A CONTAINMENT RESULT IN $P^n$
AND THE CHUDNOVSKY CONJECTURE

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Abstract. In this paper we prove the containment $I^{(nm)} \subset M^{(n-1)m}I^m$, for a radical ideal $I$ of $s$ general points in $\mathbb{P}^n$, where $s \geq 2^n$. As a corollary we get that the Chudnovsky Conjecture holds for a very general set of at least $2^n$ points in $\mathbb{P}^n$.

1. Introduction

Given any subscheme $Z \subset \mathbb{P}^n$ and its homogenous ideal $I = I_Z$ in $\mathbb{F}[\mathbb{P}^n] = \mathbb{F}[x_0, \ldots, x_n]$, we define $\alpha(I)$ as the minimal degree of a non-zero element in $I$. We will assume that $\text{char} \mathbb{F} = 0$.

In general $\alpha(I)$ is hard to compute and it behaves quite unpredictably. However there is an asymptotic counterpart of the $\alpha$-invariant, which is the Waldschmidt constant ([17]) defined as

$$\hat{\alpha}(I) = \lim_{m \to \infty} \frac{\alpha(I^{(m)})}{m},$$

where $I^{(m)}$ denotes the $m$-th symbolic power of the ideal $I$ (for the definition and basic properties of $I^{(m)}$ see [13]). It turns out that this constant is well defined and satisfies the inequality

$$\alpha(I^{(m)}) \geq m\hat{\alpha}(I)$$

for all $m$. In fact $\hat{\alpha}(I) = \inf_{m \geq 1} \frac{\alpha(I^{(m)})}{m}$.

For an ideal $I$ we have $\alpha(I^r) = r\alpha(I)$, but the behaviour of $\alpha(I^{(m)})$ is much more complicated and less understood. Skoda in 1977 [19] showed that $\alpha(I^{(m)}) \geq m\alpha(I)/n$ for an ideal $I$ of points in $\mathbb{P}^n$ (over complex numbers). Chudnovsky [4], in 1981, improved that bound for $n = 2$ to $\alpha(I^{(m)}) \geq m(\alpha(I) + 1)/2$ and conjectured the following.

Chudnovsky Conjecture. For an ideal $I$ of points in $\mathbb{P}^n$ the following inequality holds:

$$\frac{\alpha(I^{(m)})}{m} \geq \frac{\alpha(I) + n - 1}{n}.$$
In particular

\[ \hat{\alpha}(I) \geq \frac{\alpha(I) + n - 1}{n}. \]

Esnault and Viehweg [11], in 1983 showed that \( \alpha(I^{(m)}) \geq m(\alpha(I) + 1)/n \) for any set of points in \( \mathbb{P}^n \).

In 2001 Ein, Lazarsfeld, Smith in [10] and in 2002 Hochster and Huneke in [14] showed that, for any homogeneous ideal \( I \) in \( \mathbb{F}[\mathbb{P}^n] \), the containment \( I(nm) \subset I^m \) holds, thus recovering the Skoda bound more generally — for all homogeneous ideals (since \( I(nm) \subset I^m, \alpha(I^{(nm)}) \geq \alpha(I^m) = nma(I)/n \); passing with \( m \) to infinity gives \( \hat{\alpha}(I) \geq \alpha(I)/n \)). Harbourne and Huneke in [13, Lemma 3.2], observed that the Chudnovsky Conjecture would follow from a more general containment

\[ I^{(nm)} \subset M^{(n-1)m} I^m, \]

where by \( M = (x_0, \ldots, x_n) \) we denote the irrelevant maximal ideal.

The containment (3) holds (for a given \( m \)) for general points in \( \mathbb{P}^2 \) ([13], Proposition 3.10), for at most \( n + 1 \) general points in \( \mathbb{P}^n \) and also for general points in \( \mathbb{P}^3 \) ([7], [8]). As a corollary, the Chudnovsky Conjecture holds for very general points in \( \mathbb{P}^3 \).

The main result of the present paper is the following theorem.

**Theorem 1.** For a non-negative integer \( m \), and for a radical ideal \( I \) of \( s \) general points in \( \mathbb{P}^n \), where \( s \geq 2^n \), the containment

\[ I(nm) \subset M^{(n-1)m} I^m \]

holds.

As a corollary, the Chudnovsky Conjecture holds for a very general set of at least \( 2^n \) points in \( \mathbb{P}^n \).

We will work on filling the gap between \( n + 1 \) and \( 2^n \) in our future project.

### 2. A Bound for \( \hat{\alpha}(I) \)

In this section we give a bound for the Waldschmidt constant of an ideal of \( s \) very general points in \( \mathbb{P}^n \). The bound in fact easily follows from the much stronger result of Evain [12], who showed that for an ideal \( I \) of general \( kn \) points \( \alpha(I) \) is “expected”. Since the methods of Evain are highly non-trivial and very delicate, we give a short proof of our bound here to make the paper more self-contained.

**Theorem 2.** For a radical ideal \( I \) of \( s \) very general points \( P_1, \ldots, P_s \) in \( \mathbb{P}^n \) we have

\[ \hat{\alpha}(I) \geq \lfloor \sqrt[s]{s} \rfloor. \]

**Proof.** To prove the bound we have to show that the system of divisors of degree \( dm - 1 \) passing through \( P_1, \ldots, P_s \) with multiplicity \( m \) is empty if \( d = \lfloor \sqrt[s]{s} \rfloor \). By \((d, m \times s)\) we denote a system of divisors of degree \( d \), passing through \( s \) (general) points with multiplicity \( m \).

Without loss of generality we may suppose that \( s = kn, k \in \mathbb{N} \), as the emptiness of the system \((km - 1; m \times k^n)\), implies the emptiness of \((km - 1, m \times r), k^n \leq r < (k + 1)^n\).

For \( n = 1 \) the non-existence of the system \((km - 1; m \times k)\) is trivial. Then we proceed by induction. (Note that for \( n = 2 \) the non-existence of the system \((km - 1; m \times k^2)\) was also proved by Nagata in [15].)
For \( n \geq 2 \), suppose there exists a divisor in \( \mathbb{P}^n \) of degree \( km - 1 \), passing through \( P_1, \ldots, P_s \) with multiplicity \( m \). Take \( k \) general hyperplanes in \( \mathbb{P}^n \) and put \( k^{n-1} \) points on each such hyperplane (in general position). Then on the hyperplane we have the system of divisors of degree \( km - 1 \) which have to pass through \( k^{n-1} \) points with multiplicity \( m \). By inductive assumption, this system is empty. Thus, all hyperplanes must be components of the system. Repeating the procedure (of checking that all hyperplanes must be the components of the system) \( m \) times we get that the system has to have degree \( km \), not \( km - 1 \), a contradiction.

3. A combinatorial lemma

Here we prove an auxiliary combinatorial lemma, which we will use in the proof of Theorem 4.

Lemma 3. (1) For any integers \( k \geq 4 \) and \( n \geq 3 \) the following inequality holds:

\[
k^n \leq \binom{kn - n}{n}.
\]

(2) Moreover, if \( n \geq 5 \),

\[
3^n \leq \binom{2n}{n}
\]

holds.

Proof. First we prove (1). Observe that it is enough to prove the inequality for \( k = 4 \) only. Indeed, our inequality may be written as

\[
(nk - n) \cdot \ldots \cdot (nk - 2n + 1) \geq n!k^n.
\]

The left-hand side is greater than or equal to

\[
\left( n - \frac{n}{4} \right) \cdot \ldots \cdot \left( n - \frac{2n - 1}{4} \right) k,
\]

as \( k \geq 4 \). Dividing both sides by \( k^n \) and multiplying by \( 4^n \) we see that this is enough to prove

\[
(4n - n) \cdot \ldots \cdot (4n - 2n + 1) \geq n!4^n,
\]

i.e., our inequality with \( k = 4 \).

Thus, we have to prove

\[
\binom{3n}{n} \geq 4^n.
\]

Since

\[
\frac{\binom{3(n+1)}{n+1}}{\binom{3n}{n}} = \frac{(3n + 3)(3n + 2)(3n + 1)}{(n + 1)(2n + 2)(2n + 1)} > 4
\]

for all \( n \geq 1 \) and

\[
\binom{9}{3} > 4^3,
\]

the claim follows by induction.

As for the second claim of the lemma, we proceed analogously, observing that

\[
\frac{\binom{2(n+1)}{n+1}}{\binom{2n}{n}} = \frac{(2n + 2)(2n + 1)}{(n + 1)(n + 1)} > 3
\]
for all \( n \geq 1 \) and
\[
\binom{10}{5} > 3^5.
\]
\( \square \)

4. A Containment Result

Now we are able to prove our containment theorem.

**Theorem 4.** For a radical ideal \( I \) of \( s \) very general points in \( \mathbb{P}^n \), where \( s \geq 2^n \) and \( n \geq 3 \) the containment
\[ I^{(nm)} \subset M^{(n-1)m} \]
holds, for any integer \( m \geq 1 \).

**Proof.** Dumnicki in [7] showed the containment for any number of very general points in \( \mathbb{P}^3 \), so we assume \( n \geq 4 \).

Let \( \text{reg}(I) \) denote the Castelnuovo-Mumford regularity of \( I \), see e.g. [5], and let \( \sigma(I) \) denote the maximal degree of an element in a minimal set of generators of \( I \). Since in general \( \sigma(I) \leq \text{reg}(I) \), by [13, Proposition 2.3] it is enough to show that
\[
\alpha(I^{(mn)}) \geq (n-1)m + m \text{reg}(I).
\]

By (2) the above inequality follows from
\[
n\hat{\alpha}(I) \geq n - 1 + \text{reg}(I).
\]

From Theorem 2 we know that
\[
\hat{\alpha}(I) \geq \lfloor \sqrt[n]{s} \rfloor.
\]

For the ideal \( I \) as in our theorem we have that \( \text{reg}(I) = r + 1 \), where \( r \) is such an integer that
\[
\binom{r-1+n}{n} < s \leq \binom{r+n}{n},
\]
see [2, p. 1176].

Thus, it is enough to prove that
\[
n\lfloor \sqrt[n]{s} \rfloor \geq n - 1 + r + 1,
\]
for \( r \) satisfying
\[
\binom{r-1+n}{n} < s.
\]

Suppose to the contrary, that \( n\lfloor \sqrt[n]{s} \rfloor < n - 1 + r + 1 \). If we can prove that then
\[
\binom{r+n-1}{n} \geq s,
\]
we get a contradiction, and we are done.

As \( \lfloor \sqrt[n]{s} \rfloor \) is constant (equals \( k-1 \)) for all \( s = (k-1)^n, \ldots, k^n - 1 \), we may assume that the right-hand side of inequality (5) equals \( k^n - 1 \), which is the worst possible case.

Thus, we reduced the problem to proving that for \( r \geq nk - 2n + 1 \) we have
\[
k^n - 1 \leq \binom{r+n-1}{n}.
\]
This will follow if we prove that

\[ k^n \leq \binom{kn - n}{n}. \]

This last inequality is proved in Lemma 3 for \( n = 4 \) and \( k \geq 4 \) and for \( n \geq 5 \) and \( k \geq 3 \). This proves our theorem for all \( n \geq 5 \) and \( s \geq 2^n \) (remember, that we have \( s \geq (k - 1)^n \)) and also when \( n = 4 \) and \( s \geq 3^n \). Moreover, by direct computation inequality (11) holds for \( n = 4 \) with \( 2^4 \leq s \leq 70 \), so it remains only to prove our theorem only for \( n = 4 \) and \( 71 \leq s < 3^4 \). This is done in the lemma below.

**Lemma 5.** The Waldschmidt constant for an ideal of at least 71 very general points in \( \mathbb{P}^4 \) is bounded below by \( 9/4 \).

Before the proof, let us observe that using the bound \( 9/4 \) instead of \( \lfloor \sqrt[4]{s} \rfloor \) in (11) finishes the proof for \( n = 4 \) and \( s = 71, \ldots, 80 \).

**Proof.** It is enough to show that for each \( m \) the system of hypersurfaces of degree \( 9m - 1 \) with 71 \( 4m \)-fold points is empty. Since the system of degree \( 8m - 1 \) with 16 \( 4m \)-fold points is empty, by [6] (Theorem 1) it is enough to show that \( 9m - 1 \) with four \( 8m \)-fold points and seven \( 4m \)-fold points is empty. The last one, by Theorem 3 in [6], is equivalent to a system of degree 2 with at least one point of multiplicity \( 4m \), which is empty. \( \square \)

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**References**


