

## SUBGROUP GROWTH IN SOME PROFINITE CHEVALLEY GROUPS

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(Communicated by Pham Huu Tiep)

**ABSTRACT.** In this article we improve the known uniform bound for subgroup growth of Chevalley groups  $\mathbf{G}(\mathbb{F}_p[[t]])$ . We introduce a new parameter, the ridgeline number  $v(\mathbf{G})$ , and give new bounds for the subgroup growth of  $\mathbf{G}(\mathbb{F}_p[[t]])$  expressed through  $v(\mathbf{G})$ . We achieve this by deriving a new estimate for the codimension of  $[U, V]$  where  $U$  and  $V$  are vector subspaces in the Lie algebra of  $\mathbf{G}$ .

For a finitely generated group  $G$ , let  $a_n(G)$  be the number of subgroups of  $G$  of index  $n$  and  $s_n(G)$  the number of subgroups of  $G$  of index at most  $n$ . The “subgroup growth” of  $G$  is the asymptotic behaviour of the sequence  $(a_n(G))_{n \in \mathbb{N}}$ . It turns out that the subgroup growth and structure of  $G$  are not unrelated, and in fact the latter can sometimes be used as a characterisation of  $G$ . (For complete details we refer the reader to a book by Lubotzky and Segal [LSe03].)

For example, Lubotzky and Mann [LM91] show that a group  $G$  is  $p$ -adic analytic if and only if there exists a constant  $c > 0$  such that  $a_n(G) < n^c$ . This inspiring result is followed by Shalev who proves that if  $G$  is a pro- $p$  group for which  $a_n(G) \leq n^{c \log_p n}$  for some constant  $c < \frac{1}{8}$ , then  $G$  is  $p$ -adic analytic.

Answering a question of Mann, Barnea and Guralnick investigate the subgroup growth of  $SL_2^1(\mathbb{F}_p[[t]])$  for  $p > 2$ , and show that the supremum of the set of all those  $c$  that a pro- $p$  group  $G$  is  $p$ -adic analytic provided that  $a_n(G) < n^{c \log_p n}$ , is no larger than  $\frac{1}{2}$ . Thus one may see that not only the growth type, but also the precise values of the constants involved are important when studying the connection between subgroup growth and the structure of a group.

Later on Lubotzky and Shalev pioneer a study of the so-called  $\Lambda$ -standard groups [LSh94]. A particular subclass of these groups are  $\Lambda$ -perfect groups for which they show existence of a constant  $c > 0$  such that

$$a_n(G) < n^{c \log_p n}.$$

An important subclass of those groups are the congruence subgroups of Chevalley groups over  $\mathbb{F}_p[[t]]$ . Let  $\mathbf{G}$  be a simple simply-connected Chevalley group scheme,  $G(1)$  the first congruence subgroup of  $\mathbf{G}(\mathbb{F}_p[[t]])$ . Abért, Nikolov and Szegedy show

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Received by the editors November 13, 2015 and, in revised form, August 31, 2016, October 25, 2016 and November 2, 2016.

2010 *Mathematics Subject Classification.* Primary 20E07; Secondary 17B45, 17B70.

*Key words and phrases.* Subgroup growth, Lie algebras, Chevalley groups.

that if  $m$  is the dimension of  $\mathbf{G}$ , then

$$s_{p^k}(G(1)) \leq p^{\frac{7}{2}k^2 + mk},$$

that is,  $s_n(G(1)) \leq n^{\frac{7}{2} \log_p n + m}$  [ANS03].

In this article we improve their estimates (cf. [ANS03]). The Lie algebra  $\mathfrak{g}_{\mathbb{Z}}$  of  $\mathbf{G}$  is defined over integers. Let  $\mathbb{K}$  be a field of characteristic  $p$ , which could be either zero or a prime. To state the results we now introduce a new parameter of the Lie algebra  $\mathfrak{g} := \mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{K}$ . We fix an invariant bilinear form  $\eta = \langle \cdot, \cdot \rangle$  on  $\mathfrak{g}$  of maximal possible rank. Let  $\mathfrak{g}^0$  be its kernel. Notice that the nullity  $r := \dim \mathfrak{g}^0$  of  $\eta$  is independent of the choice of  $\eta$ .

**Definition 0.1.** Let  $l$  be the rank of  $\mathfrak{g}$ ,  $m$  its dimension and  $s$  the maximal dimension of the centraliser of a non-central element  $\mathbf{x} \in \mathfrak{g}$ . We define the *ridgeline number* of  $\mathfrak{g}$  as

$$v(\mathbf{G}) = v(\mathfrak{g}) := \frac{l}{m - s - r}.$$

We discuss ridgeline numbers in Section 2. The values of  $v(\mathfrak{g})$  can be found in the table in Appendix A.

**Definition 0.2.** The positive characteristic  $p$  of the field  $\mathbb{K}$  is called *good* if  $p$  does not divide the coefficients of the highest root. The positive characteristic  $p$  of the field  $\mathbb{K}$  is called *very good* if  $p$  is good and  $\mathfrak{g}$  is simple. We call the positive characteristic  $p$  *tolerable* if any proper ideal of  $\mathfrak{g}$  is contained in its centre.

We discuss these restrictions in Section 2. We may now state our main result.

**Theorem 0.3.** *Let  $\mathbf{G}$  be a simple simply-connected Chevalley group scheme of rank  $l \geq 2$ . Suppose  $p$  is a tolerable prime for  $\mathbf{G}$ . Let  $G(1)$  be the first congruence subgroup of  $\mathbf{G}(\mathbb{F}_p[[t]])$ , that is,  $G(1) = \ker(\mathbf{G}(\mathbb{F}_p[[t]]) \rightarrow \mathbf{G}(\mathbb{F}_p))$ . If  $m := \dim \mathbf{G}$ , then*

$$a_{p^k}(G(1)) \leq p^{\frac{(3+4v(\mathfrak{g}))}{2}k^2 + (m - \frac{3}{2} - 2v(\mathfrak{g}))k}.$$

*If  $l = 2$  and  $p$  is very good, then a stronger estimate holds:*

$$a_{p^k}(G(1)) \leq p^{\frac{3}{2}k^2 + (m - \frac{3}{2})k}.$$

Notice that as one can see from the table in Appendix A, with one exception (when  $\mathbf{G} = A_l$  and  $p$  divides  $l + 1$ ) the largest possible value of  $v(\mathfrak{g})$  is  $\frac{1}{2}$  ( $v(\mathfrak{g}) \leq \frac{2}{3}$  in that special case). This makes  $\frac{3+4v(\mathfrak{g})}{2} \leq \frac{5}{2}$  ( $\frac{3+4v(\mathfrak{g})}{2} \leq \frac{17}{6}$  correspondingly).

Our proof of Theorem 0.3 in many ways follows the ones of Barnea and Guralnick and of Abért, Nikolov and Szegedy. The improvement in the result is due to the following new estimates.

**Theorem 0.4.** *Let  $\mathfrak{a}$  be a Lie algebra over a field  $\mathbb{K}$ . Suppose that the Lie algebra  $\mathfrak{g} = \mathfrak{a} \otimes_{\mathbb{K}} \overline{\mathbb{K}}$  is a Chevalley Lie algebra of rank  $l \geq 2$  and that the characteristic of  $\mathbb{K}$  is zero or tolerable. Then for any two subspaces  $U$  and  $V$  of  $\mathfrak{a}$ , we have*

$$\text{codim}([U, V]) \leq (1 + v(\mathfrak{g}))(\text{codim}(U) + \text{codim}(V)).$$

*If  $l = 2$  and the characteristic of  $\mathbb{K}$  is zero or very good, a stronger result holds:*

$$\text{codim}([U, V]) \leq \text{codim}(U) + \text{codim}(V).$$

We conjecture that the second estimate holds for any Lie algebra  $\mathfrak{g}$  (if the characteristic of  $\mathbb{K}$  is zero or very good).

1. PROOF OF THEOREM 0.3

The proof of Theorem 0.3 relies on Theorem 0.4 that will be proved later. We follow Abért, Nikolov and Szegedy [ANS03, Theorem 2] and Barnea, Guralnick [BG01, Theorem 1.4].

Suppose that hypotheses of Theorem 0.3 hold. We start with the following observation (cf. [ANS03, Corollary 1 and Lemma 1] and [LSh94, Lemma 4.1]).

**Lemma 1.1.** *If  $H$  is an open subgroup of  $G(1)$  and  $d(H)$  is the minimal number of generators of  $H$ , then*

$$d(H) \leq m + (3 + 4v(\mathfrak{g})) \log_p |G(1) : H|.$$

Moreover, if  $l = 2$  and  $p$  is very good, then

$$d(H) \leq m + 3 \log_p |G(1) : H|.$$

Notice that in the second case  $\mathfrak{g} = A_2, C_2$  or  $G_2$  and  $m = 8, 10$  or  $14$  correspondingly.

*Proof.* First of all recall that  $d(H) = \log_p |H : \Phi(H)| \leq \log_p |H : H'|$  where  $\Phi(H)$  is the Frattini subgroup. Because of the correspondence between the open subgroups of  $G(1)$  and subalgebras of its graded Lie algebra  $\mathcal{L} = L(G(1))$  (see [LSh94]),  $\log_p |H : H'| \leq \dim \mathcal{H}/\mathcal{H}'$  where  $\mathcal{H} = L(H)$  is the corresponding subalgebra of  $\mathcal{L}$ . Hence it suffices to show that

$$\dim \mathcal{H}/\mathcal{H}' \leq m + (3 + 4v(\mathfrak{g})) \dim \mathcal{L}/\mathcal{H}$$

in the general case, and that

$$\dim \mathcal{H}/\mathcal{H}' \leq m + 3 \dim \mathcal{L}/\mathcal{H}$$

in the very good rank 2 case.

Recall that the graded Lie algebra  $\mathcal{L}$  is isomorphic to  $\mathfrak{g} \otimes_{\mathbb{F}} t\mathbb{F}[t]$  where  $\mathbb{F} = \mathbb{F}_p$ . Since every element  $a \in \mathcal{L}$  can be uniquely written as  $a = \sum_{i=1}^{\infty} a_i \otimes t^i$  with  $a_i \in \mathfrak{g}$ , one can define  $l(a) := a_s$  where  $s$  is the smallest integer such that  $a_s \neq 0$ , and in this case  $s := \deg(a)$ . Now set

$$H_i := \langle l(a) \mid a \in \mathcal{H} \text{ with } \deg(a) = i \rangle.$$

Observe that  $H_i = \{l(a) \mid a \in \mathcal{H} \text{ with } \deg(a) = i\} \cup \{0\}$ . Then  $\dim \mathcal{L}/\mathcal{H} = \sum_{i=1}^{\infty} \dim \mathfrak{g}/H_i$ , and this sum is finite as the left hand side is finite. Then

$$[H_i \otimes t^i, H_j \otimes t^j] \subseteq [H_i, H_j] \otimes t^{i+j} \subseteq H'_{i+j} \otimes t^{i+j}$$

where  $H'_{i+j} := \langle l(a) \mid a \in \mathcal{H}' \text{ with } \deg(a) = i + j \rangle$ , and so  $\dim \mathfrak{g}/[H_i, H_j] \geq \dim \mathfrak{g}/H'_{i+j}$ . Adding up these inequalities for  $i = j$  and  $i = j + 1$  we get

$$\dim \mathcal{L}/\mathcal{H}' = \sum_{i=1}^{\infty} \dim L/H'_i \leq \dim \mathfrak{g} + \sum_{1 \leq i < j \leq i+1} \dim \mathfrak{g}/[H_i, H_j].$$

Now we use the estimates of Theorem 0.4:

$$\dim \mathcal{L}/\mathcal{H}' \leq m + \sum_{1 \leq i < j \leq i+1} \alpha(\dim \mathfrak{g}/H_i + \dim \mathfrak{g}/H_j) \leq m + 4\alpha \dim \mathcal{L}/\mathcal{H},$$

where  $\alpha = 1 + v(\mathfrak{g})$  or  $1$  depending on the rank of  $\mathfrak{g}$  and  $p$ . The result follows immediately. □

Now we apply an estimate [LS94, Lemma 4.1]:  $a_{p^k}(G(1)) \leq p^{g_1 + \dots + g_{p^k-1}}$  where

$$g_{p^i} = g_{p^i}(G(1)) = \max\{d(H) \mid H \leq_{\text{open}} G(1), |G(1) : H| = p^i\}.$$

Using Lemma 1.1, in the general case ( $l \geq 2$ ) we have

$$a_{p^k}(G(1)) \leq p^{\sum_{i=0}^{k-1} m + (3+4v(\mathfrak{g}))i} = p^{\frac{(3+4v(\mathfrak{g}))}{2}k^2 + (m - \frac{3}{2} - 2v(\mathfrak{g}))k}.$$

For  $l = 2$  and very good  $p$ , Lemma 1.1 gives us

$$a_{p^k}(G(1)) \leq p^{\sum_{i=0}^{k-1} m + 3i} = p^{\frac{3}{2}k^2 + (m - \frac{3}{2})k}.$$

This finishes the proof of the theorem.

## 2. RIDGELINE NUMBERS AND SMALL PRIMES

We adopt the notation of Definition 0.1. We prove that  $m - s = 2(h^\vee - 1)$  where  $h^\vee$  is the dual Coxeter number of  $\mathfrak{g}$  (see Proposition 3.4). Therefore,

$$v(\mathfrak{g}) = \frac{l}{2(h^\vee - 1) - r}.$$

We present the values of  $v(\mathfrak{g})$  in Appendix A. We include only Lie algebras in tolerable characteristics (see Definition 0.2) where our method produces new results.

Let us remind the reader that the very good characteristics are  $p \nmid l + 1$  in type  $A_l$ ,  $p \neq 2$  in types  $B_l$ ,  $C_l$ ,  $D_l$ ,  $p \neq 2, 3$  in types  $E_6$ ,  $E_7$ ,  $F_4$ ,  $G_2$ , and  $p \neq 2, 3, 5$  in type  $E_8$ . If  $p$  is very good, the Lie algebra  $\mathfrak{g}$  behaves as in characteristic zero. In particular,  $\mathfrak{g}$  is simple, its Killing form is non-degenerate, etc. Let us contemplate what calamities betide the Lie algebra  $\mathfrak{g}$  in small characteristics.

Suppose that  $p$  is tolerable but not very good. If  $p$  does not divide the determinant of the Cartan matrix of  $\mathfrak{g}$ , the Lie algebra  $\mathfrak{g}$  is simple. This covers the following primes:  $p = 2$  in types  $E_6$  and  $G_2$ ,  $p = 3$  in types  $E_7$  and  $F_4$ ,  $p = 2, 3, 5$  in type  $E_8$ . In this scenario, the  $\mathfrak{g}$ -modules  $\mathfrak{g}$  and  $\mathfrak{g}^*$  are isomorphic, which immediately gives us a non-degenerate invariant bilinear form on  $\mathfrak{g}$  [H95, 0.13].

If  $p$  divides the determinant of the Cartan matrix of  $\mathfrak{g}$ , there is more than one Chevalley Lie algebra. We study the simply-connected Lie algebra  $\mathfrak{g}$ , i.e.,  $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$  and  $\mathfrak{g}/\mathfrak{z}$  is simple (where  $\mathfrak{z}$  is the centre). There is a canonical map to the adjoint Lie algebra  $\mathfrak{g}^b$ :

$$\varphi : \mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha} \mathfrak{g}_{\alpha} \rightarrow \mathfrak{g}^b = \mathfrak{h}^b \oplus \bigoplus_{\alpha} \mathfrak{g}_{\alpha}.$$

The map  $\varphi$  is the identity on the root spaces  $\mathfrak{g}_{\alpha}$ . Let us describe it on the Cartan subalgebras. The basis of the Cartan subalgebra  $\mathfrak{h}$  are the simple coroots  $\mathbf{h}_i = \alpha_i^\vee = [\mathbf{e}_{\alpha_i}, \mathbf{e}_{-\alpha_i}]$ . The basis of the Cartan subalgebra  $\mathfrak{h}^b$  are the fundamental coweights  $\mathbf{y}_i = \varpi_i^\vee$  defined by  $\alpha_i(\mathbf{y}_j) = \delta_{i,j}$ . Now the map  $\varphi$  on the Cartan subalgebras is given by

$$\varphi(\mathbf{h}_i) = \sum_j c_{j,i} \mathbf{y}_j$$

where  $c_{j,i}$  are entries of the Cartan matrix of the coroot system of  $\mathfrak{g}$ . The image of  $\varphi$  is  $[\mathfrak{g}^b, \mathfrak{g}^b]$ . The kernel of  $\varphi$  is the centre  $\mathfrak{z}$ . From our description  $\mathfrak{z}$  is the subspace of  $\mathfrak{h}$  equal to the null space of the Cartan matrix. It is equal to  $\mathfrak{g}^0$ , the kernel of  $\eta$ . The dimension of  $\mathfrak{z}$  is at most 2 (see the values of  $r$  in Appendix A).

The key dichotomy now is whether the Lie algebra  $\mathfrak{g}/\mathfrak{z}$  is simple or not. If  $\mathfrak{g}$  is simply-laced, the algebra  $\mathfrak{g}/\mathfrak{z}$  is simple. This occurs when  $p \mid l + 1$  in type  $A_l$ ,  $p = 2$  in types  $D_l$  and  $E_7$ ,  $p = 3$  in type  $E_6$ . Notice that  $A_1$  in characteristic 2 needs to

be excluded:  $\mathfrak{g}/\mathfrak{z}$  is abelian rather than simple. In this scenario the  $\mathfrak{g}$ -modules  $\mathfrak{g}/\mathfrak{z}$  and  $(\mathfrak{g}/\mathfrak{z})^*$  are isomorphic. This gives us an invariant bilinear form with the kernel  $\mathfrak{z}$  [H95, 0.13].

Let us look meticulously at  $\mathfrak{g}$  of type  $D_l$  when  $p = 2$ . The standard representation gives a homomorphism of Lie algebras

$$\rho : \mathfrak{g} \rightarrow \mathfrak{so}_{2l}(\mathbb{K}), \quad \mathbf{x} \mapsto \rho(\mathbf{x}) = \begin{pmatrix} \rho_{11}(\mathbf{x}) & \rho_{12}(\mathbf{x}) \\ \rho_{21}(\mathbf{x}) & \rho_{22}(\mathbf{x}) \end{pmatrix},$$

where  $\rho_{22}(\mathbf{x}) = \rho_{11}(\mathbf{x})^t$ , while  $\rho_{12}(\mathbf{x})$  and  $\rho_{21}(\mathbf{x})$  are skew-symmetric  $l \times l$ -matrices, which for  $p = 2$  is equivalent to symmetric with zeroes on the diagonal. The Lie algebra  $\mathfrak{so}_{2l}(\mathbb{K})$  has a 1-dimensional centre spanned by the identity matrix. If  $l$  is odd,  $\rho$  is an isomorphism, and  $\mathfrak{g}$  has a 1-dimensional centre. However, if  $l$  is even,  $\rho$  has a 1-dimensional kernel, and  $\mathfrak{g}$  has a 2-dimensional centre.

It is instructive to observe how the standard representation  $\rho$  equips  $\mathfrak{g}$  with an invariant form. A skew-symmetric matrix  $Z$  can be written uniquely as a sum  $Z = Z^L + Z^U$ , where  $Z^L$  is strictly lower triangular and  $Z^U$  is strictly upper triangular. Then the bilinear form is given by

$$\eta(\mathbf{x}, \mathbf{y}) := \langle \rho(\mathbf{x}), \rho(\mathbf{y}) \rangle := \text{Tr}(\rho_{11}(\mathbf{x})\rho_{11}(\mathbf{y}) + \rho_{12}(\mathbf{x})^L \rho_{21}(\mathbf{y})^U + \rho_{21}(\mathbf{x})^L \rho_{12}(\mathbf{y})^U).$$

This form  $\eta$  is a reduction of the form  $\frac{1}{2}\text{Tr}(\varphi(\mathbf{x})\varphi(\mathbf{y}))$  on  $\mathfrak{so}_{2l}(\mathbb{Z})$ , hence it is invariant.

Finally we suppose that  $p$  is not tolerable. This happens when  $p = 2$  in types  $B_l, C_l$  and  $F_4$  or  $p = 3$  in type  $G_2$ . In all these cases  $\mathfrak{g}$  is not simply-laced and the quotient algebra  $\mathfrak{g}/\mathfrak{z}$  is not simple. The short root vectors generate a proper non-central ideal  $I$ . This ideal sits in the kernel of any non-zero invariant form. Consequently, our method fails to produce any new result.

### 3. PROOF OF THEOREM 0.4: THE GENERAL CASE

Let  $\mathfrak{a}$  be an  $m$ -dimensional Lie algebra over a field  $\mathbb{K}$  of characteristic  $p$  (prime or zero). We consider it as a topological space in the Zariski topology. We also consider a function  $\dim \circ \mathfrak{c} : \mathfrak{a} \rightarrow \mathbb{R}$  that for an element  $\mathbf{x} \in \mathfrak{a}$  computes the dimension of its centraliser  $\mathfrak{c}(\mathbf{x})$ .

**Lemma 3.1.** *The function  $\dim \circ \mathfrak{c}$  is upper semicontinuous, i.e., for any number  $n$  the set  $\{\mathbf{x} \in \mathfrak{a} \mid \dim(\mathfrak{c}(\mathbf{x})) \leq n\}$  is Zariski open.*

*Proof.* Observe that  $\mathfrak{c}(\mathbf{x})$  is the kernel of the adjoint operator  $\text{ad}(\mathbf{x})$ . Thus,  $\dim(\mathfrak{c}(\mathbf{x})) \leq n$  is equivalent to  $\text{rank}(\text{ad}(\mathbf{x})) \geq m - n$ . This is clearly an open condition, given by the non-vanishing of one of the  $(m - n)$ -minors.  $\square$

Now we move to  $\overline{\mathbb{K}}$ , the algebraic closure of  $\mathbb{K}$ . Let  $\bar{\mathfrak{a}} = \mathfrak{a} \otimes_{\mathbb{K}} \overline{\mathbb{K}}$ . To distinguish centralisers in  $\mathfrak{a}$  and  $\bar{\mathfrak{a}}$  we denote  $\mathfrak{c}(\mathbf{x}) := \mathfrak{c}_{\mathfrak{a}}(\mathbf{x})$  and  $\bar{\mathfrak{c}}(\mathbf{x}) := \mathfrak{c}_{\bar{\mathfrak{a}}}(\mathbf{x})$ . Now we assume that  $\bar{\mathfrak{a}}$  is the Lie algebra of a connected algebraic group  $\mathcal{A}$ . Let  $\text{Orb}(\mathbf{x})$  be the  $\mathcal{A}$ -orbit of an element  $\mathbf{x} \in \bar{\mathfrak{a}}$ .

**Lemma 3.2.** *Let  $\mathbf{x}$  and  $\mathbf{y}$  be elements of  $\bar{\mathfrak{a}}$  such that  $\mathbf{x} \in \overline{\text{Orb}(\mathbf{y})}$ . Then  $\dim \bar{\mathfrak{c}}(\mathbf{x}) \geq \dim \bar{\mathfrak{c}}(\mathbf{y})$ .*

*Proof.* The orbit  $\text{Orb}(\mathbf{y})$  intersects any open neighbourhood of  $\mathbf{x}$ , and, in particular, the set  $X = \{\mathbf{z} \in \bar{\mathfrak{a}} \mid \dim(\bar{\mathfrak{c}}(\mathbf{z})) \leq \dim(\bar{\mathfrak{c}}(\mathbf{x}))\}$ , which is open by Lemma 3.1. If  $\mathbf{z} \in \text{Orb}(\mathbf{y}) \cap X$ , then  $\dim \bar{\mathfrak{c}}(\mathbf{x}) \geq \dim \bar{\mathfrak{c}}(\mathbf{z}) = \dim \bar{\mathfrak{c}}(\mathbf{y})$ .  $\square$

The stabiliser subscheme  $\mathcal{A}_x$  is, in general, non-reduced in positive characteristic. It is reduced (equivalently, smooth) if and only if the inclusion  $\mathfrak{c}(x) \supseteq \text{Lie}(\mathcal{A}_x)$  is an equality (cf. [H95, 1.10]). If  $\mathcal{A}_x$  is smooth, the orbit-stabiliser theorem implies that

$$\dim(\mathfrak{a}) = m = \dim \mathcal{A}_x + \dim \text{Orb}(x) = \dim \bar{\mathfrak{c}}(x) + \dim \text{Orb}(x).$$

In particular, Lemma 3.2 follows from the inequality  $\dim \text{Orb}(x) \leq \dim \text{Orb}(y)$ .

Let us further assume that  $\mathcal{A} = \mathcal{G}$  is a simple connected simply-connected algebraic group and  $\bar{\mathfrak{a}} = \mathfrak{g}$  is a simply-connected Chevalley Lie algebra. Let us fix a triangular decomposition  $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}$ . An element  $x \in \mathfrak{g}$  is called *semisimple* if  $\text{Orb}(x) \cap \mathfrak{h} \neq \emptyset$ . An element  $x \in \mathfrak{g}$  is called *nilpotent* if  $\text{Orb}(x) \cap \mathfrak{n} \neq \emptyset$ . We call a representation  $x = x_s + x_n$  a *quasi-Jordan decomposition* if  $x_s \in g(\mathfrak{h})$  (image of  $\mathfrak{h}$  under  $g$ ) and  $x_n \in g(\mathfrak{n})$  for the same  $g \in \mathcal{G}$ .

Recall that a *Jordan decomposition* is a quasi-Jordan decomposition  $x = x_s + x_n$  such that  $[x_s, x_n] = 0$ . A Jordan decomposition exists and is unique if  $\mathfrak{g}$  admits a non-degenerate bilinear form [KW, Theorem 4].

Notice that part (1) of the following lemma cannot be proved by the argument that the Lie subalgebra  $\mathbb{K}x$  is contained in a maximal soluble subalgebra: in characteristic 2 the Borel subalgebra  $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$  is not maximal soluble.

**Lemma 3.3.** *Assume that  $p \neq 2$  or  $\mathcal{G}$  is not of type  $C_l$  (in particular, this excludes  $C_2 = B_2$  and  $C_1 = A_1$ ). Then the following statements hold:*

- (1) *Every  $x \in \mathfrak{g}$  admits a (non-unique) quasi-Jordan decomposition  $x = x_s + x_n$ .*
- (2)  *$x_s$  belongs to the orbit closure  $\overline{\text{Orb}(x)}$ .*
- (3) *If  $\text{Orb}(x)$  is closed, then  $x$  is semisimple.*
- (4)  *$\dim \bar{\mathfrak{c}}(x_s) \geq \dim \bar{\mathfrak{c}}(x)$ .*

*Proof* (cf. [KW, Section 3]). (1) Our assumption on  $\mathfrak{g}$  assures the existence of a regular semisimple element  $h \in \mathfrak{h}$ , i.e., an element such that  $\bar{\mathfrak{c}}(h) = \mathfrak{h}$ . The differential  $d_{(e,h)}a : \mathfrak{g} \oplus \mathfrak{h} \rightarrow \mathfrak{g}$  of the action map  $a : \mathcal{G} \times \mathfrak{h} \rightarrow \mathfrak{g}$  is given by the formula

$$d_{(e,h)}a(x, k) = [x, h] + k.$$

Since the adjoint operator  $\text{ad}(h)$  is a diagonalizable operator whose 0-eigenspace is  $\mathfrak{h}$ , the kernel of  $d_{(e,x)}a$  is  $\mathfrak{h} \oplus 0$ . Hence, the image of  $a$  contains an open subset of  $\mathfrak{g}$ . Since the set  $\bigcup_{g \in \mathcal{G}} g(\mathfrak{b})$  contains the image of  $a$ , it is a dense subset of  $\mathfrak{g}$ .

Let  $\mathcal{B}$  be the Borel subgroup of  $\mathcal{G}$  whose Lie algebra is  $\mathfrak{b}$ . The quotient space  $\mathcal{F} = \mathcal{G}/\mathcal{B}$  is a flag variety. Since  $\mathcal{F}$  is projective, the projection map  $\pi : \mathfrak{g} \times \mathcal{F} \rightarrow \mathfrak{g}$  is proper. The Springer variety  $\mathcal{S} = \{(x, g(\mathcal{B})) \mid x \in g(\mathfrak{b})\}$  is closed in  $\mathfrak{g} \times \mathcal{F}$ . Hence,  $\bigcup_{g \in \mathcal{G}} g(\mathfrak{b}) = \pi(\mathcal{S})$  is closed in  $\mathfrak{g}$ . Thus,  $\bigcup_{g \in \mathcal{G}} g(\mathfrak{b}) = \mathfrak{g}$ . Choosing  $g$  such that  $x \in g(\mathfrak{b})$  gives a decomposition.

(2) Suppose  $x_s \in g(\mathfrak{h})$ . Let  $\mathcal{T}$  be the torus whose Lie algebra is  $g(\mathfrak{h})$ . We decompose  $x$  over the roots of  $\mathcal{T}$ :

$$x = x_s + x_n = x_0 + \sum_{\alpha \in Y(\mathcal{T})} x_\alpha.$$

We can choose a basis of  $Y(\mathcal{T})$  so that only positive roots appear. Hence, the action map  $a : \mathcal{T} \rightarrow \mathfrak{g}$ ,  $a(t) = t(x)$  extends along the embedding  $\mathcal{T} \hookrightarrow \mathbb{K}^l$  to a map  $\hat{a} : \mathbb{K}^l \rightarrow \mathfrak{g}$ . Observe that  $x_s = \hat{a}(0)$ .

Let  $U \ni x_s$  be an open subset of  $\mathfrak{g}$ . Then  $\hat{a}^{-1}(U)$  is open in  $\mathbb{K}^l$  and  $\mathcal{T} \cap \hat{a}^{-1}(U)$  is not empty. Pick  $t \in \mathcal{T} \cap \hat{a}^{-1}(U)$ . Then  $a(t) = t(x) \in U$ , thus,  $x_s \in \overline{\mathcal{T}(x)} \subseteq \overline{\text{Orb}(x)}$ .

- (3) This immediately follows from (1) and (2).
- (4) This immediately follows from (2) and Lemma 3.2. □

If  $\alpha$  is a long simple root, its root vector  $\mathbf{e}_\alpha \in \mathfrak{g} = \bar{\mathfrak{a}}$  is known as *the minimal nilpotent*. The dimension of  $\text{Orb}(\mathbf{e}_\alpha)$  is equal to  $2(h^\vee - 1)$  (cf. [W99]).

**Proposition 3.4.** *Suppose that  $l \geq 2$  and that the characteristic  $p$  of the field  $\mathbb{K}$  is tolerable for  $\mathfrak{g}$ . Then for any non-central  $\mathbf{x} \in \mathfrak{a}$ ,*

$$\dim \mathfrak{c}(\mathbf{x}) \leq \dim \bar{\mathfrak{c}}(\mathbf{e}_\alpha) = m - 2(h^\vee - 1).$$

*Proof.* Let  $\mathbf{x} \in \mathfrak{a}$  ( $\mathbf{y} \in \mathfrak{g}$ ) be a non-central element with  $\mathfrak{c}(\mathbf{x})$  ( $\bar{\mathfrak{c}}(\mathbf{y})$ ) correspondingly of the largest possible dimension. Observe that  $\dim \mathfrak{c}(\mathbf{x}) \leq \dim \bar{\mathfrak{c}}(\mathbf{x}) \leq \dim \bar{\mathfrak{c}}(\mathbf{y})$ .

Let us examine a quasi-Jordan decomposition  $\mathbf{y} = \mathbf{y}_s + \mathbf{y}_n$ . Since  $\mathbf{y}_s \in \overline{\text{Orb}(\mathbf{y})}$ , we conclude that  $\dim \bar{\mathfrak{c}}(\mathbf{y}_s) \geq \dim \bar{\mathfrak{c}}(\mathbf{y})$ . But  $\dim \bar{\mathfrak{c}}(\mathbf{y})$  is assumed to be maximal. There are two ways to reconcile this: either  $\dim \bar{\mathfrak{c}}(\mathbf{y}_s) = \dim \bar{\mathfrak{c}}(\mathbf{y})$ , or  $\mathbf{y}_s$  is central.

Suppose  $\mathbf{y}_s$  is central. Then  $\mathbf{y}$  and  $\mathbf{y}_n$  have the same centralisers. We may assume that  $\mathbf{y} = \mathbf{y}_n$  is nilpotent. Lemma 3.2 allows us to assume without loss of generality that the orbit  $\text{Orb}(\mathbf{y})$  is minimal, that is,  $\overline{\text{Orb}(\mathbf{y})} = \text{Orb}(\mathbf{y}) \cup \{0\}$ . On the other hand, the closure  $\overline{\text{Orb}(\mathbf{y})}$  contains a root vector  $\mathbf{e}_\beta$ .

Let us prove the last statement. First, observe that  $\mathbb{K}^\times \mathbf{y} \subseteq \text{Orb}(\mathbf{y})$ . If  $p$  is good, this immediately follows from Premet’s version of the Jacobson-Morozov Theorem [P95]. If  $\text{Orb}(\lambda \mathbf{y}) \neq \text{Orb}(\mathbf{y})$  in an exceptional Lie algebra in a bad tolerable characteristic, then we observe two distinct nilpotent orbits with the same partition into Jordan blocks. It never occurs: all the partitions are listed in the VIGRE paper [V05, section 6]. The remaining case of  $p = 2$  and  $\mathfrak{g}$  is of type  $D_l$  is also settled in the VIGRE paper [V05]. Now let  $\mathbf{y} \in \mathfrak{g}(\mathfrak{h})$ , and  $\mathcal{T}_0$  be the torus whose Lie algebra is  $\mathfrak{g}(\mathfrak{h})$ . Consider  $\mathcal{T} := \mathcal{T}_0 \times \mathbb{K}^\times$  with the second factor acting on  $\mathfrak{g}$  via the vector space structure. Write  $\mathbf{y} = \sum_{\beta \in Y(\mathcal{T}_0)} \mathbf{y}_\beta$  using the roots of  $\mathcal{T}_0$ . The closure of the orbit  $\overline{\mathcal{T}(\mathbf{y})}$  is contained in  $\overline{\text{Orb}(\mathbf{y})}$ . Let us show that  $\overline{\mathcal{T}(\mathbf{y})}$  contains one of  $\mathbf{y}_\beta$ . Let us write  $\mathcal{T}_0 = G_m \times G_m \times \dots \times G_m$  and decompose  $\mathbf{y} = \mathbf{y}_k + \mathbf{y}_{k+1} + \dots + \mathbf{y}_n$  using the weights of the first factor  $G_m$  with  $\mathbf{y}_k \neq 0$ . Then

$$\mathcal{T}(\mathbf{y}) \supseteq \{(\lambda, 1, 1, \dots, 1, \lambda^{-k}) \cdot \mathbf{y} \mid \lambda \in \mathbb{K}^\times\} = \{\mathbf{y}_k + \lambda^1 \mathbf{y}_{k+1} + \dots + \lambda^{n-k} \mathbf{y}_n \mid \lambda \in \mathbb{K}^\times\}.$$

Hence,  $\mathbf{y}_k \in \overline{\mathcal{T}(\mathbf{y})}$ . Repeat this argument with  $\mathbf{y}_k$  instead of  $\mathbf{y}$  for the second factor of  $\mathcal{T}_0$ , and so on. At the end we arrive at non-zero  $\mathbf{y}_\beta$ , hence,  $\mathbf{e}_\beta \in \overline{\text{Orb}(\mathbf{y})}$ .

Without loss of generality we now assume that  $\mathbf{y} = \mathbf{e}_\beta$  for a simple root  $\beta$ . If  $p$  is good, then  $\dim(\bar{\mathfrak{c}}(\mathbf{e}_\beta))$  does not depend on the field:

$$\bar{\mathfrak{c}}(\mathbf{e}_\beta) = \ker(d\beta : \mathfrak{h} \rightarrow \mathbb{K}) \oplus \bigoplus_{\gamma+\beta \text{ is not a root}} \mathfrak{g}_\gamma.$$

In particular, it is as in characteristic zero: the long root vector has a larger centraliser than the short root vector and  $\dim \bar{\mathfrak{c}}(\mathbf{y}) = \dim \bar{\mathfrak{c}}(\mathbf{e}_\alpha) = m - 2(h^\vee - 1)$  [W99]. If  $p = 2$  and  $\mathfrak{g}$  is of type  $D_l$ , then a direct calculation gives the same formula for  $\dim \bar{\mathfrak{c}}(\mathbf{e}_\alpha)$ . In the exceptional cases in bad characteristics, the orbits and their centralisers are computed in the VIGRE paper [V05]. One goes through their tables and establishes the formula for  $\dim \bar{\mathfrak{c}}(\mathbf{y})$  in all the cases.

Now suppose  $\dim \bar{\mathfrak{c}}(\mathbf{y}_s) = \dim \bar{\mathfrak{c}}(\mathbf{y})$ . We may assume that  $\mathbf{y} = \mathbf{y}_s$  is semisimple. Then  $\mathbf{y}$  is in some Cartan subalgebra  $g^{-1}(\mathfrak{h})$  and  $\dim \bar{\mathfrak{c}}(g(\mathbf{y})) = \dim \bar{\mathfrak{c}}(\mathbf{y})$ . Moreover,

$$\bar{\mathfrak{c}}(g(\mathbf{y})) = \mathfrak{h} \oplus \bigoplus_{\{\alpha | \alpha(g(\mathbf{y}))=0\}} \mathfrak{g}_\alpha$$

is a reductive subalgebra. If  $\varphi : \mathfrak{g} \rightarrow \mathfrak{g}^b$  is the canonical map (see Section 2), then  $\dim \bar{\mathfrak{c}}(g(\mathbf{y})) = \dim \mathfrak{c}_{\mathfrak{g}^b}(\varphi(g(\mathbf{y})))$ . It remains to examine the Lie algebras case by case and exhibit a non-zero element in  $\mathfrak{h}^b$  with the maximal dimension of centraliser. This is done in Appendix A. □

Now we can give a crucial dimension estimate for the proof of Theorem 0.4.

**Proposition 3.5.** *Let  $\mathfrak{a}$  be an  $m$ -dimensional Lie algebra with an associative bilinear form  $\eta$ , whose kernel  $\mathfrak{a}^0$  is the centre of  $\mathfrak{a}$ . Suppose  $r = \dim(\mathfrak{a}^0)$  and  $k \geq \dim(\mathfrak{c}(\mathbf{x}))$  for any non-central element  $\mathbf{x} \in \mathfrak{a}$ . Finally, let  $U, V$  be subspaces of  $\mathfrak{a}$  such that  $\dim(U) + \dim(V) > m + k + r$ . Then  $[U, V] = \mathfrak{a}$ .*

*Proof.* Suppose not. Let us consider the orthogonal complement  $W = [U, V]^\perp \neq \mathfrak{a}^0$  under the form  $\eta$ . Observe that  $U \subseteq [V, W]^\perp$  since  $\eta$  is associative. But  $W$  admits a non-central element  $\mathbf{x} \in W$  so that  $\dim(\mathfrak{c}(\mathbf{x})) \leq k$ . Hence

$$\dim([V, W]) \geq \dim(V) - k \quad \text{and} \quad \dim([V, W]^\perp) \leq m + k + r - \dim(V).$$

Inevitably,  $\dim(U) \leq m + k + r - \dim(V)$ . □

We may now prove the first part of Theorem 0.4. We use  $m, l, r$  and  $s$  as in Definition 0.1. If  $\dim(U) + \dim(V) > m + s + r$ , we are done by Proposition 3.5:

$$\text{codim}([U, V]) = 0 \leq (1 + v(\mathfrak{g}))(\text{codim}(U) + \text{codim}(V)).$$

Now we assume that  $\dim(U) + \dim(V) \leq m + s + r$ . It is known [ANS03] that

$$\text{codim}([U, V]) \leq l + \text{codim}(U) + \text{codim}(V).$$

It remains to notice that  $l = v(\mathfrak{g})(m - s - r) \leq v(\mathfrak{g})(\text{codim}(U) + \text{codim}(V))$ . The theorem is proved.

#### 4. PROOF OF THEOREM 0.4: RANK 2

In this section  $\mathbf{G}$  is a Chevalley group scheme of rank 2. The characteristic  $p$  of the field  $\mathbb{K}$  is zero or very good for  $\mathfrak{g}$ . Let  $\{\alpha, \beta\}$  be the set of simple roots of  $\mathfrak{g}$  with  $|\beta| \leq |\alpha|$ . If  $\mathfrak{g}$  is of type  $A_2$ , then  $\alpha$  and  $\beta$  have the same length. The group  $\mathcal{G} = \mathbf{G}(\overline{\mathbb{K}})$  acts on  $\mathfrak{g}$  via the adjoint action. By  $\mathfrak{c}(\mathbf{x})$  we denote the centraliser  $\mathfrak{c}_{\mathfrak{g}}(\mathbf{x})$  in this section. Let us summarise some standard facts about this adjoint action (cf. [H95]).

- (1) If  $\mathbf{x} \in \mathfrak{g}$ , the stabiliser  $\mathcal{G}_{\mathbf{x}}$  is smooth, i.e., its Lie algebra is the centraliser  $\mathfrak{c}(\mathbf{x})$ .
- (2) The dimensions  $\dim(\text{Orb}(\mathbf{x})) = \dim(\mathcal{G}) - \dim(\mathfrak{c}(\mathbf{x}))$  and  $\dim(\mathfrak{c}(\mathbf{x}))$  are even.
- (3) If  $\mathbf{x} \neq 0$  is semisimple,  $\dim(\mathfrak{c}(\mathbf{x})) \in \{2, 4\}$ . Hence,  $\dim(\text{Orb}(\mathbf{x})) \in \{m - 2, m - 4\}$ .
- (4) A truly mixed element  $\mathbf{x} = \mathbf{x}_s + \mathbf{x}_n$  (with non-zero semisimple and nilpotent parts) is regular, i.e.,  $\dim(\mathfrak{c}(\mathbf{x})) = 2$  (cf. Lemma 3.3).
- (5)  $\mathbf{x}$  is nilpotent if and only if  $\text{Orb}(\mathbf{x})$  contains 0.
- (6) There is a unique orbit of regular nilpotent elements  $\text{Orb}(\mathbf{e}_r)$  where  $\mathbf{e}_r = \mathbf{e}_\alpha + \mathbf{e}_\beta$ . In particular,  $\dim(\mathfrak{c}(\mathbf{e}_r)) = 2$  and  $\dim(\text{Orb}(\mathbf{e}_r)) = m - 2$ .



(7) For two nilpotent elements  $\mathbf{x}$  and  $\mathbf{y}$  we write  $\mathbf{x} \succeq \mathbf{y}$  if  $\overline{\text{Orb}(\mathbf{x})} \supseteq \text{Orb}(\mathbf{y})$ . The following are representatives of all the nilpotent orbits in  $\mathfrak{g}$  (in brackets we report  $[\dim(\text{Orb}(\mathbf{x})), \dim(\mathfrak{c}(\mathbf{x}))]$ ):

(a) If  $\mathbf{G}$  is of type  $A_2$ , then

$$\mathbf{e}_r [6, 2] \succeq \mathbf{e}_\alpha [4, 4] \succeq 0 [0, 8].$$

(b) If  $\mathbf{G}$  is of type  $C_2$ , then  $\mathbf{e}_\alpha$  and  $\mathbf{e}_\beta$  are no longer in the same orbit and so we have

$$\mathbf{e}_r [8, 2] \succeq \mathbf{e}_\beta [6, 4] \succeq \mathbf{e}_\alpha [4, 6] \succeq 0 [0, 10].$$

(c) If  $\mathbf{G}$  is of type  $G_2$ , there is an additional subregular nilpotent orbit of an element  $\mathbf{e}_{sr} = \mathbf{e}_{2\alpha+3\beta} + \mathbf{e}_\beta$ . In this case we have

$$\mathbf{e}_r [12, 2] \succeq \mathbf{e}_{sr} [10, 4] \succeq \mathbf{e}_\beta [8, 6] \succeq \mathbf{e}_\alpha [6, 8] \succeq 0 [0, 14].$$

We will now prove Theorem 0.4 for groups of type  $A_2$ ,  $C_2$  and  $G_2$ . We need to show that if  $U$  and  $V$  are subspaces of  $\mathfrak{g}$ , then

$$(1) \quad \dim([U, V]) \geq \dim(U) + \dim(V) - \dim \mathfrak{g}.$$

We will use the following device repeatedly:

**Lemma 4.1.** *The inequality*

$$(2) \quad \dim([U, V]) \geq \dim(V) - \dim(V \cap \mathfrak{c}(\mathbf{x}))$$

*holds for any  $\mathbf{x} \in U$ . In particular, if there exists  $\mathbf{x} \in U$  such that*

$$\dim(U) + \dim(V \cap \mathfrak{c}(\mathbf{x})) \leq \dim \mathfrak{g},$$

*then inequality (1) holds.*

*Proof.* It immediately follows from the observation  $[U, V] \supseteq [\mathbf{x}, V] \cong V / (V \cap \mathfrak{c}(\mathbf{x}))$ . □

Now we give a case-by-case proof of inequality (1). Without loss of generality we assume that  $1 \leq \dim(U) \leq \dim(V)$  and that the field  $\mathbb{K}$  is algebraically closed.

4.1.  $\mathbf{G} = A_2$ . Using the standard facts, observe that if  $\mathbf{x} \in \mathfrak{g} \setminus \{0\}$ , then  $\dim(\mathfrak{c}(\mathbf{x})) \leq 4$ . Moreover, if  $\dim(\mathfrak{c}(\mathbf{x})) = 4$ , then either  $\mathbf{x} \in \text{Orb}(\mathbf{e}_\alpha)$ , or  $\mathbf{x}$  is semisimple. Since  $\dim \mathfrak{g} = 8$ , we need to establish that

$$\dim([U, V]) \geq \dim(U) + \dim(V) - 8.$$

Now we consider various possibilities.

*Case 1.* If  $\dim(U) \leq 4$ , then  $\dim(V \cap \mathfrak{c}(\mathbf{x})) \leq \dim(\mathfrak{c}(\mathbf{x})) \leq 4 \leq 8 - \dim(U)$  for any non-zero  $\mathbf{x} \in U$ . We are done by Lemma 4.1.

*Case 2.* If  $\dim(U) + \dim(V) > 12$ , then the hypotheses of Proposition 3.5 hold. Hence,  $[U, V] = \mathfrak{g}$  that obviously implies the desired conclusion.

Therefore we may suppose that  $\dim(U) + \dim(V) \leq 12$  and  $\dim U \geq 5$ . This leaves us with the following two cases.

*Case 3.*  $\dim(U) = 5$  and  $\dim(V) \leq 7$ . We need to show that

$$\dim([U, V]) \geq \dim(U) + \dim(V) - 8 = \dim(V) - 3.$$

As  $\dim(\overline{\text{Orb}(\mathbf{e}_\alpha)}) = 4$ , we may pick  $\mathbf{x} \in U$  with  $\mathbf{x} \notin \overline{\text{Orb}(\mathbf{e}_\alpha)}$ . If  $\mathbf{x}$  is regular, we are done by Lemma 4.1 since  $\dim(V \cap \mathfrak{c}(\mathbf{x})) \leq \dim(\mathfrak{c}(\mathbf{x})) = 2$ . If  $\mathbf{x}$  is not regular, then  $\dim(\mathfrak{c}(\mathbf{x})) = 4$  and  $\mathbf{x}$  is semisimple. In particular, its centraliser  $\mathfrak{c}(\mathbf{x})$  contains a Cartan subalgebra  $\mathfrak{g}(\mathfrak{h})$  of  $\mathfrak{g}$ .

Let us consider the intersection  $V \cap \mathfrak{c}(\mathbf{x})$ . If  $\dim(V \cap \mathfrak{c}(\mathbf{x})) \leq 3$ , we are done by Lemma 4.1. Otherwise,  $V \supseteq \mathfrak{c}(\mathbf{x})$  and  $V$  contains a regular semisimple element  $\mathbf{y} \in \mathfrak{g}(\mathfrak{h}) \subseteq V$ . If  $U \supseteq \mathfrak{c}(\mathbf{y}) = \mathfrak{g}(\mathfrak{h})$ , then  $U \ni \mathbf{y}$  and we are done by Lemma 4.1 as in the previous paragraph. Otherwise,  $\dim(U \cap \mathfrak{c}(\mathbf{y})) \leq 1$  and we finish the proof using Lemma 4.1:

$$\dim([U, V]) \geq \dim(U) - \dim(U \cap \mathfrak{c}(\mathbf{y})) \geq 5 - 1 = 4 \geq \dim(V) - 3.$$

*Case 4.*  $\dim(U) = \dim(V) = 6$ . This time we must show that

$$\dim([U, V]) \geq 4 = \dim(V) - 2.$$

By Lemma 4.1 it suffices to find a regular element in  $\mathbf{x} \in U$  (or in  $V$ ) since

$$\dim(V \cap \mathfrak{c}(\mathbf{x})) \leq \dim(\mathfrak{c}(\mathbf{x})) = 2.$$

Observe that

$$\dim(U \cap V) \geq \dim(U) + \dim(V) - 8 = 4 = \dim(\overline{\text{Orb}(\mathbf{e}_\alpha)}).$$

Since  $\overline{\text{Orb}(\mathbf{e}_\alpha)}$  is an irreducible algebraic variety and not an affine space, there exists  $\mathbf{x} \in U \cap V$  such that  $\mathbf{x} \notin \overline{\text{Orb}(\mathbf{e}_\alpha)}$ . If  $\mathbf{x}$  is regular, we are done. If  $\mathbf{x}$  is not regular,  $\mathbf{x}$  is semisimple and its centraliser  $\mathfrak{c}(\mathbf{x}) = \mathbb{K}\mathbf{x} \oplus \mathfrak{l}$ , a direct sum of Lie algebras  $\mathbb{K}\mathbf{x} \cong \mathbb{K}$  and  $\mathfrak{l} \cong \mathfrak{sl}_2(\mathbb{K})$ .

Consider the intersection  $V \cap \mathfrak{c}(\mathbf{x})$ . If  $\dim(V \cap \mathfrak{c}(\mathbf{x})) \leq 2$ , we are done by Lemma 4.1 as before. Assume that  $\dim(V \cap \mathfrak{c}(\mathbf{x})) \geq 3$ . If  $\dim(V \cap \mathfrak{c}(\mathbf{x})) = 4$ ,  $V$  contains  $\mathfrak{c}(\mathbf{x})$  and consequently a regular semisimple element  $\mathbf{y}$ .

Finally, consider the case  $\dim(V \cap \mathfrak{c}(\mathbf{x})) = 3$ . Let  $\pi_2$  be the natural projection  $\pi_2 : \mathfrak{c}(\mathbf{x}) \rightarrow \mathfrak{l}$  and set  $W := \pi_2(V \cap \mathfrak{c}(\mathbf{x}))$ . Since  $\mathbb{K}\mathbf{x} \subseteq V \cap \mathfrak{c}(\mathbf{x})$ , the subspace  $W$  of  $\mathfrak{sl}_2(\mathbb{K})$  is 2-dimensional. Clearly,  $V \cap \mathfrak{c}(\mathbf{x}) \subseteq \mathbb{K}\mathbf{x} \oplus W$ . Since both spaces have dimension 3,  $V \cap \mathfrak{c}(\mathbf{x}) = \mathbb{K}\mathbf{x} \oplus W$ . Then  $W = \mathbf{a}^\perp$  (with respect to the Killing form), where  $0 \neq \mathbf{a} \in \mathfrak{sl}_2(\mathbb{K})$  is either semisimple or nilpotent. In both cases  $W$  contains a non-zero nilpotent element  $\mathbf{z}$ . Thus, we have found a regular element  $\mathbf{x} + \mathbf{z} \in V \cap \mathfrak{c}(\mathbf{x})$ . This finishes the proof for  $A_2$ .

**4.2.  $\mathbf{G} = C_2$ .** Notice that this time  $\dim(\mathfrak{c}(\mathbf{x})) \leq 6$  for all  $0 \neq \mathbf{x} \in \mathfrak{g}$ . Moreover, if  $\dim(\mathfrak{c}(\mathbf{x})) = 6$ ,  $\mathbf{x} \in \overline{\text{Orb}(\mathbf{e}_\alpha)}$ . Finally, the set  $\overline{\text{Orb}(\mathbf{e}_\alpha)} = \text{Orb}(\mathbf{e}_\alpha) \cup \{0\}$  is a 4-dimensional cone, and the set  $\overline{\text{Orb}(\mathbf{e}_\beta)} = \text{Orb}(\mathbf{e}_\beta) \cup \text{Orb}(\mathbf{e}_\alpha) \cup \{0\}$  is a 6-dimensional cone.

As  $\dim \mathfrak{g} = 10$ , this time we need to show that

$$\dim([U, V]) \geq \dim(U) + \dim(V) - 10 = \dim(V) - (10 - \dim(U)).$$

*Case 1.*  $\dim(U) \leq 4$ . We are done by Lemma 4.1 since for any  $0 \neq \mathbf{x} \in U$ ,

$$\dim(V \cap \mathfrak{c}(\mathbf{x})) \leq \dim(\mathfrak{c}(\mathbf{x})) \leq 6 \leq 10 - \dim(U).$$

*Case 2.*  $5 \leq \dim(U) \leq 6$ . Hence, we may choose  $\mathbf{x} \in U$  such that  $\mathbf{x} \notin \overline{\text{Orb}(\mathbf{e}_\alpha)}$ . We are done by Lemma 4.1 since

$$\dim(V \cap \mathfrak{c}(\mathbf{x})) \leq \dim(\mathfrak{c}(\mathbf{x})) \leq 4 \leq 10 - \dim(U).$$

*Case 3.* If  $\dim(U) + \dim(V) > 16$ , then the hypotheses of Proposition 3.5 hold. Hence,  $[U, V] = \mathfrak{g}$ , which implies the desired conclusion.

Therefore, we may assume that  $\dim(U) + \dim(V) \leq 16$  and  $\dim(U) \geq 7$ . This leaves us with the remaining two cases.

*Case 4.*  $\dim(U) = 7, \dim(V) \leq 9$ . Now we must show that  $\dim([U, V]) \geq \dim(V) - 3$ . By Lemma 4.1 it suffices to pick  $\mathbf{x} \in U$  with  $\dim(V \cap \mathfrak{c}(\mathbf{x})) \leq 3$ . In particular, a regular element will do.

Let us choose  $\mathbf{x} \in U$  such that  $\mathbf{x} \notin \overline{\text{Orb}(\mathbf{e}_\beta)}$ . If  $\mathbf{x}$  is regular, we are done. If  $\mathbf{x}$  is not regular,  $\mathbf{x}$  is semisimple. Hence, its centraliser  $\mathfrak{c}(\mathbf{x})$  contains a Cartan subalgebra  $\mathfrak{g}(\mathfrak{h})$ . Let us consider the intersection  $V \cap \mathfrak{c}(\mathbf{x})$ . If  $\dim(V \cap \mathfrak{c}(\mathbf{x})) \leq 3$ , we are done again. Assume that  $\dim(V \cap \mathfrak{c}(\mathbf{x})) = 4$ . Consequently,  $V \supseteq \mathfrak{c}(\mathbf{x})$  and  $V$  contains a regular semisimple element  $\mathbf{y} \in \mathfrak{g}(\mathfrak{h}) \subseteq V$ . Now if  $U \supseteq \mathfrak{c}(\mathbf{y}) = \mathfrak{g}(\mathfrak{h})$ , then we have found a regular element  $\mathbf{y} \in U$ . Otherwise,  $\dim(U \cap \mathfrak{c}(\mathbf{y})) \leq 1$ , and so, as  $\mathbf{y} \in V$ , we finish using inequality (2) of Lemma 4.1:

$$\dim([U, V]) \geq \dim(U) - \dim(U \cap \mathfrak{c}(\mathbf{y})) \geq 7 - 1 = 6 \geq \dim(V) - 3.$$

*Case 5.*  $\dim(U) = \dim(V) = 8$ . Let us observe that

$$\dim(U \cap V) \geq \dim(U) + \dim(V) - 10 = 6 = \dim(\overline{\text{Orb}(\mathbf{e}_\beta)}).$$

Since  $\overline{\text{Orb}(\mathbf{e}_\beta)}$  is an irreducible algebraic variety and not an affine space, there exists  $\mathbf{x} \in U \cap V$  such that  $\mathbf{x} \notin \overline{\text{Orb}(\mathbf{e}_\beta)}$ . If  $\mathbf{x}$  is regular, we are done by Lemma 4.1:

$$\dim([U, V]) \geq \dim(V) - \dim(V \cap \mathfrak{c}(\mathbf{x})) \geq 8 - 2 = 6 = \dim(U) + \dim(V) - 10.$$

If  $\mathbf{x}$  is not regular, then  $\mathbf{x}$  is semisimple and its centraliser  $\mathfrak{c}(\mathbf{x}) = \mathbb{K}\mathbf{x} \oplus \mathfrak{l}$ , a direct sum of Lie algebras  $\mathbb{K}$  and  $\mathfrak{l} \cong \mathfrak{sl}_2(\mathbb{K})$ . If  $\dim(V \cap \mathfrak{c}(\mathbf{x})) \leq 2$ , then by Lemma 4.1

$$\dim([U, V]) \geq \dim(V) - \dim(V \cap \mathfrak{c}(\mathbf{x})) \geq 8 - 2 = 6.$$

Thus we may assume that  $\dim(V \cap \mathfrak{c}(\mathbf{x})) \geq 3$ . We now repeat the argument from the last paragraph of Section 4.1. This concludes Section 4.2.

**4.3.  $\mathbf{G} = G_2$ .** In this case  $\dim(\mathfrak{c}(\mathbf{x})) \leq 8$  for all  $0 \neq \mathbf{x} \in \mathfrak{g}$ . Moreover, if  $\dim(\mathfrak{c}(\mathbf{x})) = 8$ , then  $\mathbf{x} \in \text{Orb}(\mathbf{e}_\alpha)$ . The centre of  $\mathfrak{c}(\mathbf{e}_\alpha)$  is  $\overline{\mathbb{K}\mathbf{e}_\alpha}$ . Finally, the set  $\overline{\text{Orb}(\mathbf{e}_\alpha)} = \text{Orb}(\mathbf{e}_\alpha) \cup \{0\}$  is a 6-dimensional cone, the set  $\overline{\text{Orb}(\mathbf{e}_\beta)} = \text{Orb}(\mathbf{e}_\beta) \cup \text{Orb}(\mathbf{e}_\alpha) \cup \{0\}$  is an 8-dimensional cone and the set  $\overline{\text{Orb}(\mathbf{e}_{sr})} = \text{Orb}(\mathbf{e}_{sr}) \cup \text{Orb}(\mathbf{e}_\beta) \cup \text{Orb}(\mathbf{e}_\alpha) \cup \{0\}$  is a 10-dimensional cone.

As  $\dim \mathfrak{g} = 14$ , our goal now is to show that

$$\dim([U, V]) \geq \dim(U) + \dim(V) - 14.$$

In order to do so, as before, we are going to consider several mutually exclusive cases.

*Case 1.*  $\dim(U) \leq 6$ . We are done by Lemma 4.1 since for any  $0 \neq \mathbf{x} \in U$ ,

$$\dim(V \cap \mathfrak{c}(\mathbf{x})) \leq \dim(\mathfrak{c}(\mathbf{x})) \leq 8 \leq 14 - \dim(U).$$

*Case 2.*  $7 \leq \dim(U) \leq 8$ . In this case we may choose  $\mathbf{x} \in U$  such that  $\mathbf{x} \notin \overline{\text{Orb}(\mathbf{e}_\alpha)}$ . We are done by Lemma 4.1 since

$$\dim(V \cap \mathfrak{c}(\mathbf{x})) \leq \dim(\mathfrak{c}(\mathbf{x})) \leq 6 \leq 14 - \dim(U).$$

*Case 3.*  $9 \leq \dim(U) \leq 10$ . Now we may pick  $\mathbf{x} \in U$  such that  $\mathbf{x} \notin \overline{\text{Orb}(\mathbf{e}_\beta)}$ . Again we are done by Lemma 4.1 since

$$\dim(V \cap \mathfrak{c}(\mathbf{x})) \leq \dim(\mathfrak{c}(\mathbf{x})) \leq 4 \leq 14 - \dim(U).$$

*Case 4.* If  $\dim(U) + \dim(V) > 22$ , then  $[U, V] = \mathfrak{g}$  by Proposition 3.5. This leaves us with a single last possibility.

*Case 5.*  $\dim(U) = \dim(V) = 11$ . It remains to show that

$$\dim([U, V]) \geq 8 = \dim(V) - 3.$$

By dimension considerations we can choose  $\mathbf{x} \in U$  such that  $\mathbf{x} \notin \overline{\text{Orb}(\mathbf{e}_{sr})}$ . Then  $\dim(\mathfrak{c}(\mathbf{x})) \leq 4$ . If  $\dim(V \cap \mathfrak{c}(\mathbf{x})) \leq 3$ , we are done by Lemma 4.1. Thus we may assume that  $\dim(\mathfrak{c}(\mathbf{x})) = 4$  and  $\mathfrak{c}(\mathbf{x}) \subseteq V$ . Since  $\mathbf{x}$  is not nilpotent,  $\mathbf{x}$  must be semisimple. Hence,  $\mathfrak{c}(\mathbf{x}) \subseteq V$  contains a Cartan subalgebra  $\mathfrak{g}(\mathfrak{h})$  and, therefore, a regular semisimple element  $\mathbf{y} \in \mathfrak{g}(\mathfrak{h})$ . We are done by Lemma 4.1:

$$\dim(U \cap \mathfrak{c}(\mathbf{y})) \leq \dim(\mathfrak{c}(\mathbf{y})) \leq 2.$$

We have finished the proof of Theorem 0.4.

#### APPENDIX A. RIDGELINE NUMBERS AND MAXIMAL DIMENSIONS OF CENTRALISERS

Column 3 contains the nullity  $r$  of an invariant form. It is equal to  $\dim \mathfrak{z}$ . Column 4 contains the dual Coxeter number. Column 5 contains the ridgeline number. Column 6 contains dimension of the centraliser of the minimal nilpotent. Column 7 contains a minimal non-central semisimple element in  $\mathfrak{g}^b$ , using simple coweights  $\mathbf{y}_i$  and the enumeration of roots in Bourbaki [Bo68]. Column 8 contains the dimension of the centraliser of the minimal semisimple element in  $\mathfrak{g}^b$ .

type of $\mathfrak{g}$	$p$	$r$	$h^\vee$	$v(\mathfrak{g})$	$m - 2(h^\vee - 1)$	$\mathbf{y}$	$\dim \mathfrak{c}(\mathbf{y})$
$A_l, l \geq 2$	$(p, l + 1) = 1$	0	$l + 1$	$\frac{1}{2}$	$l^2$	$\mathbf{y}_1$	$l^2$
$A_l, l \geq 2$	$p \mid (l + 1)$	1	$l + 1$	$\frac{l}{2l-1}$	$l^2$	$\mathbf{y}_1$	$l^2$
$B_l, l \geq 3$	$p \neq 2$	0	$2l - 1$	$\frac{1}{4}(1 + \frac{1}{l-1})$	$2l^2 - 3l + 4$	$\mathbf{y}_1$	$2l^2 - 3l + 2$
$C_l, l \geq 2$	$p \neq 2$	0	$l + 1$	$\frac{1}{2}$	$2l^2 - l$	$\mathbf{y}_1$	$2l^2 - 3l + 2$
$D_l, l \geq 4$	$p \neq 2$	0	$2l - 2$	$\frac{1}{4}(1 + \frac{3}{2l-3})$	$2l^2 - 5l + 6$	$\mathbf{y}_1$	$2l^2 - 5l + 4$
$D_l, l = 2l_0 \geq 4$	2	2	$2l - 2$	$\frac{1}{4}(1 + \frac{2}{l-2})$	$2l^2 - 5l + 6$	$\mathbf{y}_1$	$2l^2 - 5l + 4$
$D_l, l = 2l_0 + 1 \geq 4$	2	1	$2l - 2$	$\frac{1}{4}(1 + \frac{7}{4l-7})$	$2l^2 - 5l + 6$	$\mathbf{y}_1$	$2l^2 - 5l + 4$
$G_2$	$p > 3$	0	4	$\frac{1}{3}$	8	$\mathbf{y}_1$	4
$G_2$	$p = 2$	0	4	$\frac{1}{3}$	8	$\mathbf{y}_1$	6
$F_4$	$p \neq 2$	0	9	$\frac{1}{4}$	36	$\mathbf{y}_1$	22
$E_6$	$p \neq 3$	0	12	$\frac{3}{11}$	56	$\mathbf{y}_1$	46
$E_6$	3	1	12	$\frac{2}{7}$	56	$\mathbf{y}_1$	46
$E_7$	$p \neq 2$	0	18	$\frac{7}{34}$	99	$\mathbf{y}_7$	79
$E_7$	$p = 2$	1	18	$\frac{7}{33}$	99	$\mathbf{y}_7$	79
$E_8$	$p \neq 2$	0	30	$\frac{4}{29}$	190	$\mathbf{y}_8$	134
$E_8$	$p = 2$	0	30	$\frac{4}{29}$	190	$\mathbf{y}_3$	136

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