor space  $T^r(k^n)$  is replaced by its intersection  $L^r(k^n)$  with the free Lie algebra  $L(k^n)$  on  $k^n$ .

Recall that the free Lie algebra  $L(k^n)$  is the Lie subalgebra of the tensor algebra  $T(k^n)$  generated by  $k^n$ , where  $T(k^n)$  is regarded as a Lie algebra via the Lie bracket  $[a, b] = ab - ba$ . Fixing a basis  $X$  of  $k^n$ , we may identify the tensor algebra  $T(k^n)$  with the free associative algebra  $k\langle X \rangle$  on  $X$ . In this point of view, elements of  $k\langle X \rangle$  are regarded as noncommutative polynomials in “variables”  $X$ , and polynomials in the subspace  $L(k^n)$  are known as Lie polynomials. The grading  $T(k^n) = \bigoplus_{r \geq 0} T^r(k^n)$  induces a corresponding grading

$$L(k^n) = \bigoplus_{r \geq 0} L^r(k^n), \quad \text{where } L^r(k^n) = L(k^n) \cap T^r(k^n)$$

on  $L(k^n)$ . The  $r^{\text{th}}$  graded component  $L^r(k^n)$  in the above decomposition is the space of homogeneous Lie polynomials of degree  $r$ .

Since  $g \cdot [a, b] = [g \cdot a, g \cdot b]$  for any  $g \in \text{GL}_n(k)$ ,  $a, b \in k^n$ , it is clear that  $L(k^n)$  is invariant under the action of  $\text{GL}_n(k)$ , hence is a left  $k \text{GL}_n(k)$ -module. It follows that the same holds for  $L^r(k^n)$ . A natural problem is to describe the centralizer algebra  $\text{End}_{\text{GL}_n(k)}(L^r(k^n))$  as a subquotient of  $k\Sigma_r$ ; ideally one would like to identify a subalgebra  $B_r$  of  $k\Sigma_r$  which maps onto this centralizer. Furthermore, having identified such a subalgebra  $B_r$ , it is natural to ask whether the triple  $(k \text{GL}_n(k), L^r(k^n), B_r)$  satisfies Schur-Weyl duality. When it does, we have an analogue of Schur-Weyl duality for the module  $L^r(k^n)$  of homogeneous Lie polynomials of degree  $r$ .

Our main results explain how to identify the appropriate subalgebra  $B_r$  and establish that, indeed,  $(k \text{GL}_n(k), L^r(k^n), B_r)$  satisfies Schur-Weyl duality, provided only that the characteristic of  $k$  is strictly larger than  $r$ . (We agree that characteristic zero is infinite and hence always larger than any  $r$ .) Furthermore, under our assumption on the characteristic, it is well-known that an idempotent  $e \in k\Sigma_r$  exists such that  $L^r(k^n) = T^r(k^n)e$ . Then  $B_r$  may be taken to be the subalgebra  $ek\Sigma_re$  of  $k\Sigma_r$ . The more general question of whether Schur-Weyl duality holds whenever  $|k| > r$  and the characteristic of  $k$  does not divide  $r$  remains open.

The paper is organized as follows. In §2 we state the main theorem, Theorem 2.2, show that the triple  $(k \text{GL}_n(k), L^r(k^n), ek\Sigma_re)$  satisfies Schur-Weyl duality when the characteristic of  $k$  is strictly larger than  $r$ , and draw some general conclusions.

The proof of Theorem 2.2 is given in §3 as an application of more general results. We consider a triple  $(A, V, B)$  that satisfies Schur-Weyl duality and an idempotent  $e \in B$ , and we ask when  $(A, Ve, eBe)$  satisfies Schur-Weyl duality. It turns out that  $e\overline{B}e$  is always equal to  $\text{End}_A(Ve)$ . The equality  $\overline{A} = \text{End}_{eBe}(Ve)$  seems to be a more delicate question. We show that if  $V$  is a completely reducible  $A$ -module whose irreducible constituents are absolutely irreducible, then in fact  $(A, Ve, eBe)$  satisfies Schur-Weyl duality.

In the case of classical Schur-Weyl duality,  $L^r(k^n)$  is a tilting module, and it is tempting to try to use the theory of tilting modules and a  $p$ -modular system to derive results in positive characteristic from known results in characteristic zero.

This approach can be used to yield a uniform proof in all characteristics of some of the known properties of the algebra  $ek\Sigma_r e$ , but we have been unable to show that  $(k\text{GL}_n(k), L^r(k^n), ek\Sigma_r e)$  satisfies Schur-Weyl duality in general.

Finally, in §4 we describe the commuting algebra  $ek\Sigma_r e$  in the favorable case when the characteristic of  $k$  is strictly larger than  $r$ , and we show that, in general, whether  $(k\text{GL}_n(k), L^r(k^n), ek\Sigma_r e)$  satisfies Schur-Weyl duality may be reduced to a statement about permutation representations of  $\Sigma_r$  arising from Young subgroups.

2. NOTATION AND MAIN RESULTS

In this section we establish our basic notation and formulate the main results.

Recall that  $k$  denotes a field. Throughout the paper, we set  $T^{n,r} = T^r(k^n)$  and  $L^{n,r} = L^r(k^n)$  for ease of notation, and we denote by  $\Phi$  and  $\Psi$  the  $k$ -algebra homomorphisms

$$k\text{GL}_n(k) \xrightarrow{\Phi} \text{End}_k(T^{n,r}) \xleftarrow{\Psi} k\Sigma_r$$

induced by the commuting actions of  $\text{GL}_n(k)$  and  $\Sigma_r$  described in the introduction. Note that because  $\Sigma_r$  acts on the right, the homomorphism  $\Psi$  is given by  $\Psi(\sigma)(x) = x \cdot \sigma^{-1}$  for  $\sigma \in \Sigma_r$  and  $x \in T^{n,r}$ . Then classical Schur-Weyl duality is the pair of equalities

$$(3) \quad \Phi(k\text{GL}_n(k)) = \text{End}_{\Sigma_r}(T^{n,r}) \quad \text{and} \quad \Psi(k\Sigma_r) = \text{End}_{\text{GL}_n(k)}(T^{n,r}).$$

It will be convenient to define a Lie idempotent to be any idempotent  $e$  in  $k\Sigma_r$  such that  $L^{n,r} = T^{n,r} \cdot e$ . This agrees with the usual definition (see e.g. [11, §8.4]) when  $n \geq r$ . Lie idempotents exist whenever the characteristic of  $k$  does not divide  $r$ . Proofs of the following well-known result may be found in [5, §2] and [11, §8.4].

**Lemma 2.1.** *Assume that the characteristic of  $k$  does not divide  $r$ . Then the Dynkin-Specht-Wever idempotent  $e_r = \frac{1}{r}(1 - \gamma_2) \cdots (1 - \gamma_r)$ , where for  $2 \leq i \leq r$ ,  $\gamma_i$  is the descending  $i$ -cycle  $(i \cdots 2 1)$  in  $\Sigma_r$ , is a Lie idempotent.*

Suppose that  $e$  is a Lie idempotent. Then  $ek\Sigma_r e$  acts on  $L^{n,r}$  on the right and  $\text{GL}_n(k)$  acts on  $L^{n,r}$  on the left. Thus there are  $k$ -algebra homomorphisms

$$k\text{GL}_n(k) \xrightarrow{\Phi_e} \text{End}_k(L^{n,r}) \xleftarrow{\Psi_e} ek\Sigma_r e$$

such that the images of  $\Phi_e$  and  $\Psi_e$  commute, so  $L^{n,r}$  is a  $(k\text{GL}_n(k), ek\Sigma_r e)$ -bimodule. Our main theorem is the following analogue of classical Schur-Weyl duality for this bimodule. The theorem is proved in §3.

**Theorem 2.2.** *Suppose that  $k$  is a field of cardinality strictly larger than  $r$  such that the characteristic of  $k$  does not divide  $r$  and that  $e$  is a Lie idempotent in  $k\Sigma_r$ . Then*

$$\Psi_e(ek\Sigma_r e) = \text{End}_{\text{GL}_n(k)}(L^{n,r}).$$

*If in addition  $T^{n,r}$  is a direct sum of absolutely irreducible  $k\text{GL}_n(k)$ -modules, then*

$$\Phi_e(k\text{GL}_n(k)) = \text{End}_{ek\Sigma_r e}(L^{n,r}).$$

Recall that the Schur algebra over  $k$  is the algebra

$$\mathcal{S}(n, r) = \Phi(k\text{GL}_n(k)) = \text{End}_{\Sigma_r}(T^{n,r})$$

appearing in (3). Suppose that the characteristic of  $k$  is larger than  $r$ , so  $|k| > r$  as well. It is well-known that in this case  $k\Sigma_r$  is a split, semisimple  $k$ -algebra (see [8])

and so  $\Psi(k\Sigma_r)$  is a split, semisimple  $k$ -algebra. By classical Schur-Weyl duality the centralizer of  $\Psi(k\Sigma_r)$  in  $\text{End}_k(T^{n,r})$  is  $\mathcal{S}(n,r)$ . It is easy to see that the centralizer of a split, semisimple subalgebra of  $\text{End}_k(T^{n,r})$  is again split, semisimple (see §3), so  $\mathcal{S}(n,r)$  is a split, semisimple  $k$ -algebra. Thus,  $T^{n,r}$  is a direct sum of absolutely irreducible  $k \text{GL}_n(k)$ -modules, and so both equalities in the theorem hold.

**Corollary 2.3.** *The triple  $(k \text{GL}_n(k), L^{n,r}, e k \Sigma_r e)$  satisfies Schur-Weyl duality whenever the characteristic of  $k$  is strictly larger than  $r$ .*

Since classical Schur-Weyl duality is known to hold whenever  $|k| > r$  (see [2]), it is natural to ask whether  $(k \text{GL}_n(k), L^{n,r}, e k \Sigma_r e)$  satisfies Schur-Weyl duality whenever  $|k| > r$  and the characteristic of  $k$  does not divide  $r$ . Based on small rank examples, it seems likely that this is indeed the case. As a step in this direction, in §4 it is shown that the second equality in the theorem is equivalent to a statement about intertwining operators between certain transitive permutation representations arising from Young subgroups of  $\Sigma_r$ .

Assume for a moment that  $n \geq r$ . Then  $T^{n,r}$  is a faithful  $k\Sigma_r$ -module and the Schur functor  $\mathfrak{f}: M \mapsto \epsilon M$  from left  $\mathcal{S}(n,r)$ -modules to left  $k\Sigma_r$ -modules may be defined, where  $\epsilon \in \mathcal{S}(n,r)$  is an idempotent that projects  $T^{n,r}$  onto its  $(1^r, 0^{n-r})$ -weight space. By [7, (6.3d)] we have that  $\mathfrak{f}(T^{n,r}) = \epsilon T^{n,r}$  is isomorphic to the left regular  $k\Sigma_r$ -module  ${}_{k\Sigma_r} k\Sigma_r$ , and so (3) takes the form

$$(4) \quad \mathcal{S}(n,r) = \text{End}_{k\Sigma_r}(T^{n,r})$$

and

$$(5) \quad \text{End}_{k\Sigma_r}(\mathfrak{f}(T^{n,r})) = \text{End}_{k\Sigma_r}({}_{k\Sigma_r} k\Sigma_r) \cong \text{End}_{\mathcal{S}(n,r)}(T^{n,r}).$$

Assume further that the characteristic of  $k$  does not divide  $r$ , so that a Lie idempotent  $e$  exists. Because  $L^{n,r} = T^{n,r}e$  is a  $\text{GL}_n(k)$ -stable subspace of  $T^{n,r}$ , it has a natural  $\mathcal{S}(n,r)$ -module structure. The Lie module,

$$\text{Lie}(r) = \mathfrak{f}(L^{n,r}),$$

is the  $k\Sigma_r$ -module obtained by applying the Schur functor to  $L^{n,r}$ . Then

$$\mathfrak{f}(L^{n,r}) = \epsilon L^{n,r} = \epsilon(T^{n,r}e) = (\epsilon T^{n,r})e \cong {}_{k\Sigma_r} e,$$

and so the Lie module is isomorphic to the left  $k\Sigma_r$ -module  ${}_{k\Sigma_r} e$ . Let  $\overline{\mathcal{S}(n,r)} = \overline{\Phi_e(k \text{GL}_n(k))}$ . If  $\varphi \in \mathcal{S}(n,r)$ , then  $\varphi$  is  $\Sigma_r$ -equivariant and so  $\varphi(L^{n,r}) \subseteq L^{n,r}$ . Hence restriction defines an algebra homomorphism from  $\mathcal{S}(n,r)$  to  $\text{End}_{e k \Sigma_r e}(L^{n,r})$  with image equal to  $\overline{\mathcal{S}(n,r)}$ . With this notation, Theorem 2.2 immediately implies the following analogue of classical Schur-Weyl duality expressed by (4) and (5) above, with the  $r^{\text{th}}$  graded piece of the free associative algebra on  $n$  letters replaced by the  $r^{\text{th}}$  graded piece of the free Lie algebra on  $n$  letters, and with the left regular  $k\Sigma_r$ -module replaced by the  $r^{\text{th}}$  Lie module.

**Corollary 2.4.** *Suppose that  $|k| > r$ , the characteristic of  $k$  does not divide  $r$ , and  $n \geq r$ . Then*

$$\overline{\mathcal{S}(n,r)} = \text{End}_{e k \Sigma_r e}(L^{n,r})$$

and

$$\text{End}_{e k \Sigma_r e}(\mathfrak{f}(L^{n,r})) = \text{End}_{e k \Sigma_r e}(\text{Lie}(r)) \cong \text{End}_{\overline{\mathcal{S}(n,r)}}(L^{n,r}),$$

where  $e$  is any Lie idempotent.

Returning to the case of general  $n$  and  $r$ , suppose the field  $k$  contains a primitive  $r^{\text{th}}$  root of unity  $\zeta$ . Then the right ideal  $ek\Sigma_r$  and the algebra  $ek\Sigma_re$  arise in a surprisingly different context. To describe this connection further, fix an  $r$ -cycle  $\gamma$  in  $\Sigma_r$  and let  $\Gamma = \langle \gamma \rangle$ . Let  $f = (1/r) \sum_{i=1}^r \zeta^{-i} \gamma^i$ . Then  $f$  is the primitive idempotent in  $k\Gamma$  corresponding to a faithful character of  $\Gamma$ . The right ideal  $fk\Sigma_r$  of  $k\Sigma_r$  affords the induced representation  $\text{Ind}_{\Gamma}^{\Sigma_r} \zeta$ , and the subalgebra  $fk\Sigma_rf$  is isomorphic to the endomorphism algebra of the induced module  $\text{Ind}_{\Gamma}^{\Sigma_r} \zeta$ . There is a Lie idempotent  $\kappa$ , the Klyachko idempotent (see §4), such that  $e\kappa = e$ ,  $\kappa f = f$ , and  $f\kappa = \kappa$ . It follows that

$$ek\Sigma_r \cong \kappa k\Sigma_r = fk\Sigma_r \quad \text{and so} \quad ek\Sigma_re \cong fk\Sigma_rf \cong \text{End}_{k\Sigma_r}(\text{Ind}_{\Gamma}^{\Sigma_r} \zeta).$$

On the other hand, suppose that  $k = \mathbb{C}$  and let  $M$  denote the subset of  $\mathbb{C}^n$  consisting of vectors with distinct coordinates. Then  $M$  is the complement of the union of the hyperplanes in the braid arrangement on  $r$  strands. Arnold [1] has described the cohomology ring  $H^*(M)$ . The group  $\Sigma_r$  acts on  $M$  by permuting the coordinates and hence acts on the cohomology spaces  $H^p(M)$ . Lehrer and Solomon [10] have described these representations of  $\Sigma_r$  as direct sums of representations induced from linear characters of centralizers. A special case is the  $r$ -cycle  $\gamma$  and its centralizer  $\Gamma$ . In this case, it follows from the results in [4, §5] that the representation of  $\Sigma_r$  afforded by the highest non-vanishing cohomology space  $H^{r-1}(M)$  is isomorphic to the representation afforded by  $\mathbb{C}_{\text{sgn}} \otimes f\mathbb{C}\Sigma_r \cong \mathbb{C}_{\text{sgn}} \otimes e\mathbb{C}\Sigma_r$ , where  $\mathbb{C}_{\text{sgn}}$  denotes the sign representation of  $\Sigma_r$ .

### 3. GENERALIZED SCHUR-WEYL DUALITY

We will now prove Theorem 2.2. It turns out that our result is a special case of a more general result, as formulated below.

Suppose that  $A$  and  $B$  are  $k$ -algebras and  $V$  is an  $(A, B)$ -bimodule. Then  $V$  is a left  $B^{\text{op}}$ -module and there are  $k$ -algebra homomorphisms

$$(6) \quad A \xrightarrow{\Phi} \text{End}_k(V) \xleftarrow{\Psi} B^{\text{op}},$$

where  $\Phi(a)(v) = av$  and  $\Psi(b)(v) = vb$  for  $a \in A$ ,  $v \in V$ , and  $b \in B$ . Assume that the triple  $(A, V, B)$  satisfies Schur-Weyl duality, so

$$\Phi(A) = \text{End}_B(V) \quad \text{and} \quad \Psi(B) = \text{End}_A(V).$$

Suppose that  $e$  in  $B$  is an idempotent such that  $Ve \neq 0$ . Clearly  $Ve$  is an  $(A, eBe)$ -bimodule, and we can ask under what conditions  $(A, Ve, eBe)$  satisfies Schur-Weyl duality. In this situation, the commuting actions induce  $k$ -algebra homomorphisms

$$A \xrightarrow{\Phi_e} \text{End}_k(Ve) \xleftarrow{\Psi_e} (eBe)^{\text{op}}$$

such that

$$(7) \quad \Phi_e(A) \subseteq \text{End}_{eBe}(Ve) \quad \text{and} \quad \Psi_e(eBe) \subseteq \text{End}_A(Ve).$$

We wish to find conditions under which the above inclusions are equalities; that is, we wish to prove that  $(A, Ve, eBe)$  satisfies Schur-Weyl duality under appropriate hypotheses. That the second inclusion in (7) is an equality is an easy general fact, requiring no additional hypothesis.

**Lemma 3.1.** *Suppose that  $(A, V, B)$  satisfies Schur-Weyl duality,  $e$  is an idempotent in  $B$  such that  $Ve \neq 0$ , and  $\Psi_e : eBe \rightarrow \text{End}_k(Ve)$  is the  $k$ -algebra homomorphism induced by the right  $eBe$ -module structure on  $Ve$ . Then*

$$\Psi_e(eBe) = \text{End}_A(Ve).$$

*Proof.* Set  $\pi = \Psi(e)$ . Then  $\pi(v) = ve$  for  $v$  in  $V$ ,  $\pi$  is an idempotent in  $\text{End}_A(V)$ , and  $Ve$  is the image of  $\pi$ .

Suppose  $\varphi$  is in  $\text{End}_k(V)$ . Then  $\pi\varphi\pi(Ve) \subseteq Ve$ . We denote the restriction of  $\pi\varphi\pi$  to  $Ve$  by  $\pi\varphi\pi|_{Ve}$ . Then  $\pi\varphi\pi|_{Ve}$  is in  $\text{End}_k(Ve)$ . Define

$$\Pi : \text{End}_k(V) \rightarrow \text{End}_k(Ve) \quad \text{by} \quad \Pi(\varphi) = \pi\varphi\pi|_{Ve}.$$

Clearly  $Ve$  is an  $A$ -submodule of  $V$ . If  $\varphi$  is  $A$ -linear, then so is  $\pi\varphi\pi|_{Ve}$ . Therefore,  $\Pi(\text{End}_A(V)) \subseteq \text{End}_A(Ve)$ . The  $A$ -module decomposition  $V \cong Ve \oplus V(1 - e)$  of  $V$  determines a canonical decomposition

$$\begin{aligned} &\text{End}_A(V) \\ &\cong \text{End}_A(Ve) \oplus \text{Hom}_A(Ve, V(1 - e)) \oplus \text{Hom}_A(V(1 - e), Ve) \oplus \text{End}_A(V(1 - e)) \end{aligned}$$

under which the linear map  $\Pi$  is identified with the projection onto  $\text{End}_A(Ve)$ . In particular,

$$(8) \quad \Pi(\text{End}_A(V)) = \text{End}_A(Ve).$$

It is straightforward to check that  $\Pi\Phi = \Phi_e$ , and so we can extend (6) to a commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{\Phi} & \text{End}_k(V) & \xleftarrow{\Psi} & B \\ & \searrow \Phi_e & \downarrow \Pi & & \downarrow \Pi_e \\ & & \text{End}_k(Ve) & \xleftarrow{\Psi_e} & eBe \end{array}$$

of  $k$ -linear maps, where  $\Pi_e : B \rightarrow eBe$  is given by  $\Pi_e(b) = ebe$ . Thus,

$$\Psi_e(eBe) = \Psi_e\Pi_e(B) = \Pi\Psi(B) = \Pi(\text{End}_A(V)) = \text{End}_A(Ve),$$

where the penultimate equality follows from the assumption that  $(A, V, B)$  satisfies Schur-Weyl duality, and the final equality follows from (8). □

**The semisimple case.** Showing that the containment  $\Phi_e(A) \subseteq \text{End}_{eBe}(Ve)$  is an equality is not so easy. We consider first the case when  $\Phi(A)$  is a split, semisimple  $k$ -algebra.

Precisely, assume that  $V$  is a finite dimensional  $k$ -vector space, and suppose that  $(A, V, B)$  satisfies Schur-Weyl duality and that  $V$  is a completely reducible  $A$ -module whose irreducible constituents are absolutely irreducible. In this case we show that if  $e$  is an idempotent in  $B$  with  $Ve \neq 0$ , then  $(A, Ve, eBe)$  satisfies Schur-Weyl duality and both the  $eBe$ -module structure of  $Ve$  and the algebra structure of  $\Psi_e(eBe)$  are determined by the  $A$ -module structure of  $V$ . The argument is based on a version of the double centralizer theorem in the form given below. We include a sketch of the proof in order to establish notation and because of the lack of a suitable reference.

**Theorem 3.2.** *Suppose  $k$  is a field,  $V$  is a finite dimensional  $k$ -vector space, and  $X$  is a subalgebra of  $\text{End}_k(V)$  such that  $V$  is a completely reducible  $X$ -module whose irreducible constituents are absolutely irreducible. Define  $Y = \text{End}_X(V)$  and suppose  $\{L_1, \dots, L_p\}$  is a complete set of non-isomorphic, irreducible  $X$ -modules. For  $1 \leq i \leq p$  define  $M_i = \text{Hom}_X(L_i, V)$ . Then the following statements hold.*

- (1)  $X$  and  $Y$  are split, semisimple  $k$  algebras and  $\{M_1, \dots, M_p\}$  is a complete set of non-isomorphic, irreducible  $Y$ -modules.
- (2) The natural evaluation map  $\bigoplus_{i=1}^p L_i \otimes_k M_i \rightarrow V$  is an  $(X, Y^{\text{op}})$ -bimodule isomorphism.
- (3)  $X = \text{End}_Y(V)$  and the triple  $(X, V, Y^{\text{op}})$  satisfies Schur-Weyl duality.

*Proof.* It follows from [3, (3.31)] that  $X$  is semisimple and that each  $L_i$  occurs as an irreducible constituent of  $V$ . Let  $V_1, \dots, V_p$  be the homogeneous components of  $V$  where  $V_i \cong L_i^{m_i}$ , so  $V \cong \bigoplus_{i=1}^p V_i \cong \bigoplus_{i=1}^p L_i^{m_i}$ . Set  $\dim L_i = l_i$ . Then  $X \cong \bigoplus_{i=1}^p M_{l_i}(k)$  is a split, semisimple  $k$ -algebra.

Now let  $Y = \text{End}_X(V) \cong \bigoplus_{i=1}^p \text{End}_X(V_i)$  be the centralizer of  $X$  in  $\text{End}_k(V)$ . For  $1 \leq i \leq p$  set  $Y_i = \text{End}_X(V_i)$ ,  $M'_i = \text{Hom}_X(L_i, V_i)$ , and  $M_i = \text{Hom}_X(L_i, V)$ . Then  $Y_i \cong M_{m_i}(k)$  is a simple  $k$ -algebra,  $Y \cong \bigoplus_{i=1}^p M_{m_i}(k)$  is a semisimple  $k$ -algebra with the property that every irreducible  $Y$ -module is absolutely irreducible,  $M'_i$  is an irreducible  $Y_i$ -module,  $M_i$  is an irreducible  $Y$ -module on which the factor  $Y_i$  acts non-trivially, and  $M'_i \cong M_i$  as  $Y$ -modules, where  $Y$  acts on  $M'_i$  via the projection to  $Y_i$ . It is straightforward to check that the natural evaluation map  $\varphi_i: L_i \otimes_k M'_i \rightarrow V_i$  with  $\varphi_i(l \otimes f) = f(l)$  is an isomorphism of  $(X, Y_i^{\text{op}})$ -bimodules, where  $X \times Y_i^{\text{op}}$  acts on  $L_i \otimes_k M'_i$  with  $(x, y) \cdot l \otimes f = x \cdot l \otimes y \circ f$ , and on  $V_i$  by  $(x, y) \cdot v = x(y(v))$ , for all  $x \in X$ ,  $y \in Y_i$ ,  $l \in L_i$ ,  $f \in M'_i$ , and  $v \in V_i$ . This proves the first statement in the theorem.

Using the isomorphisms  $\varphi_i$ , it is straightforward to show that the natural evaluation map  $\bigoplus_{i=1}^r L_i \otimes_k M_i \rightarrow V$  is an isomorphism of  $(X, Y^{\text{op}})$ -bimodules.

It follows that as a  $Y$ -module,  $V$  is isomorphic to  $\bigoplus_{i=1}^p M_i^{l_i}$ . Because each  $M_i$  is absolutely irreducible we see that  $\text{End}_Y(V) \cong \bigoplus_{i=1}^p M_{l_i}(k)$ . Finally, because  $X \subseteq \text{End}_Y(V)$ , it follows that  $X = \text{End}_Y(V)$ . Thus  $(X, V, Y^{\text{op}})$  satisfies Schur-Weyl duality.  $\square$

Now suppose  $(A, V, B)$  satisfies Schur-Weyl duality,  $V$  is a completely reducible  $A$ -module whose irreducible constituents are absolutely irreducible, and  $e$  is an idempotent in  $B$  such that  $Ve \neq 0$ . Set  $X = \Phi(A)$  and  $Y = \Psi(B^{\text{op}})$ . Then  $X$  and  $Y$  are subalgebras of  $\text{End}_k(V)$ ,  $V$  is a completely reducible  $X$ -module whose irreducible constituents are absolutely irreducible,  $Y = \text{End}_X(V)$ , and  $X = \text{End}_Y(V)$ .

**Theorem 3.3.** *With the assumptions and notation above, the following statements hold.*

- (1) The subalgebra  $\Psi_e(eBe)$  of  $\text{End}_k(Ve)$  is a split, semisimple  $k$ -algebra and  $\{M_i e \mid M_i e \neq 0\}$  is a complete set of non-isomorphic, irreducible right  $\Psi_e(eBe)$ -modules.
- (2)  $Ve$  is a completely reducible right  $eBe$ -module and  $\{M_i e \mid M_i e \neq 0\}$  is a complete set of non-isomorphic, irreducible right  $eBe$ -modules that occur as constituents of  $Ve$ .
- (3) The triple  $(A, Ve, eBe)$  satisfies Schur-Weyl duality.

*Proof.* Set  $\pi = \Psi(e)$ , so  $\pi$  is a non-zero idempotent in  $Y$ . By Theorem 3.2 and the general theory of split, semisimple algebras,  $\pi Y \pi$  is a split, semisimple  $k$ -algebra and  $\{\pi M_i \mid \pi M_i \neq 0\}$  is a complete set of non-isomorphic, irreducible  $\pi Y \pi$ -modules. Now  $\pi(V) = Ve$  and the image of the homomorphism  $\Psi_e: (eBe)^{\text{op}} \rightarrow \text{End}_k(Ve)$  coincides with the image of the natural homomorphism  $\Psi_\pi: \pi Y \pi \rightarrow \text{End}_k(\pi(V))$ . Moreover, by definition  $\pi M_i = M_i e$  for  $1 \leq i \leq p$ . Therefore,  $\Psi_e(eBe)$  is a split, semisimple  $k$ -algebra and  $\{M_i e \mid M_i e \neq 0\}$  is a complete set of non-isomorphic, irreducible right  $\Psi_e(eBe)$ -modules.

The algebra  $A$  acts on  $\bigoplus_{i=1}^p L_i \otimes_k M_i$  through its left action on each  $L_i$ , and the algebra  $B$  acts on  $\bigoplus_{i=1}^p L_i \otimes_k M_i$  through its right action on each  $M_i$ . Therefore, the  $(A, B)$ -bimodule isomorphism  $V \cong \bigoplus_{i=1}^p L_i \otimes_k M_i$  induces an  $(A, eBe)$ -bimodule isomorphism

$$(9) \quad Ve \cong \bigoplus_{M_i e \neq 0} L_i \otimes_k M_i e.$$

If  $M_i e \neq 0$ , then  $M_i e$  is an absolutely irreducible  $\Psi_e(eBe)$ -module and hence an absolutely irreducible  $eBe$ -module. The bimodule isomorphism (9) induces an isomorphism of right  $eBe$ -modules

$$Ve \cong \bigoplus_{M_i e \neq 0} (M_i e)^{\dim L_i},$$

which proves the second statement in the theorem.

Finally, set  $X_1 = \Phi_e(A)$ . Then  $Ve$  is a completely reducible  $X_1$ -module, and so by Theorem 3.2, if  $Y_1 = \text{End}_{X_1}(Ve)$ , then  $X_1 = \text{End}_{Y_1}(Ve)$ . Clearly  $\text{End}_{X_1}(Ve) = \text{End}_A(Ve)$  and by Lemma 3.1,  $\text{End}_A(Ve) = \Psi_e(eBe)$ . Moreover,  $\Phi_e(A) = X_1 = \text{End}_{Y_1}(Ve) = \text{End}_{eBe}(Ve)$ , and hence  $(A, Ve, eBe)$  satisfies Schur-Weyl duality, as claimed.  $\square$

We can now complete the proof of Theorem 2.2.

*Proof of Theorem 2.2.* Suppose that  $k$  is a field such that the cardinality of  $k$  is larger than  $r$  and the characteristic of  $k$  does not divide  $r$ . Then by classical Schur-Weyl duality, the triple  $(k \text{GL}_n(k), T^{n,r}, k\Sigma_r)$  satisfies Schur-Weyl duality. Moreover,  $L^{n,r} = T^{n,r}e$ . Thus, it follows from Lemma 3.1 that  $\Psi_e(ek\Sigma_r e) = \text{End}_{\text{GL}_n(k)}(L^{n,r})$ .

If in addition  $T^{n,r}$  is a direct sum of absolutely irreducible  $k \text{GL}_n(k)$ -modules, then it follows from Theorem 3.3 that  $(k \text{GL}_n(k), L^{n,r}, ek\Sigma_r e)$  satisfies Schur-Weyl duality, and so in particular,  $\Phi_e(k \text{GL}_n(k)) = \text{End}_{ek\Sigma_r e}(L^{n,r})$ . This completes the proof of the theorem.  $\square$

**Idempotents.** We now return to the general situation considered at the beginning of this section where the triple  $(A, V, B)$  satisfies Schur-Weyl duality and  $e \in B$  is an idempotent with  $Ve \neq 0$ . We give various conditions that are equivalent to the assertion that the triple  $(A, Ve, eBe)$  satisfies Schur-Weyl duality.

To start, consider the  $k$ -algebra homomorphisms

$$\Phi': A \rightarrow \text{End}_B(V) \quad \text{and} \quad \Phi'_e: A \rightarrow \text{End}_{eBe}(Ve)$$

induced by  $\Phi$  and  $\Phi_e$ , respectively. Because  $(A, V, B)$  satisfies Schur-Weyl duality,  $\Phi'$  is surjective, and by Lemma 3.1,  $(A, Ve, eBe)$  satisfies Schur-Weyl duality if and only if  $\Phi'_e$  is surjective.

Suppose  $\varphi$  is in  $\text{End}_B(V)$ . Then  $\varphi(Ve) = \varphi(V)e \subseteq Ve$ , and the restriction of  $\varphi$  to  $Ve$  induces an  $eBe$ -linear homomorphism  $\bar{\varphi}: Ve \rightarrow Ve$ . Define

$$\Theta_e: \text{End}_B(V) \rightarrow \text{End}_{eBe}(Ve) \quad \text{by} \quad \Theta_e(\varphi) = \bar{\varphi}.$$

It follows immediately from the definitions that  $\Theta_e\Phi' = \Phi'_e$ . Because  $\Phi'$  is surjective, it then follows that  $\Phi'_e$  is surjective if and only if  $\Theta_e$  is surjective. Clearly  $\Theta_e$  is surjective if and only if every  $eBe$ -linear endomorphism of  $Ve$  extends to a  $B$ -linear endomorphism of  $V$ . This proves the next lemma.

**Lemma 3.4.** *Suppose that  $(A, V, B)$  satisfies Schur-Weyl duality and  $e$  is an idempotent in  $B$  such that  $Ve \neq 0$ . Then the following are equivalent.*

- (1) *The triple  $(A, Ve, eBe)$  satisfies Schur-Weyl duality.*
- (2) *Every  $eBe$ -linear endomorphism of  $Ve$  extends to a  $B$ -linear endomorphism of  $V$ .*

Note that the second condition in the lemma depends only on  $B$  and the right  $B$ -module structure on  $V$ , and not on the algebra  $A$ . This observation can be used to replace the idempotent  $e$  by any suitably equivalent idempotent  $f$ , as we now explain. Suppose  $f$  is an idempotent in  $B$  such that

$$ef = f \quad \text{and} \quad fe = e.$$

Then the maps  $\rho_f: Ve \rightarrow Vf$  and  $\rho_e: Vf \rightarrow Ve$  given by  $\rho_f(x) = xf$  and  $\rho_e(x) = xe$  are mutual inverses. It is straightforward to check that

$$\Xi: \text{End}_{eBe}(Ve) \rightarrow \text{End}_{fBf}(Vf) \quad \text{by} \quad \Xi(\varphi) = \rho_f\varphi\rho_e$$

is an algebra isomorphism, with inverse  $\Xi^{-1}(\psi) = \rho_e\psi\rho_f$  for  $\psi$  in  $\text{End}_{fBf}(Vf)$ . It is also straightforward to check that  $\Xi\Theta_e = \Theta_f$ . This proves the next lemma.

**Lemma 3.5.** *With the notation as above,  $\Theta_e$  is surjective if and only if  $\Theta_f$  is surjective.*

The next theorem follows from Lemma 3.4 and Lemma 3.5.

**Theorem 3.6.** *Suppose that  $(A, V, B)$  satisfies Schur-Weyl duality, and  $e$  and  $f$  are idempotents in  $B$  such that  $Ve \neq 0$ ,  $ef = f$ , and  $fe = e$ . Then the following are equivalent.*

- (1)  *$(A, Ve, eBe)$  satisfies Schur-Weyl duality.*
- (2) *Every  $eBe$ -linear endomorphism of  $Ve$  extends to a  $B$ -linear endomorphism of  $V$ .*
- (3)  *$(A, Vf, fBf)$  satisfies Schur-Weyl duality.*
- (4) *Every  $fBf$ -linear endomorphism of  $Vf$  extends to a  $B$ -linear endomorphism of  $V$ .*

#### 4. COMPLEMENTS

In this section we use the results in the previous section first to investigate the commuting algebra of the  $\text{GL}_n(k)$ -action on  $L^{n,r}$  when everything is semisimple, and second to characterize when the triple  $(k\text{GL}_n(k), L^{n,r}, e k \Sigma_r e)$  satisfies Schur-Weyl duality, in terms of certain permutation representations of  $\Sigma_r$ . Throughout this section we assume that  $|k| > r$ , that the characteristic of  $k$  does not divide  $r$ , and that  $k$  contains a primitive  $r^{\text{th}}$  root of unity  $\zeta$ .

If  $e = \sum_{\sigma \in \Sigma_r} a_\sigma \sigma$  is any idempotent in  $k\Sigma_r$ , then a result of Littlewood (see [3, Exercise 9.16]) shows that the character of the right  $k\Sigma_r$ -module  $e k \Sigma_r$ , evaluated

at a permutation  $\tau$ , is the sum  $\sum_{\sigma \in \mathcal{C}} a_\sigma$ , where  $\mathcal{C}$  is the conjugacy class of  $\tau$ . When  $e$  is a Lie idempotent, Garsia [5, Theorem 5.2] gives a formula for the sums  $\sum_{\sigma \in \mathcal{C}} a_\sigma$ . This formula does not depend on the choice of Lie idempotent. Therefore, up to isomorphism, the right ideal  $ek\Sigma_r$  does not depend on the choice of  $e$ , and so the following lemma follows from Lemma 3.4.

**Lemma 4.1.** *Suppose  $e$  and  $e'$  are Lie idempotents. Then  $(k\text{GL}_n(k), L^{n,r}, ek\Sigma_r e)$  satisfies Schur-Weyl duality if and only if  $(k\text{GL}_n(k), L^{n,r}, e'k\Sigma_r e')$  does.*

By the lemma, we may choose  $e$  to be any convenient Lie idempotent. In this section we use a Lie idempotent found by Klyachko.

Given a permutation  $\sigma$ , an integer  $i$  is a descent of  $\sigma$  if  $\sigma(i) > \sigma(i + 1)$ . Let  $\mathcal{D}(\sigma)$  denote the set of descents of  $\sigma$ . By definition, the major index of  $\sigma$  is

$$\text{maj}(\sigma) = \sum_{i \in \mathcal{D}(\sigma)} i.$$

The Klyachko idempotent is the element

$$\kappa = \frac{1}{r} \sum_{\sigma \in \Sigma_r} \zeta^{\text{maj}(\sigma)} \sigma$$

in  $k\Sigma_r$ . Klyachko [9] has shown that  $\kappa$  is a Lie idempotent. Furthermore, if  $\gamma$  is any fixed  $r$ -cycle in  $\Sigma_r$  and we define

$$(10) \quad f = \frac{1}{r} \sum_{i=1}^r \zeta^{-i} \gamma^i$$

as in §2, then  $\kappa f = f$  and  $f \kappa = \kappa$  (see [11, §8.4]). Set

$$H = f k \Sigma_r f.$$

Then for any Lie idempotent  $e$  we have

$$(11) \quad ek\Sigma_r e \cong \kappa k \Sigma_r \kappa \cong H \cong \text{End}_{k\Sigma_r}(\text{Ind}_{\Gamma}^{\Sigma_r} \zeta)$$

where  $\Gamma = \langle \gamma \rangle$  is the subgroup generated by  $\gamma$ .

**The semisimple case.** Now assume that the characteristic of  $k$  is greater than  $r$ , so  $k\Sigma_r$  and  $H$  are split, semisimple  $k$ -algebras, and consider the commuting algebra  $\text{End}_{\text{GL}_n(k)}(L^{n,r})$ . By Corollary 2.3, the triple  $(k\text{GL}_n(k), L^{n,r}, \kappa k \Sigma_r \kappa)$  satisfies Schur-Weyl duality, and so by Theorem 3.6,  $(k\text{GL}_n(k), T^{n,r} f, H)$  does as well. Note that right multiplication by  $\kappa$  defines a  $\text{GL}_n(k)$ -equivariant isomorphism between  $T^{n,r} f$  and  $L^{n,r}$  that intertwines the right actions of  $H$  and  $\kappa k \Sigma_r \kappa$ , and that by (11),

$$\text{End}_{\text{GL}_n(k)}(L^{n,r}) \cong H \cong \text{End}_{k\Sigma_r}(\text{Ind}_{\Gamma}^{\Sigma_r} \zeta).$$

In the following, we consider the algebra  $H$  instead of  $\text{End}_{\text{GL}_n(k)}(L^{n,r})$ .

Recall that a partition is a sequence  $\lambda = (\lambda_1, \lambda_2, \dots)$  of non-negative integers such that (1)  $\lambda_1 \geq \lambda_2 \geq \dots$  and (2)  $\lambda_i \neq 0$  for only finitely many  $i$ . If  $\lambda_i > 0$ , then  $\lambda_i$  is a part of  $\lambda$ . Define  $\ell(\lambda)$  to be the number of parts of  $\lambda$ . If  $\sum_{i>0} \lambda_i = r$ , then say  $\lambda$  is a partition of  $r$  and write  $\lambda \vdash r$ . When  $\ell(\lambda) = a$  we generally abuse notation and write  $\lambda = (\lambda_1, \dots, \lambda_a)$  instead of  $\lambda = (\lambda_1, \dots, \lambda_a, 0, \dots)$ .

For a partition  $\lambda$  of  $r$  with at most  $n$  parts let  $V^\lambda$  be the irreducible representation of  $\text{GL}_n(k)$  with highest weight  $\lambda$  and let  $S^\lambda$  be the Specht module indexed by  $\lambda$ . For example, if  $\lambda = (r)$ , then  $V^\lambda$  is the natural module for  $\text{GL}_n(k)$  and  $S^\lambda$  is the

trivial representation of  $\Sigma_r$ , and if all the parts of  $\lambda$  are equal to 1, then  $V^\lambda$  is the determinant representation of  $GL_n(k)$  and  $S^\lambda$  is the sign representation of  $\Sigma_r$ . Semisimplicity implies (see e.g. [6, Proposition 3.3.2]) that there is an isomorphism of  $(k GL_n(k), k\Sigma_r)$ -bimodules

$$(12) \quad T^{n,r} \cong \bigoplus_{\substack{\lambda \vdash r \\ \ell(\lambda) \leq n}} V^\lambda \otimes S^\lambda.$$

Applying the map  $\rho_f$  to (12) gives

$$T^{n,r} f \cong \bigoplus_{\substack{\lambda \vdash r \\ \ell(\lambda) \leq n}} V^\lambda \otimes S^\lambda f$$

where  $\{S^\lambda f \mid S^\lambda f \neq 0\}$  is a set of non-isomorphic, irreducible, right  $H$ -modules. For a partition  $\lambda$  of  $r$  with  $S^\lambda f \neq 0$ , let  $H_\lambda$  denote the minimal two-sided ideal of  $H$  with the property that  $S^\lambda f H_\lambda \neq 0$ .

The decomposition of  $k\Sigma_r f$  into irreducible constituents is given in [11, Chapter 8] and [5]. This decomposition determines the algebra structure of  $H$  as follows.

Let  $SYT$  denote the set of standard Young tableaux with  $r$  boxes. For a partition  $\lambda$  of  $r$  let  $SYT^\lambda$  be the set of standard Young tableaux with shape  $\lambda$ . Suppose  $\mathbf{t}$  is a standard Young tableau. An integer  $i$  is a descent of  $\mathbf{t}$  if  $i + 1$  occurs in a lower row of  $\mathbf{t}$  than  $i$ . Let  $\mathcal{D}(\mathbf{t})$  denote the set of descents of  $\mathbf{t}$ . The major index of  $\mathbf{t}$  is

$$\text{maj}(\mathbf{t}) = \sum_{i \in \mathcal{D}(\mathbf{t})} i.$$

Define

$$SYT_{\equiv 1}^\lambda = \{ \mathbf{t} \in SYT^\lambda \mid \text{maj}(\mathbf{t}) \equiv 1 \pmod r \}.$$

Then the multiplicity of  $S^\lambda$  in  $k\Sigma_r f$  is  $|SYT_{\equiv 1}^\lambda|$ . Thus,

- (H1) the simple  $H$ -modules are parametrized by the set of partitions  $\lambda$  of  $r$  for which  $|SYT_{\equiv 1}^\lambda| \neq \emptyset$ ,
- (H2) the dimension of  $S^\lambda f$  is  $|SYT_{\equiv 1}^\lambda|$ ,
- (H3) the dimension of  $H_\lambda$  is  $|SYT_{\equiv 1}^\lambda|^2$ , and
- (H4)  $\dim H = \sum_{\lambda \vdash r} |SYT_{\equiv 1}^\lambda|^2$ .

Obviously  $|SYT_{\equiv 1}^\lambda|$  depends on only the integer  $r$ , and not the field  $k$ , so one might hope that statements (H1)–(H4) hold whenever the characteristic of  $k$  does not divide  $r$ .

The partitions  $\lambda$  such that  $S^\lambda f \neq 0$  have been determined by Klyachko, as follows.

**Theorem 4.2.** *Suppose that  $\lambda$  is a partition of  $r$ . Then there is a standard Young tableau with shape  $\lambda$  and major index congruent to 1 mod  $r$  if and only if  $\lambda$  is not equal to  $(1, 1, \dots, 1)$ ,  $(r)$ ,  $(2, 2)$  (in case  $r = 4$ ), or  $(2, 2, 2)$  (in case  $r = 6$ ).*

We can use the Robinson-Schensted correspondence between  $\Sigma_r$  and the set of pairs of standard Young tableaux with the same shape to obtain formulas for the dimensions of  $H_\lambda$  and  $H$  in terms of permutations instead of tableaux. Let  $P: \Sigma_r \rightarrow SYT$  be the map given by the Schensted (row) insertion algorithm. Then the Robinson-Schensted correspondence is given by the assignment  $\sigma \mapsto (P(\sigma), P(\sigma^{-1}))$ .

**Proposition 4.3.** *Suppose that the characteristic of  $k$  is greater than  $r$  and that  $k$  contains a primitive  $r^{\text{th}}$  root of unity. Then*

$$\dim H_\lambda = |\{ \sigma \in \Sigma_r \mid P(\sigma) \in \text{SYT}^\lambda, \text{maj}(\sigma) \equiv 1 \pmod r, \text{maj}(\sigma^{-1}) \equiv 1 \pmod r \}|$$

and

$$\dim H = |\{ \sigma \in \Sigma_r \mid \text{maj}(\sigma) \equiv 1 \pmod r, \text{maj}(\sigma^{-1}) \equiv 1 \pmod r \}|.$$

Thus, the dimension of  $H$  is the number of permutations  $\sigma$  such that  $\sigma$  and  $\sigma^{-1}$  both have major index congruent to 1 modulo  $r$ .

*Proof.* Clearly, the preimage of  $\text{SYT}_{\equiv 1}^\lambda \times \text{SYT}_{\equiv 1}^\lambda$  in  $\Sigma_r$  is

$$\{ \sigma \in \Sigma_r \mid P(\sigma) \in \text{SYT}^\lambda, \text{maj}(P(\sigma)) \equiv 1 \pmod r, \text{maj}(P(\sigma^{-1})) \equiv 1 \pmod r \}.$$

A standard property of the row insertion algorithm is that  $\mathcal{D}(\sigma) = \mathcal{D}(P(\sigma^{-1}))$  and so  $\text{maj}(\sigma) = \text{maj}(P(\sigma^{-1}))$ . Thus, the preimage of  $\text{SYT}_{\equiv 1}^\lambda \times \text{SYT}_{\equiv 1}^\lambda$  is the set of permutations  $\sigma$  such that  $P(\sigma)$  has shape  $\lambda$ ,  $\text{maj}(\sigma) \equiv 1 \pmod r$ , and  $\text{maj}(\sigma^{-1}) \equiv 1 \pmod r$ . Thus the proposition follows (H3) and (H4).  $\square$

**The non-semisimple case.** Now we return to the general situation, assuming only that the characteristic of  $k$  does not divide  $r$ , so  $T^{n,r}$  is not necessarily completely reducible. Our goal is to characterize when the triple  $(k \text{GL}_n(k), L^{n,r}, ek\Sigma_r e)$  satisfies Schur-Weyl duality in terms of certain permutation representations of  $\Sigma_r$ . We continue to use the idempotent  $f \in k\Gamma$  defined in (10).

It was shown in Theorem 3.6 that the triple  $(k \text{GL}_n(k), L^{n,r}, \kappa k\Sigma_r \kappa)$  satisfies Schur-Weyl duality if and only if the restriction homomorphism

$$\Theta_f : \text{End}_{k\Sigma_r}(T^{n,r}) \rightarrow \text{End}_H(T^{n,r}f)$$

is surjective. To streamline the notation, set  $\Theta = \Theta_f$ . To find conditions under which  $\Theta$  is surjective we need to consider some standard notation and constructions.

Denote the set  $\{1, 2, \dots, n\}$  simply by  $[n]$ . The group  $\Sigma_r$  acts on the set  $[n]^r$  on the right by  $(a\sigma)_j = a_{\sigma(j)}$  for  $a$  in  $[n]^r$  and  $\sigma$  in  $\Sigma_r$ . Let  $\{v_1, \dots, v_r\}$  be the standard basis of  $V$ . For  $a = (a_1, \dots, a_r)$  in  $[n]^r$  define

$$v_a = v_{a_1} \otimes \dots \otimes v_{a_r}$$

in  $T^{n,r}$ . Then

$$\mathcal{B} = \{ v_a \mid a \in [n]^r \}$$

is a  $k$ -basis of  $T^{n,r}$ . Clearly  $v_a\sigma = v_{a\sigma}$  for  $a$  in  $[n]^r$  and  $\sigma$  in  $\Sigma_r$ , so  $\mathcal{B}$  is a  $\Sigma_r$ -stable subset of  $T^{n,r}$ .

Technically, an element  $a$  in  $[n]^r$  is a function  $a : [r] \rightarrow [n]$ . In particular, if  $\sigma$  is in  $\Sigma_r$ , then  $a \circ \sigma : [r] \rightarrow [n]$ . Thus, the right action of  $\Sigma_r$  on  $[n]^r$  is simply the natural right action of  $\Sigma_r$  of the set of functions  $[r] \rightarrow [n]$  given by  $(a, \sigma) \mapsto a \circ \sigma$ . Similarly, the group  $\Sigma_n$  acts naturally on  $[n]^r$  on the left. Namely, if  $a = (a_1, a_2, \dots, a_r)$  is in  $[n]^r$  and  $\tau$  is in  $\Sigma_n$ , then

$$\tau((a_1, a_2, \dots, a_r)) = (\tau(a_1), \tau(a_2), \dots, \tau(a_r)).$$

Clearly the left  $\Sigma_n$ -action and the right  $\Sigma_r$ -action commute.

For  $a$  in  $[n]^r$ , define the content of  $a$  to be the  $n$ -tuple

$$\text{ct}(a) = (|a^{-1}(1)|, |a^{-1}(2)|, \dots, |a^{-1}(n)|).$$

Then  $\text{ct}(a) = (m_1, m_2, \dots, m_n)$ , where  $m_j$  is the multiplicity of  $j$  in  $a$ . It is easy to see that  $\text{ct}$  is an orbit map for the right action of  $\Sigma_r$  on  $[n]^r$ . In other words,

$\text{ct}(a) = \text{ct}(b)$  if and only if there is a  $\sigma$  in  $\Sigma_r$  such that  $b = a\sigma$ . Define  $\Lambda(n, r)$  to be the image of  $\text{ct}$ . Then

$$\Lambda(n, r) = \{ (m_1, m_2, \dots, m_n) \in \mathbb{N}^n \mid m_1 + m_2 + \dots + m_n = r \},$$

and so  $\Lambda(n, r)$  is the set of compositions of  $r$  into at most  $n$  parts, with parts of length zero allowed. It is well-known and straightforward to check that  $\Lambda(n, r)$  may be identified with the set of weights of the group of diagonal matrices in  $\text{GL}_n(k)$  acting on the space  $T^{n,r}$ . Elements of  $\Lambda(n, r)$  are thus referred to as “weights”. In the following, we fix  $n$  and  $r$  and set  $\Lambda = \Lambda(n, r)$ .

For a weight  $\alpha$  in  $\Lambda$ , define

$$[n]_\alpha^r = \text{ct}^{-1}(\alpha) = \{ a \in [n]^r \mid \text{ct}(a) = \alpha \},$$

and define the weight space  $T_\alpha^{n,r}$  of  $T^{n,r}$  by

$$T_\alpha^{n,r} = \text{span}\{ v_a \in \mathcal{B} \mid a \in [n]_\alpha^r \} = \text{span}\{ v_a \in \mathcal{B} \mid \text{ct}(a) = \alpha \}.$$

Because  $\text{ct}(a\sigma) = \text{ct}(a)$  for all  $a \in [n]^r$  and  $\sigma \in \Sigma_r$ , it follows that for each weight  $\alpha$ ,  $T_\alpha^{n,r}$  is a right  $\Sigma_r$ -submodule of  $T^{n,r}$  and that

$$(13) \quad T^{n,r} \cong \bigoplus_{\alpha \in \Lambda} T_\alpha^{n,r}$$

as right  $\Sigma_r$ -modules.

Now consider the subspace  $T^{n,r}f$  of  $T^{n,r}$ . Because the left action of  $\text{GL}_n(k)$  on  $T^{n,r}$  commutes with the right action of  $\Sigma_r$ , it follows that  $T^{n,r}f$  is a  $\text{GL}_n(k)$ -stable subspace of  $T^{n,r}$ , and hence that for  $\alpha \in \Lambda$ , the  $\alpha$  weight space of  $T^{n,r}f$  is equal to  $T_\alpha^{n,r}f$ . Then  $T_\alpha^{n,r}f$  is an  $H$ -submodule of  $T^{n,r}f$  and

$$(14) \quad T^{n,r}f \cong \bigoplus_{\alpha \in \Lambda} T_\alpha^{n,r}f$$

as right  $H$ -modules.

Recall that  $\Theta: \text{End}_{\Sigma_r}(T^{n,r}) \rightarrow \text{End}_H(T^{n,r}f)$  is given by  $\Theta(\varphi) = \bar{\varphi}$ , where  $\bar{\varphi}: T^{n,r} \rightarrow T^{n,r}f$  is the restriction of  $\varphi$ . The decompositions (13) and (14) induce isomorphisms of  $k$ -vector spaces

$$(15) \quad \text{End}_{\Sigma_r}(T^{n,r}) \cong \bigoplus_{\alpha, \beta \in \Lambda} \text{Hom}_{\Sigma_r}(T_\alpha^{n,r}, T_\beta^{n,r})$$

and

$$(16) \quad \text{End}_H(T^{n,r}f) \cong \bigoplus_{\alpha, \beta \in \Lambda} \text{Hom}_H(T_\alpha^{n,r}f, T_\beta^{n,r}f).$$

Suppose  $\alpha$  and  $\beta$  are in  $\Lambda$ . If  $\psi$  is in  $\text{Hom}_{\Sigma_r}(T_\alpha^{n,r}, T_\beta^{n,r})$ , then  $\psi(T_\alpha^{n,r}f) \subseteq T_\beta^{n,r}f$  and the restriction of  $\psi$  to  $T_\alpha^{n,r}$  is in  $\text{Hom}_H(T_\alpha^{n,r}f, T_\beta^{n,r}f)$ . Define

$$\Theta_\beta^\alpha: \text{Hom}_{\Sigma_r}(T_\alpha^{n,r}, T_\beta^{n,r}) \rightarrow \text{Hom}_H(T_\alpha^{n,r}f, T_\beta^{n,r}f) \quad \text{by} \quad \Theta_\beta^\alpha(\psi) = \bar{\psi},$$

where  $\bar{\psi}: T_\alpha^{n,r}f \rightarrow T_\beta^{n,r}f$  is the restriction of  $\psi$ . The maps  $\Theta$  and  $\Theta_\beta^\alpha$  are compatible with the decompositions (15) and (16) in the sense that the diagram

$$\begin{array}{ccc} \text{End}_{\Sigma_r}(T^{n,r}) & \xrightarrow{\cong} & \bigoplus_{\alpha, \beta \in \Lambda} \text{Hom}_{\Sigma_r}(T_\alpha^{n,r}, T_\beta^{n,r}) \\ \downarrow \Theta & & \downarrow \bigoplus \Theta_\beta^\alpha \\ \text{End}_H(T^{n,r}f) & \xrightarrow{\cong} & \bigoplus_{\alpha, \beta \in \Lambda} \text{Hom}_H(T_\alpha^{n,r}f, T_\beta^{n,r}f) \end{array}$$

commutes. Therefore,  $\Theta$  is surjective if and only if  $\Theta_\beta^\alpha$  is surjective for all  $\alpha$  and  $\beta$  in  $\Lambda$ . The next proposition thus follows from Lemma 4.1, Lemma 3.4, and Theorem 3.6.

**Proposition 4.4.** *Suppose  $e$  is a Lie idempotent. Then  $(k \text{GL}_n(k), L^{n,r}, ek\Sigma_r e)$  satisfies Schur-Weyl duality if and only if the maps*

$$\Theta_\beta^\alpha : \text{Hom}_{\Sigma_r}(T_\alpha^{n,r}, T_\beta^{n,r}) \rightarrow \text{Hom}_H(T_\alpha^{n,r} f, T_\beta^{n,r} f)$$

are surjections for all  $\alpha$  and  $\beta$  in  $\Lambda(n, r)$ .

Next, suppose that  $\alpha$  is a weight in  $\Lambda$ . Up to the left action of  $\Sigma_n$ , we may assume that  $\alpha = (m_1, m_2, \dots, m_p, 0, \dots, 0)$  where  $(m_1, m_2, \dots, m_p)$  is a composition of  $r$  with no parts that equal zero. Let

$$\Sigma_\alpha \cong \Sigma_{m_1} \times \Sigma_{m_2} \times \dots \times \Sigma_{m_p}$$

be the corresponding Young subgroup of  $\Sigma_r$ . The transitive action of  $\Sigma_r$  on  $[n]_\alpha^r$  induces an isomorphism of right  $k\Sigma_r$ -modules  $T_\alpha^{n,r} \cong k_\alpha \otimes_{k\Sigma_\alpha} k\Sigma_r$ , where  $k_\alpha$  is the trivial right  $k\Sigma_\alpha$ -module, as follows.

Given  $a = (a_1, a_2, \dots, a_r)$  in  $[n]_\alpha^r$ , replace the occurrences of 1 in  $a$  from left to right with  $1, 2, \dots, m_1$ , then replace the occurrences of 2 in  $a$  from left to right by  $m_1 + 1, m_1 + 2, \dots, m_1 + m_2$ , and so on. Define  $\sigma_a$  to be the permutation given in one line notation by the resulting  $r$ -tuple. For example if  $r = 8$ ,  $\alpha = (4, 2, 2, 0, \dots, 0)$ , and  $b = (2, 1, 1, 3, 2, 1, 1, 3)$ , then in one line notation  $\sigma_b = (5, 1, 2, 7, 6, 3, 4, 8)$ ; that is,  $\sigma_b(1) = 5$ ,  $\sigma_b(2) = 1$ , and so on. It is easy to see that the assignment  $a \mapsto \sigma_a$  defines a bijection between  $[n]_\alpha^r$  and the set  $\Sigma_\alpha$  of minimal length right coset representatives of  $\Sigma_\alpha$  in  $\Sigma_r$ , and that the assignment  $a \mapsto \Sigma_\alpha \sigma_a$  defines a  $\Sigma_r$ -equivariant bijection between  $[n]_\alpha^r$  and the set of right cosets  $\Sigma_\alpha \backslash \Sigma_r$ . Thus, the assignment  $v_a \mapsto 1 \otimes \sigma_a$  defines an isomorphism of right  $k\Sigma_r$ -modules

$$h_\alpha : T_\alpha^{n,r} \xrightarrow{\cong} k_\alpha \otimes_{k\Sigma_\alpha} k\Sigma_r.$$

To simplify the notation, set  $M^\alpha = k_\alpha \otimes_{k\Sigma_\alpha} k\Sigma_r$ .

Now suppose that  $\alpha, \beta \in \Lambda$ . The assignment  $\varphi \mapsto h_\beta \varphi h_\alpha^{-1}$  defines isomorphisms of  $k$ -vector spaces

$$\text{Hom}_{\Sigma_r}(T_\alpha^{n,r}, T_\beta^{n,r}) \xrightarrow{\cong} \text{Hom}_{\Sigma_r}(M^\alpha, M^\beta)$$

and

$$\text{Hom}_H(T_\alpha^{n,r} f, T_\beta^{n,r} f) \xrightarrow{\cong} \text{Hom}_H(M^\alpha f, M^\beta f),$$

such that the diagram

$$\begin{CD} \text{Hom}_{\Sigma_r}(T_\alpha^{n,r}, T_\beta^{n,r}) @>\Theta_\beta^\alpha>> \text{Hom}_H(T_\alpha^{n,r} f, T_\beta^{n,r} f) \\ @V\cong VV @VV\cong V \\ \text{Hom}_{\Sigma_r}(M^\alpha, M^\beta) @>\theta_\beta^\alpha>> \text{Hom}_H(M^\alpha f, M^\beta f) \end{CD}$$

commutes, where the map  $\theta_\beta^\alpha$  on the bottom is again given by restriction. Obviously  $\theta_\beta^\alpha$  is surjective if and only if  $\Theta_\beta^\alpha$  is. Combining this observation with Proposition 4.4 we obtain the following corollary.

**Corollary 4.5.** *Suppose  $e$  is a Lie idempotent. Then  $(k\mathrm{GL}_n(k), L^{n,r}, ek\Sigma_r e)$  satisfies Schur-Weyl duality if and only if the restriction maps*

$$\theta_\beta^\alpha: \mathrm{Hom}_{\Sigma_r}(M^\alpha, M^\beta) \rightarrow \mathrm{Hom}_H(M^\alpha f, M^\beta f)$$

*are surjections for all  $\alpha$  and  $\beta$  in  $\Lambda(n, r)$ .*

Thus we arrive at the following problem.

**Problem 4.6.** Find a combinatorially defined basis for  $\mathrm{Hom}_H(M^\alpha f, M^\beta f)$  and hence show that  $\dim \mathrm{Hom}_H(M^\alpha f, M^\beta f)$  does not depend on the field  $k$ .  $\square$

A solution to this problem should show that  $(k\mathrm{GL}_n(k), L^{n,r}, ek\Sigma_r e)$  satisfies Schur-Weyl duality whenever  $k$  is a field of characteristic not dividing  $r$  that contains a primitive  $r^{\mathrm{th}}$  root of unity.

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