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MULTIPLICITY AND REGULARITY OF LARGE PERIODIC SOLUTIONS WITH RATIONAL FREQUENCY FOR A CLASS OF SEMILINEAR MONOTONE WAVE EQUATIONS

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Abstract. We prove the existence of infinitely many classical large periodic solutions for a class of semilinear wave equations with periodic boundary conditions:

$$u_{tt} - u_{xx} + f(x,u) = 0,$$
$$u(0,t) = u(\pi,t) \ \ , u_x(0,t) = u_x(\pi,t).$$

Our argument relies on some new estimates for the linear problem with periodic boundary conditions, the Hausdorff-Young theorem of harmonic analysis and a variational formulation due to Rabinowitz. We also develop a new approach to the regularity of the distributional solutions by differentiating the equations and employing Gagliardo-Nirenberg estimates.

1. Introduction

In this paper we construct infinitely many large classical time-periodic solutions for the following semilinear wave equation:

$$u_{tt} - u_{xx} + f(x,u) = 0, \tag{1.1}$$

$$u(0,t) = u(\pi,t), \ \ u_x(0,t) = u_x(\pi,t), \tag{1.2}$$

where f is $C^{2,1}$, has polynomial growth, depends on x, u and $f(x,0) = 0$. The existence of large periodic solutions with periodic boundary conditions is not well understood. As $u = 0$ is a trivial solution we seek here nontrivial solutions of (1.1), (1.2). When the frequency is irrational the method of Craig and Wayne in [14], extended to higher dimension by Bourgain [10] and Berti and Bolle [4], proves the existence of small periodic solutions for typical potentials, but the existence of classical periodic solutions for rational frequency is not known. Note that typical constant potentials in [14], [10], [4] are satisfied for

$$u_{tt} - \Delta u - mu + f(x,u) = 0 \tag{1.3}$$

for *typical* m which excludes $m = 0$. The so-called *resonant* case $m = 0$ and with $f(x,u)$ independent of x for periodic boundary conditions has been studied by Berti and Procesi in [9]. The lack of x dependence in [9] allows one to employ ordinary differential equation techniques, and they showed the existence of quasi-periodic solutions where the frequency vector depends on two frequencies $(\omega_1, \omega_2(\epsilon))$. While

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they consider $\omega_1 \in \mathbb{Q}$, their results do not imply the existence of periodic solutions with rational frequency, as $\omega_2(\epsilon)$ there is never rational. Chierchia and You in [12] study the problem with periodic boundary conditions and a potential

$$u_{tt} - u_{xx} - v(x)u + f(u) = 0, \tag{1.4}$$

where f only depends on u; however their method excludes the constants potentials $v(x) = m$. Bricmont, Kupiainen and Schenkel in [7] prove the existence of quasi-periodic solutions with periodic boundary conditions in the nonresonant case $m > 0$ and f depending only on u. In [7] they find quasi-periodic solutions for a set of positive measure of frequencies and hence prove the existence of quasi-periodic solutions for irrational frequencies.

On the other hand there exists a substantial amount of literature for semilinear wave equations with Dirichlet boundary conditions. See for instance [22], [21], [8] for rational frequencies and the proofs of the existence of classical solutions with f having some spatial dependence rely on a fundamental solution discovered by Lovicarová in [19]. The existence of periodic solutions with irrational frequencies with Dirichlet boundary conditions in the resonant case ($m = 0$) was shown by Lidskiĭ and Schul′man [18], by Bambusi in [1], Bambusi and Paleari in [2], Berti and Bolle [6], [5] and for quasi-periodic solutions by Yuan in [26]. Quasi-periodic solutions with Dirichlet boundary conditions via KAM techniques has been shown by Pöschel, Wayne and Kuksin; see [20] and the references therein.

de Simon and Torelli in [15] do not employ Lovicarová's formula, but their C^0 estimate relies on L^2 a priori estimates on $f(x, u)$ which are not readily available for distributional solutions of (1.1). The difficulty in proving regularity of the distributional solution of (1.1) stems from the kernel of $\Box$ which is infinite dimensional. In absence of a fundamental solution for the d'Alembertian under periodic boundary conditions problem we develop an approach based on tools from harmonic analysis such as Littlewood-Paley techniques, the Hausdorff-Young theorem and Gagliardo-Nirenberg estimates. The Hausdorff-Young theorem had been employed earlier by Willem in [28] to get a L^∞ a priori estimate on solutions, by Coron to prove a Sobolev embedding in [13] and by Zhou in [29]. The argument we give here to prove the Sobolev embedding in [13] follows the Fourier approach to the Sobolev embedding as in the notes by Chemin [11]. We do prove a stronger estimate than the one in [13], which provides information about the best constant of the Sobolev embedding. Our argument also shows that the Sobolev embedding in [13] is *compact*. In this paper the existence of a *classical solution for time periodic solutions with periodic boundary conditions of the semilinear wave equation* (1.1) *will be shown by proving stronger C^γ Hölder estimates than the L^∞ in* [28], and our approach also gives an alternative proof of the existence of classical periodic solutions in the case of Dirichlet boundary conditions with semilinear term with some spatial dependence for $f(x, u)$ sufficiently smooth in both arguments x and u.

In section 1 we prove the linear estimates needed to prove the regularity of the solution. In section 2 we follow the scheme of [22] and [21] to construct weak solutions, and in section 3 we show the regularity of the solution by repeated differentiation of the equations, the linear estimates proved in section 1 and Gagliardo-Nirenberg inequalities. Since our proof is of variational nature, it is natural to ask if there is a notion of a critical exponent or critical growth for this equation. An open question is then whether there are semilinear terms $f(x, u)$ of say exponential or super exponential type (as this paper deals with semilinear terms of polynomial type) for which there are large amplitude distributional solutions which are not classical ($f(x,u)$ being assumed to be smooth). We seek time-periodic solutions satisfying periodic boundary conditions, so we seek functions $u \in \mathbb{R}$ with expansions of the form

$$u(x,t) = \sum_{(j,k)\in\mathbb{Z}\times\mathbb{Z}} \widehat{u}(j,k)e^{i2jx}e^{ikt}$$

and define the function space $E \oplus N \subset L^2$:

$$||u||^2_{E\oplus N} = \sum_{2j\neq\pm k} \frac{|Q|}{4}|k^2-4j^2||\widehat{u}(j,k)|^2 + \sum_{2j=\pm k} |4j^2||\widehat{u}(j,k)|^2 + |\widehat{u}(0,0)|^2,$$

where $Q = [0,\pi]\times[0,2\pi]$. We define the functions spaces E^+, E^-, N as follows:

$$N = \{u \in L^2,\ \widehat{u}(j,k) = 0 \text{ for } 2|j| \neq |k|\},$$

$$E^+ = \{u \in E,\ \widehat{u}(j,k) = 0 \text{ for } |k| \leq 2|j|\},$$

$$E^- = \{u \in E,\ \widehat{u}(j,k) = 0 \text{ for } |k| \geq 2|j|\},$$

where $E = E^+ \oplus E^-$.

The spectrum of the linear operator $\partial_{tt} - \partial_{xx}$ under periodic boundary conditions and 2π periodic in time consists of $-k^2+4j^2$, where the eigenfunctions are $\sin 2jx \sin kt\ \cos 2jx \cos kt, \sin 2jx \cos kt, \cos 2jx \sin kt$. When $-k^2+4j^2 \neq 0$, $-k^2+4j^2 = p$ has only finitely many solutions for any $p \neq 0$, hence we can number the nonzero eigenvalues as $\ldots < \mu_{-2} < \mu_{-1} < 0 < \mu_1 < \mu_2 < \ldots$ and the corresponding eigenfunctions $\ldots, e_{-2}, e_{-1}, e_1, e_2, \ldots$; moreover $\mu_l \to +\infty$ as $l \to +\infty$. Note that in the case of periodic boundary conditions the structure of the kernel N of $\square$ is slightly different than in the case of Dirichlet boundary conditions. Here $v \in N$ and we have

$$\begin{aligned} v(x,t) &= \sum_{j=\pm k} \widehat{v(j,k)}e^{i2jx+ikt} \\ &= \sum_{j\neq 0} v\widehat{(j,2j)}e^{i2j(x+t)} + \sum_{j} v\widehat{(j,-2j)}e^{i2j(x-t)} \end{aligned} \tag{1.5}$$

and

$$v(x,t) = p_1(x+t) + p_2(x-t) = v^+(x,t) + v^-(x,t), \tag{1.6}$$

where $v^+(x,t) = p_1(x,t)$, $v^-(x,t) = p_2(x,t)$ and where the $p_1, p_2 \in H^1(0,\pi)$ π-periodic functions and are defined as $p_1(s) = \sum_j \widehat{p_1(j)}e^{i2js}$, $p_2(s) = \sum_j \widehat{p_2(j)}e^{i2js}$ and $\widehat{p_1(0)} = 0$ (this normalizes p), $\widehat{p_1(j)} = v\widehat{(j,2j)}$,$\widehat{p_2(j)} = v\widehat{(j,-2j)}$.

We have $u = v + w$, $w = w^+ + w^-$, where $w \in E$, $w^+ \in E^+$,$w^- \in E^-$ and $v \in N$ and define the norm on $E \oplus N$:

$$||u||^2_{\beta,E} = ||w^+||^2_E + ||w^-||^2_E + \beta||v||^2_{H^1},$$

$$I_\beta(u) = \int_Q [\frac{1}{2}(u_t^2 - u_x^2 - \beta(v^2+v_t^2)] - F(x,u)dxdt. \tag{1.7}$$

Our argument relies on the Galerkin procedure for which we define the spaces

$$E_m = span\{\sin 2jx\cos kt, \sin 2jx\sin kt, \cos 2jx\cos kt,\ \cos 2jx\sin kt,\ \ 2j+k\le m, 2j\neq k\},$$

$$E_{-m} = span\{\sin 2jx\cos kt, \sin 2jx\sin kt, \cos 2jx\cos kt,\ \cos 2jx\sin kt,\ \ 2j+k\le m\ \ 2j>k\},$$

$$E_{+l} = span\{\sin 2jx\cos kt, \sin 2jx\sin kt, \cos 2jx\cos kt,\ \cos 2jx\sin kt,\ \ 4j^2-k^2\le l\ \ 2j<k\},$$

$$N_m = span\{\sin 2jx\cos jt, \sin 2jx\sin jt,\ \cos 2jx\cos jt,\ \cos 2jx\sin jt,\ \ 2j\le m\},$$

which are employed in the minimax procedure. We denote by P_m the projection of $E\oplus N$ into $E_m\oplus N_m$.

When u is a trigonometric polynomial, I_β can also be represented in $E_m\oplus N_m$ as

$$I_\beta(u) = \frac{1}{2}(||w^+||_E^2 - ||w^-||_E^2 - \beta(||v||_{L^2}^2 + ||v_t||_{L^2}^2) - \int_Q F(x,u)dxdt,$$

where $\frac{\partial F(x,u)}{\partial u} = f(x,u)$. First seek the weak solution of the modified equation

$$\square u = \beta v_{tt} - f(x,u) - \beta v \tag{1.8}$$

and then send the parameter β to zero.

Assumptions on $f(u)$**:** Let $s>1$ and we assume that there are positive constants

$$c_0^1 \le c_0^2, c_1^1, c_1^2 \tag{1.9}$$

such that

$$c_0^1|u|^{s-1}u + c_1^1 \le f(x,u) \le c_0^2|u|^{s-1}u + c_2^1 \tag{1.10}$$

with $\frac{c_0^1}{2} > \frac{c_0^2}{s+1}$. These assumptions are satisfied by some nonlinearities of polynomial type. $f(x,u)$ must also be strongly monotone increasing:

$$\frac{\partial f(x,u)}{\partial u} \ge \alpha > 0. \tag{1.11}$$

Theorem 1.1. *Under assumptions* (1.10), (1.11) *and* $f\in C^{2,1}$, (1.1), (1.2) *admit infinitely many* 2π *periodic classical solutions.*

2. Estimates

We start by a calculus lemma needed for the minimax procedure:

Lemma 2.1. *There are constants* c_F, C_F, c_1, c_2, d *depending on* s *such that*

$$c_1|u|^{s+1} - d(s) \le \frac{1}{2}f(x,u)u - F(x,u) \le c_2|u|^{s+1} + d(s), \tag{2.12}$$

where

$$F(x,u) = \int_0^u f(x,s)ds \tag{2.13}$$

and

$$c_F(s)|u|^{s+1} - d(s) \le F(x,u) \le C_F(s)|u|^{s+1} + d(s). \tag{2.14}$$

This is possible because of the assumptions (1.10) *we have imposed on* f.

Proof. Integrating (1.10) we obtain upper and lower estimates on F,

$$c_0^1 \frac{|u|^{s+1}}{s+1}, \tag{2.15}$$

and deduce (2.14) as the interval $[0, \pi]$ is compact. We now prove (2.12). Let $u > 0$. Then we have

$$\frac{1}{2}c_0^1|u|^{s+1} + \frac{1}{2}c_1^1 u \leq \frac{1}{2}f(x,u)u \leq \frac{1}{2}c_0^2|u|^{s+1} + \frac{1}{2}c_2^1 u, \tag{2.16}$$

and by subtracting with (2.16) we obtain

$$(\frac{1}{2}c_0^1 - \frac{1}{s+1}c_0^2)|u|^{s+1} + \frac{1}{2}c_1^1 u - c_2^1 u \leq \frac{1}{2}f(x,u)u - F(x,u) \tag{2.17}$$

and

$$\frac{1}{2}f(x,u)u - F(x,u) \leq \frac{1}{2}c_0^2|u|^{s+1} + \frac{1}{2}c_1^1 u + d(s); \tag{2.18}$$

hence (2.12) can be deduced. The case $u \leq 0$ follows similarly.

Define $l^q = \{\hat{u}(j,k)\ s.t.\ \sum_{(j,k)\in\mathbb{Z}\times\mathbb{Z}} |\hat{u}(j,k)|^q < +\infty\}$.

Theorem 2.1. *Let $1 < p \leq 2$. The function $u = \sum_{2j\neq\pm k} \hat{u}(j,k)e^{2ijx+ikt} \in C^\gamma$, where $\gamma < 1 - \frac{1}{p}$, if*

$$\hat{u}(j,k) = \frac{\hat{f}(j,k)}{4j^2 - k^2} \tag{2.19}$$

for $2j \neq \pm k$, $\hat{f} \in l^q$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Let B_m be the set

$$B_m = \{(j,k) \in \mathbb{Z}\times\mathbb{Z}\ 2|j| + |k| \leq 2.2^m\}$$

and Δ_m,$\Delta_m = B_m \setminus B_{m-1}$, so we have in Δ_m, $2^m \leq 2|j| + |k| \leq 2.2^m$, and the C^γ norm will be estimated by $\sup_m 2^{\gamma m}||\Delta_m u||_{C^0}$; see [25] or [17]:

$$\begin{aligned}
2^{\gamma m}||\Delta_m u||_{C^0} &= 2^{\gamma m}||\sum_{(j,k)\in\Delta_m} \hat{u}(j,k)e^{i2jx}e^{ikt}||_{C^0} \\
&= ||\sum_{(j,k)\in\Delta_m} \frac{2^m \hat{f}(j,k)}{4j^2-k^2}e^{i2jx}e^{ikt}||_{C^0} \\
&\leq [\sum_{(j,k)\in\Delta_m} \frac{2^{\gamma mp}}{||2j|^2-k^2|^p}]^{\frac{1}{p}}[\sum_{(j,k)\in\mathbb{Z}\times\mathbb{Z}} |\hat{f}(j,k)|^q]^{\frac{1}{q}} \\
&\leq [\sum_{(j,k)\in\Delta_m} \frac{|2|j|+|k||^{\gamma p}}{|2|j|+|k||^p||2j|-|k|)^2|^p}]^{\frac{1}{p}}[\sum_{(j,k)\in\mathbb{Z}\times\mathbb{Z}} |\hat{f}(j,k)|^q]^{\frac{1}{q}} \\
&\leq [\sum_{(j,k)\in\Delta_m} \frac{1}{|2|j|+|k||^{(1-\gamma)p}||2j|-|k|)|^p}]^{\frac{1}{p}}[\sum_{(j,k)\in\mathbb{Z}\times\mathbb{Z}} |\hat{f}(j,k)|^q]^{\frac{1}{q}}.
\end{aligned}$$

Now consider the system with $(r,q) \in \mathbb{Z}\times\mathbb{Z}$,$(j,k) \in \mathbb{N}\times\mathbb{N}$,

$$\begin{cases} 2j+k &= r, \\ 2j-k &= q. \end{cases}$$

Whenever it has a solution it is unique. Hence

$$\begin{aligned} 2^{\gamma m}||\Delta_m u||_{C^0} &\leq [\sum_{(r,q)\in\mathbb{Z}\times\mathbb{Z},r\neq 0,q\neq 0} \frac{1}{r^{(1-\gamma)p}q^p}]^{\frac{1}{p}}[\sum_{(j,k)\in\mathbb{Z}\times\mathbb{Z}} |\hat{f}(j,k)|^q]^{\frac{1}{q}} \\ &\leq [\sum_{r\in\mathbb{Z},r\neq 0} \frac{1}{r^{(1-\gamma)p}} \sum_{q\in\mathbb{Z},q\neq 0} \frac{1}{q^p}]^{\frac{1}{p}}[\sum_{(j,k)\in\mathbb{Z}\times\mathbb{Z}} |\hat{f}(j,k)|^q]^{\frac{1}{q}} \\ &\leq c||\hat{f}||_{l^q} \leq c||f||_{L^p} \end{aligned} \tag{2.20}$$

as long as $\gamma < 1 - \frac{1}{p}$ or equivalently $(1-\gamma)p > 1$ and $p > 1$, and the last inequality follows from the Hausdorff-Young theorem.

Remark. The argument here provides an alternate proof of the Hölder continuity of weak solutions of $\Box w = f$ where $f \in L^p \cap N^\perp$, where $N^\perp$ denotes the weak orthogonal of the kernel of $\Box$ with Dirichlet boundary conditions. This was proved by Brezis, Coron and Nirenberg in [8] via Lovicarova's fundamental solution for $1 < p \leq 2$.

We prove a bootstrapping estimate in the next lemma. It follows step by step the proof Theorem 4 in [23] established for Dirichlet boundary conditions. We do not provide a proof.

Lemma 2.2 ([23])**.** *Let $f, w \in L^2(Q)$ such that*

$$\widehat{f}(j,k) = 0 = \widehat{w}(j,k) \ \text{ for } 2j = \pm k \tag{2.21}$$

and

$$(-k^2 + 4j^2)\widehat{w}(j,k) = \widehat{f}(j,k), \ \text{ for } 2j \neq \pm k. \tag{2.22}$$

Then $w \in H^1$.

Let E^s be the closure of $\{e^{i2jx+ikt},\ 2j \neq \pm k\}$ under the norm $||u||^2_{E^s} = \sum_{2j\neq\pm k} |\widehat{u}(j,k)|^2|k^2 - 4j^2|^s$. Then we have the Sobolev estimate:

Theorem 2.2. *For $0 < s < 1$ the space E^s is continuously embedded in L^p where $p = \frac{2-s}{1-s}$.*

This theorem implies that the embedding in [13], $E^1 \subset L^p$, is compact, as $E^1 \subset E^s$ is compact for $s < 1$. We will show that it also implies a Gagliardo-Nirenberg inequality of the type

$$||u||_{L^p} \leq c(p)||u||^{1-s(p)}_{L^2}||u||^{s(p)}_{E^1}, \tag{2.23}$$

where $c(p)$ will be computed explicitly.

Proof.

$$f = f_{1,A} + f_{2,A}, \tag{2.24}$$

where

$$f_{1,A} = \sum_{2j\neq\pm k, 2|j|+|k|\leq A} \hat{f}(j,k)e^{i2jx}e^{ikt} \tag{2.25}$$

and

$$f_{2,A} = \sum_{2j\neq\pm k, 2|j|+|k|> A} \hat{f}(j,k)e^{i2jx}e^{ikt} \tag{2.26}$$

$$
\begin{aligned}
|f_{1,A}| &\leq \sum_{2j\neq\pm k, 2|j|+|k|\leq A} |\hat{f}(j,k)| \\
&\leq \sum_{2j\neq\pm k, 2|j|+|k|\leq A} |4j^2-k^2|^{-\frac{s}{2}}|4j^2-k^2|^{\frac{s}{2}}|\hat{f}(j,k)|, \qquad (2.27)
\end{aligned}
$$

and applying Cauchy-Schwarz we have

$$
\begin{aligned}
|f_{1,A}| &\leq (\sum_{2j\neq\pm k, 2|j|+|k|\leq A} \frac{1}{|4j^2-k^2|^s})^{\frac{1}{2}}(\sum_{2j\neq\pm k, 2|j|+|k|\leq A} |4j^2-k^2|^s|\hat{f}(j,k)|^2)^{\frac{1}{2}} \\
&\leq ||f||_{E^s}(\sum_{m,n\in\mathbb{N}\leq A} \frac{4}{m^s}\frac{1}{n^s})^{\frac{1}{2}} \\
&\leq c||f||_{E^s}A^{-s+1}. \qquad (2.28)
\end{aligned}
$$

Now we seek A_λ such that $|f_{1,A}| \leq \frac{\lambda}{4}$ and $c||f||_E A^{1-s} \leq \frac{\lambda}{4}$; this leads to the inequality $A^{1-s} \leq \frac{\lambda}{4c||f||_{E^s}}$. So let $A_\lambda := (\frac{\lambda}{4c||f||_{E^s}})^{\frac{1}{1-s}}$. Now

$$
\int_{[0,\pi][0,2\pi]} |f(x,t)|^p dxdt = p\int_0^\infty y^{p-1}w(y)dy,
$$

where $w_f(y) = |\{(x,t) \in [0,\pi][0,2\pi] : |f(x,t| > y\}|$. Now $|f(x,t)| > \lambda$ implies $|f_{1,A}| > \frac{\lambda}{2}$ or $|f_{2,A}| > \frac{\lambda}{2}$. Recalling the definition of A_λ above we have $|f_{2,A_\lambda}| > \frac{\lambda}{2}$ and $w_f(\lambda) \leq w_{f_{2,A_\lambda}}(\frac{\lambda}{2})$; hence

$$
\begin{aligned}
\int_{[0,\pi][0,2\pi]} |f(x,t)|^p dxdt &= p\int_0^\infty \lambda^{p-1}w_f(\lambda)d\lambda \\
&\leq p\int_0^\infty \lambda^{p-1}w_{f_{2,A_\lambda}}(\frac{\lambda}{2})d\lambda.
\end{aligned}
$$

Since

$$
w(\lambda) \leq \frac{1}{\lambda^2}\int_{|f|\geq\lambda} |f(x,t)|^2 dxdt, \qquad (2.29)
$$

then

$$
\begin{aligned}
\int_{[0,\pi][0,2\pi]} |f(x,t)|^p dxdt &\leq 4\int_0^\infty \lambda^{p-3}\int_{|f_{2,A_\lambda}(x,t)|>\frac{\lambda}{2}} |f_{2,A_\lambda}(x,t)|^2 dxdtd\lambda \\
&\leq 4\int_0^\infty \lambda^{p-3}\int_{[0,\pi][0,2\pi]} |f_{2,A_\lambda}(x,t)|^2 dxdtd\lambda. \qquad (2.30)
\end{aligned}
$$

Then we can invoke the Parseval formula to deduce

$$
\begin{aligned}
\int_{[0,\pi][0,2\pi]} |f(x,t)|^p dxdt &\leq 4\int_0^\infty \lambda^{p-3}\sum_{2j\neq\pm k} |\hat{f}_{2,A_\lambda}(j,k)|^2 d\lambda \\
&= 4\int_0^\infty \lambda^{p-3}\sum_{2j\neq\pm k, 2|j|+|k|>A_\lambda} |\hat{f}(j,k)|^2 d\lambda. \qquad (2.31)
\end{aligned}
$$

Now $2|j|+|k| \geq A_\lambda = (\frac{\lambda}{4c||f||_{E^s}})^{\frac{1}{1-s}}$ implies $\lambda \leq 4c||f||_{E^s}((2|j|+|k|))^{1-s}$. We continue the estimate from (2.31):

$$\begin{aligned}
&\int_{[0,\pi][0,2\pi]} |f(x,t)|^p dxdtd\lambda \\
&\quad\leq 4 \sum_{2j\neq\pm k} \int_0^\infty \lambda^{p-3}|\hat{f}(j,k)|^2 1_{\{(\lambda,j,k)\ s.t.\ 2|j|+|k|\geq A_\lambda\}} d\lambda \\
&\quad\leq 4 \sum_{2j\neq\pm k} \int_0^{4c||f||_{E^s}(2|j|+|k|)^{1-s}} \lambda^{p-3}|\hat{f}(j,k)|^2 d\lambda \\
&\quad\leq 4 \sum_{2j\neq\pm k} |\hat{f}(j,k)|^2 \int_0^{4c||f||_{E^s}(2|j|+|k|)^{1-s}} \lambda^{p-3} d\lambda \\
&\quad\leq 4 \sum_{2j\neq\pm k, 2|j|+|k|} |\hat{f}(j,k)|^2 [\frac{\lambda^{p-2}}{p-2}]_0^{4c||f||_{E^s}(2|j|+|k|)^{1-s}} \\
&\quad\leq 4 \sum_{2j\neq\pm k, 2|j|+|k|} |\hat{f}(j,k)|^2 \frac{1}{p-2}[4c||f||_{E^s}((2|j|+|k|))^{1-s}]^{p-2}.
\end{aligned}$$

Now if $s=(1-s)(p-2)$, i.e. $s(p)=\frac{p-2}{p-1}$, then

$$\int_{[0,\pi][0,2\pi]} |f(x,t)|^p dxdt \leq \frac{4(4c)^{p-2}}{p-2}||f||^p_{E^{s(p)}}, \tag{2.32}$$

and we have the following Gagliardo-Nirenberg inequality for $p>2$ by applying Holder's inequality:

$$||u||_{L^p} \leq c(p)||u||_{E^{s(p)}} \leq c(p)||u||_{L^2}^{1-s(p)}||u||_{E^1}^{s(p)}. \tag{2.33}$$

The argument goes as follows:

$$\begin{aligned}
||u||^2_{E^s} &= \sum_{2j\neq\pm k} |\widehat{u}(j,k)|^2|k^2-4j^2|^s \\
&= \sum_{2j\neq\pm k} |\widehat{u}(j,k)|^{2(1-s)}|\widehat{u}(j,k)|^{2s}|k^2-4j^2|^s \\
&\leq [\sum_{2j\neq\pm k} |\widehat{u}(j,k)|^2]^{1-s}[\sum_{2j\neq\pm k} |\widehat{u}(j,k)|^2|k^2-4j^2|]^s \\
&\leq ||u||^{2(1-s)}_{L^2}||u||^{2s}_{E^1}. \qquad (2.34)
\end{aligned}$$

Let i_N be the projection from E^s to $E^{s'}$ for $s>s'$ onto the subspace spanned by $e^{ijx+2ikt}$ for $2j\neq\pm k, |k^2-4j^2|\leq N$, and i be the embedding from E^s onto $E^{s'}$. Then i is the uniform limit of i_N, and hence the embedding i is compact. □

3. Construction of the weak solution

The functional I_β satisfies the Palais-Smale condition. The arguments follow as in [22], and we do not repeat them here.

Then we can define $g_\theta(u)=u(x,t+\theta)$. Define

$$\mathcal{G}=\{g_\theta\ \ s.t.\ \theta\in[0,2\pi)\}, \tag{3.35}$$

$$V_l = N_m \oplus E_{-m} \oplus E_{+8l}, \tag{3.36}$$

$$G_l = \{h \in C(V_l, E_m \oplus N_m) \ \text{ such that } \ h \text{ satisfies } \ \gamma_1 - \gamma_4\}, \tag{3.37}$$

$Fix\mathcal{G} = \{u \in E \ s.t. \ g(u) = u \ \forall g \in \mathcal{G}\} = span\{\cos 2jx, \ \sin 2jx, \ \ j \in \mathbb{Z}\} \subset E^-$. Define P_{0m}, P_{-m} as the orthogonal projections from $E_m \oplus N_m$ onto respectively, and let $N^m (= E_{0m}), E_{-m}$ and P_l be the orthogonal projection from $E_m \oplus N_m$ onto V_l.

$$\begin{cases} \gamma_1 - & h \text{ is equivariant } \ h(g_\theta(u)) = g_\theta(h(u)), \\ \gamma_2 - & h(u) = u \ \text{ if } \ u \in Fix\mathcal{G}, \\ \gamma_3 - & \text{there exists } \ r = r(h) \ \ h(u) = u \text{ if } u \in V_l \setminus B_{r(h)}, \\ \gamma_4 - & u = w^+ + w^- + v \in V_l \ \ (P_{0m} + P_{-m})h(u) = \alpha(u)v + \alpha^-(u)w^- + \phi(u), \\ & \text{where } \ \alpha, \overline{\alpha} \in C(V_l, [1, \overline{\alpha}]), \end{cases}$$

and $1 < \overline{\alpha}$ depends on h, ϕ continuous. Define

$$c_l(\beta) = \inf_{h \in G_l} \sup_{u \in V_l} I_\beta(h(u)) \tag{3.38}$$

and $c_l(\beta) \to +\infty$ as $l \to \infty$ independently of m, β.

To prove the next lemma we will need a topological corollary from [16], which relies on an explicit dependence on the dimension of the space V_l. For this purpose we need a relabelling of the subspace E_{+l} by increasing eigenvalues and accordingly the corresponding eigenfunctions. Since $-k^2 + 4j^2 = p$ has only finitely many solutions for any $p \neq 0$, we can number the positive eigenvalues as

$$0 < \mu_1 < \mu_2 < \dots \ ,$$

and also the corresponding eigenfunctions

$$e_1, e_2, \dots \ .$$

Now E_{+8l} can be relabelled as $E_{\mathbf{i}}$, where $\mathbf{i}$ is the dimension of E_{+8l}. V_l can be relabelled as $V_{\mathbf{i}}$, where

$$V_{\mathbf{i}} = N_m \oplus E_{-m} \oplus E_{\mathbf{i}}. \tag{3.39}$$

Lemma 3.1. *$c_l(\beta) \to +\infty$ as $l \to +\infty$.*

Proof.

$$I_\beta(u) \ = \ \frac{1}{2}||w^+||_E^2 - \frac{1}{2}||w^-||_E^2 - \frac{\beta}{2}||v||_{H^1}^2 - \int_Q F(u)dxdt, \tag{3.40}$$

and there exist by assumption (1.10) $c(s), d(s) > 0$ such that

$$I_\beta(u) \ \geq \ \frac{1}{2}||w^+||_E^2 - \frac{1}{2}||w^-||_E^2 - \frac{\beta}{2}||v||_{H^1}^2 - c(s)\int_Q |u|^{s+1}dxdt - d(s). \tag{3.41}$$

Also, if $u \in \partial B_\rho \cap V_{\mathbf{i-1}}^\perp$, then $u = w^+ \in E^+$, and we have

$$I_\beta(w^+) \ \geq \ \frac{1}{2}||w^+||_E^2 - c(s)\int_Q |w^+|^{s+1}dxdt - d(s)$$

and

$$8l||w^+||_{L^2} \leq ||w^+||_E = \rho. \tag{3.42}$$

Now by the Sobolev embedding theorem, Theorem 2.2, there is $\theta(s) < 1$ such that if $\widehat{u}(j,k) = 0$ for $2j = \pm k$ we have $||w^+||_{L^{s+1}} \leq ||w^+||_{E^{\theta(s)}}$. Hence

$$\begin{aligned} I_\beta(u) &\geq \frac{1}{2}||w^+||_E^2 - c(s)(||w^+||_{L^2}^{1-\theta(s)}||w^+||_{E^1}^{\theta(s)})^{s+1} - d(s) \\ &\geq \frac{1}{2}\rho^2 - \rho^{s+1}(8l)^{-(1-\theta(s))(s+1)} - d(s). \end{aligned} \tag{3.43}$$

By recalling $||w^+|| \leq \frac{\rho}{8l}$ form (3.42), denoting $||w^+||_E = \rho$, choosing a constant $C(s)$ large, and setting $\rho = \frac{1}{8}(8l)^{(1-\theta(s))\frac{s+1}{s-1}}$, we have $I_\beta(u) \geq \frac{1}{4}\rho^2 - d(s)$. Applying Corollary 2.4 in [16] to $P_{\mathbf{i-1}}h \in C(\partial B_\rho, V_{\mathbf{i-1}})$ we have

$$h(V_{\mathbf{i}}) \cap \partial B_\rho \cap V_{\mathbf{i-1}}^\perp \neq \emptyset; \text{ hence} \tag{3.44}$$

$$\sup_{V_{\mathbf{i}}} I_\beta(h(u)) \geq \inf_{u \in \partial B_\rho \cap V_{\mathbf{i-1}}^\perp} I(u) \geq \frac{1}{4}\rho^2 - d(s) \to +\infty. \tag{3.45}$$

The $c_l(\beta)$ are critical values of I_β on $E_m \oplus N_m$. This is obtained by a standard argument; see Propositions 2.33 and 2.37 in [22]. □

Lemma 3.2. *Forall $u \in E_{+8l}$, there is a constant $M(l)$ independent of β, m such that*

$$I_\beta(u) \leq M(l). \tag{3.46}$$

Proof. Let $u \in E_{+8l}$ such that

$$\begin{aligned} I_\beta(u) &= \frac{1}{2}||w^+||_E^2 - \int_Q F(w^+)dxdt \\ &\leq \frac{1}{2}||w^+||_E^2 - c(s)\int_Q |w^+|^{s+1}dxdt + d(s) \\ &\leq c(s) + \sup_{u \in E_{+8l}} \frac{1}{2}||w^+||_E^2 - c(s,Q)||w^+||_{L^2}^{s+1}. \end{aligned} \tag{3.47}$$

Now E_{+8l} $||u||_E^2 \leq (8l)||u||_{L^2}^2$ and $\sup_{u \in E_{+8l}} \frac{1}{2}||u||_E^2 - c(s,Q)||u||_{L^2}^{s+1} > 0$, while, as $||u||_E \to +\infty$, E_{+8l} is dominated by $||u||_{L^2}^{s+1}$ as $s+1 > 2$ and is attained at say $\overline{u}$. Hence we have $c(s,Q)||\overline{u}||_{L^2}^{s+1} \leq ||\overline{u}||_E^2 \leq (8l)||\overline{u}||_{L^2}^2$ and we can conclude there is $M(l)$ depending on l but independent of β such that $I_\beta(u) \leq M(l)$. □

Lemma 3.3. *There is $R_l \to +\infty$ such that $I_\beta(u) \to -\infty$ uniformly as $||u||_{\beta,E} = R_l \to +\infty$ for $u \in V_l$.*

Proof. If $||u||_{\beta,E}^2 = R_l^2$, then

$$||w^+||_{\beta,E}^2 + ||w^-||_{\beta,E}^2 + \beta||v||_{H^1}^2 = R_l^2, \tag{3.48}$$

and we have two possibilities: either $||w^+||_{\beta,E}^2 \geq \frac{R_l^2}{3}$ or $||w^-||_{\beta,E}^2 + \beta||v||_{H^1}^2 \geq \frac{2R_l^2}{3}$.

Case 1. $||w^+||^2_{\beta,E} \geq \frac{R_l^2}{3}$.

$$\begin{aligned} I_\beta(u) &= \frac{1}{2}||w^+||^2_E - \frac{1}{2}||w^-||^2_E - \frac{\beta}{2}||v||^2_{H^1} - \int_Q F(u)dxdt \\ &\leq \frac{1}{2}||w^+||^2_E - \frac{1}{2}||w^-||^2_E - \frac{\beta}{2}||v||^2_{H^1} - c(s)\int_Q |u|^{s+1}dxdt + d(s) \\ &\leq \frac{1}{2}||w^+||^2_E - c(s)||u||^{s+1}_{L^{s+1}} + d(s), \end{aligned}$$

and as $w^+ \in E_{+8l} \oplus E_{-m} \oplus N_m$ and $s > 1$ we also have

$$\frac{1}{8l}||w^+||_E \leq ||w^+||_{L^2} \leq ||u||_{L^2} \leq c(Q)||u||_{L^{s+1}}. \tag{3.49}$$

Thus

$$I_\beta(u) \leq \frac{1}{2}||w^+||^2_E - c(s)(\frac{||w^+||_E}{c(Q)8l})^{s+1} + c(f), \tag{3.50}$$

and for $R_l(s)$ large enough $I_\beta(u) \to -\infty$ uniformly as $s > 1$.

Case 2. $||w^-||^2_{\beta,E} + \beta||v||^2_{H^1} \geq \frac{2R_l^2}{3}$ and $||w^+||^2_{\beta,E} \leq \frac{R_l^2}{3}$ follows similarly.

Lemma 3.4. *Let u_l be a critical point corresponding to c_l in $E_m \oplus N_m$, and there is a subsequence $p(l)$ such that $u_{p(l_1)} \neq u_{p(l_2)}$ for $p(l_1) \neq p(l_2)$.*

Proof. By Lemmas 3.1 and 3.2 there are constants c_4, c_5 independent of m, β such that $c_4(l) \leq c_l \leq c_5(l)$ and such that $c_4(l) \to +\infty$. Hence we can construct a subsequence denoted by $p(l)$ such that

$$c_{p(l_1)} < c_{p(l_2)} \tag{3.51}$$

for $p(l_1) < p(l_2)$, and thus the critical points corresponding to the critical values $c_{p(l)}$ are distinct.

Theorem 3.1. *Let f be C^1; for l large enough there is a distributional solution $u = v + w$ of the modified problem* (1.8).

Proof. In this proof the constants may dependent on β and f but are independent of m. The proof of this theorem here is slightly simpler from the one in [22] as we take advantage of the polynomial growth of the nonlinear term. We also employ the Galerkin approximation.

Let $u_l^m = w^m + v^m \in E_m \oplus N_m$ be a distributional solution corresponding to the critical value c_l and any $\phi \in E_m \oplus N_m$:

$$I'_\beta(u_l^m)\phi = 0. \tag{3.52}$$

Now taking $\phi = v_{tt}^m \in N_m$ we have

$$(\beta v_{tt}^m, v_{tt}^m)_{L^2} = (f(x, u_l^m), v_{tt}^m)_{L^2} + \beta(v_t^m, v_t^m)$$

and by (1.10) there are positive constants c, d such that

$$\beta||v_{tt}^m||^2_{L^2} \leq c||u^s||_{L^2}||v_{tt}^m||_{L^2} + d||v_{tt}^m||_{L^2}.$$

Note that the estimate on $||u^s||_{L^2}$ which depends on β follows from the one of $||u||_E$ and the Sobolev embedding in the Palais-Smale proof!!!

$$\beta||v_{tt}^m||^2_{L^2} \leq c||v_{tt}^m||_{L^2},$$

and hence

$$||v_{tt}^m||_{L^2} \leq c(\beta).$$

We now have

$$w_{tt}^m - w_{xx}^m = \beta v_{tt}^m + P^m f(x, u_l^m) \in L^2;$$

hence $w^m \in H^1 \cap C^\gamma$, $\gamma < \frac{1}{2}$ by Theorem 2.1 and Lemma 2.2. This now implies $w^m \in H^2$, $w^m \to w$ as $m \to +\infty$ pointwise and $w \in H^1 \cap C^\gamma$. Then if $\phi = v_{tttt}^m$, then

$$(\beta v_{tt}^m, v_{tttt}^m)_{L^2} = (f(x, u_l^m), v_{tttt}^m)_{L^2} - \beta(v_t^m, v_{ttt}^m)$$
$$(\beta v_{ttt}^m, v_{ttt}^m)_{L^2} = (f_u(x, u_l^m)u_{lt}^m, v_{ttt}^m)_{L^2} - \beta(v_t^m, v_{ttt}^m),$$

so there exists c independent of m so that we deduce $||v_{ttt}^m||_{L^2} \leq c(\beta)$. Hence $v_{ttt}^m \to v_{tt} \in C^0$ as $m \to +\infty$ and v is C^2 and w is C^γ by applying Theorem 2.1 to (1.8). We now have

$$u_l^m \to u \in C^\gamma \text{ as } m \to +\infty, \tag{3.53}$$

and since (3.52) holds for any $\phi \in E_m \oplus N_m$ we can deduce

$$I'(u)\phi = 0 \ \forall \phi \in E_m \oplus N_m. \tag{3.54}$$

Now sending $m \to \infty$, u is a weak solution of (1.8).

Let $u^m = w^m + v^m$ be the approximate solution on $E_m \oplus N_m$; then

$$\widehat{\Box w^m}(j,k) = \widehat{f(u^m)}(j,k) \tag{3.55}$$

$\forall 2j \neq k \in E_m$. Hence by Lemma 2.1 and the Hausdorff-Young theorem we have

$$||w^m||_{C^\gamma} \leq M_1, \tag{3.56}$$

with M_1 independent of m, β. Hence we can conclude that $w = \lim_{m \to +\infty} w^m \in C^\gamma$ for any $\gamma < 1 - \frac{s}{s+1}$.

The following lemma follows step by step the method of [21] to get an a priori estimate on $||v||_{C^0}$ independently of β. We omit the proof.

Lemma 3.5. *There is a constant c independent of β such that*

$$||v(\beta)||_{C^0} \leq c. \tag{3.57}$$

4. Regularity of the solution

Here we prove that if $f \in C^{2,1}$, then the weak solution u is C^2. Since $||v||_{C^0}, ||w||_{C^0}$ are bounded independently of β, $f(u) \in C^0$. We also have

$$(-4j^2 + k^2)\widehat{w}(j,k) = f\widehat{(x, v + w)}(j,k) \quad 2j \neq \pm k. \tag{4.58}$$

Then by Lemma 2.2 we have $w \in H^1$. Since f is smooth, then too $f(w + v) \in H^1$. Then (4.58) implies $w \in H^2$ and iterating once again leads to $w \in H^3$. Now going back to the original equation,

$$-\beta v_{tt} = \Box w - f(x, u) - \beta v, \tag{4.59}$$

and recalling that $v \in C^2$, we deduce $v \in H^3$ which with (4.58) implies $w \in H^4$. Iterating once more implies $v \in H^4$; then again $w \in H^5$ and $v \in H^5$. Thus differentiating with respect to t in the weak sense we have

$$\Box w_t - \beta v_{ttt} = -f'_u(x, u)(w_t + v_t) - \beta v_t \tag{4.60}$$

in the Fourier space. We now want to get estimates independently of β, pass to the limit and find solutions of (1.1). Now multiplying by $v_t(\beta)$ and integrating we have

$$\beta(v_t, v_t) + \beta(v_{tt}, v_{tt}) + (f'_u(x,u)v_t, v_t) = -(f'_u(x,u)w_t, v_t) \tag{4.61}$$

and

$$\alpha||v_t||^2_{L^2} < (f'_u(x,u)v_t, v_t) \leq -(f'_u(x,u)w_t, v_t). \tag{4.62}$$

Since $f'_u > \alpha > 0$ and $||w||_{H^1} \leq c$ with c independent of β, there is a constant c independent of β such that $||v_t||_{L^2} \leq c$. This combined with (4.58) implies $||w||_{H^2} \leq c$, where c is independent of β. Differentiating (4.60) with reference to t we get

$$\begin{aligned}\beta v_{tt} + \Box w_{tt} - \beta v_{tttt} + f'_u(x,u)v_{tt} = &- f''_{uu}(x,u)v_t^2 - f''_{uu}(x,u)w_t^2 - 2f''_{uu}(x,u)w_t v_t \\ &- f'_u(x,u)w_{tt}.\end{aligned} \tag{4.63}$$

Now we multiply (4.63) by v_{tt} and estimate the L^2 norm of the first term of the RHS:

$$\begin{aligned}(f''_{uu}(x,u)v_t^2, v_{tt}) &\leq c(f)\int_0^\pi\int_0^{2\pi} v_t^2|v_{tt}|dxdt \\ &\leq c(f)(\int_0^\pi\int_0^{2\pi} v_t^4 dxdt)^{\frac{1}{2}}(\int_0^\pi\int_0^{2\pi}|v_{tt}|^2 dxdt).\end{aligned}$$

We then deduce

$$\begin{aligned}(f''_{uu}(x,u)v_t^2, v_{tt}) &\leq c(f)||v_t||^{\frac{3}{2}}_{L^2}||v_t||^{\frac{1}{2}}_{H^1} \\ &\leq c(f)||v_t||^{\frac{3}{2}}_{H^1},\end{aligned} \tag{4.64}$$

where the constant $c(f)$ is independent of β and the inequalities in the previous argument stem from the Gagliardo-Nirenberg inequality.

The L^1 norms of the terms in the RHS of (4.63) multiplied by v_{tt} can be estimated by noting that $f(u) \in H^1$,$w \in C^{1,\gamma}$,$0 < \gamma < \frac{1}{2}$, and that the respective norms can be estimated independently of β:

$$(f''_{uu}(x,u)w_t^2, v_{tt}) \leq c||v_{tt}||_{L^2}, \tag{4.65}$$

$$(2f''_{uu}(x,u)w_t, v_{tt}) \leq c||v_{tt}||_{L^2}, \tag{4.66}$$

$$-(f'_u(x,u)w_{tt}, v_{tt}) \leq c||w_{tt}||_{L^2}||v_{tt}||_{L^2}. \tag{4.67}$$

Recalling (4.63) and multiplying by v_{tt},

$$\begin{aligned}\beta(v_{tt}, v_{tt}) + \beta(v_{ttt}, v_{ttt}) + (f'_u(x,u)v_{tt}, v_{tt}) = &(-f''_{uu}(x,u)v_t, v_{tt}) \\ &+(-f''_{uu}(x,u)w_t, v_{tt}) \\ &+(-f'_u(x,u)w_{tt}, v_{tt}).\end{aligned} \tag{4.68}$$

We can now continue from (4.63), (4.65), (4.65), (4.66), and (4.67), and we have

$$\beta(v_{tt}, v_{tt}) + \beta(v_{ttt}, v_{ttt}) + (f'_u(x,u)v_{tt}, v_{tt}) \leq c||v_t||^{\frac{3}{2}}_{H^1}. \tag{4.69}$$

Thus there exists c independent of β such that $||v_{tt}||_{L^2} \leq c$, where c is independent of β. At this stage we can conclude that there is a constant c independent of β such that $||f(u)||_{H^2} \leq c$. Combining this with (4.58) we have $||w||_{H^3} \leq c$ with c independent of β, and $w \in C^{2,\frac{1}{2}}$ and $v \in C^1$ with upper bounds independent of β.

We have now proved that if f is C^2, then the solution is that $u \in H^2 \cap C^1$ is a weak solution of the equation. We now differentiate (4.63) and have

$$\begin{aligned}\beta v_{ttt} + \Box w_{ttt} + f'(u)v_{ttt} \quad = \quad & -f''_{uu}(x,u)v_{tt} - f'''_{uuu}(x,u)(v_t + w_t)v_t^2 \\ & -f''_{uu}(x,u)2v_t w_{tt} - f'''_{uuu}(x,u)w_t^2 \\ & -f''_{uu}(x,u)2w_t w_{tt} - 2f'''_{uuu}(x,u)(v_t + w_{tt})w_t v_t \\ & -2f''_{uu}(x,u)w_{tt}v_t - 2f''_{uu}(x,u)w_t v_{tt} \\ & -f''_{uu}(x,u)(v_t + w_t)w_{tt} - f''_{uu}(x,u)w_{ttt}.\end{aligned}$$

Multiplying both sides of the preceding equality by v_{ttt} and integrating we conclude that $||v_{ttt}||_{L^2} \leq c$, where c is independent of β; thus v is C^2. Now recalling the Hölder regularity bootstrap and (4.58) we get $w \in C^{3,\gamma}$, $0 < \gamma < \frac{1}{2}$.

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