

## GROWTH AND COGROWTH OF NORMAL SUBGROUPS OF A FREE GROUP

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ABSTRACT. We give a sufficient condition for a sequence of normal subgroups of a free group to have the property that both their growths tend to the upper bound and their cogrowth tend to the lower bound. The condition is represented by planarity of the quotient graphs of the tree.

### 1. INTRODUCTION AND STATEMENTS OF RESULT

We denote by  $F_n$  the free group of rank  $n \geq 2$  with a free set of generators  $S$ . Let  $T_n$  denote the Cayley graph of  $F_n$  with respect to  $S$ . We equip  $T_n$  with the word metric  $d$ . Let  $G < F_n$  be a subgroup of  $F_n$ . The Poincaré exponent  $\delta(G)$  of  $G$  is given by

$$\delta(G) := \limsup_{R \rightarrow \infty} \frac{1}{R} \log N_G(R), \quad \text{where } N_G(R) := \text{card} \{g \in G \mid d(\text{id}, g) \leq R\}.$$

It is well known that  $\delta(G)$  is given by the exponent of convergence of the Poincaré series

$$\delta(G) = \inf \left\{ s > 0 \mid \sum_{g \in G} e^{-sd(\text{id}, g)} < \infty \right\}.$$

For a normal subgroup  $\{1\} \neq N \triangleleft F_n$  the ratio

$$\eta(F_n/N) := \frac{\delta(N)}{\delta(F_n)} = \frac{\delta(N)}{\log(2n-1)}$$

is known as the cogrowth of the group presentation  $F_n/N$ , which was introduced by Grigorchuk [Gri80]. We have  $\eta(F_n/N) \leq 1$  and by a well-known result of Grigorchuk [Gri80] we have that  $\eta(F_n/N) = 1$  if and only if  $F_n/N$  is amenable. This criterion is deduced by combining Grigorchuk's cogrowth formula ([Gri80]; see (2.1) below) and results of Kesten on random walks on countable groups ([Kes59b, Kes59a]; see also Cohen [Coh82]).

In this paper we focus on Grigorchuk's lower bound of the cogrowth  $\eta(F_n/N) > 1/2$  [Gri80]. We have the following more general results due to Roblin [Rob05]. We say that  $G < F_n$  is of divergence type if  $\sum_{g \in G} e^{-\delta(G)d(\text{id}, g)} = \infty$ .

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**Proposition 1.1** ([Rob05]). *Let  $G < F_n$  and  $\{1\} \neq N \triangleleft G$ . Then we have the following:*

- (1)  $\delta(N) \geq \delta(G)/2$ .
- (2) *If  $G$  is of divergence type, then  $\delta(N) > \delta(G)/2$ .*

*Remark.* For Kleinian groups the first assertion was proved by Falk and Stratmann in [FS04]. An alternative proof of (1) can be given by following the arguments in [Jae15]. An ergodic-theoretic proof of  $\delta(N) > \delta(F_n)/2$  was recently given in [Jae16]. Results in a more general setting have also been obtained in [MYJ15].

Another important notion we consider is the growth of graphs. By a graph we mean an unoriented graph with countable vertex set, bounded vertex degree and loops as well as multiple edges allowed. For a connected graph  $\Gamma$  with path metric  $d$  and for some/any  $\gamma_0 \in \Gamma$ , the growth is given by

$$\text{growth}(\Gamma) = \limsup_{n \rightarrow \infty} (\text{card}\{\gamma \in \Gamma \mid d(\gamma, \gamma_0) \leq n\})^{1/n}.$$

Let  $H < F_n$  be a subgroup. We consider the action of  $H$  on  $T_n$  by left-multiplication. We denote by  $\Gamma_H := T_n/H$  the quotient graph with vertex set

$$\mathcal{V}(\Gamma_H) := \{Hg \mid g \in F_n\},$$

where  $Hx$  and  $Hy \in \mathcal{V}(\Gamma_H)$  are connected by an edge if and only if there exists  $s \in S$  such that  $Hxs = Hy$  or  $Hys = Hx$ .

The following result on growth tightness of  $F_n$  was proved in [GdlH97].

**Proposition 1.2** ([GdlH97]). *Let  $N$  be a normal subgroup of  $F_n$  with infinite index. Then*

$$\text{growth}(\Gamma_N) < \text{growth}(T_n) = 2n - 1.$$

A generalization of the growth tightness to hyperbolic groups was obtained by Arzhantseva and Lysenok in [AL02].

The main result of this paper is the following, which gives a sufficient condition on a sequence of normal subgroups of  $F_n$  under which both the growth and the cogrowth converge to their bounds simultaneously. Recall that a graph is called planar if there exists an embedding in the sphere. The condition for a finite graph to be planar is known as Wagner’s theorem [Wag37].

**Theorem 1.3.** *Let  $N_k \triangleleft F_n$ ,  $k \in \mathbb{N}$ , be a sequence of normal subgroups such that  $\Gamma_{N_k}$  is planar. Let  $\ell_k := \min\{\ell \in \mathbb{N} \mid \exists h \in N_k, d(\text{id}, h) = \ell\}$ . If  $\ell_k \rightarrow \infty$ , as  $k \rightarrow \infty$ , then we have*

$$\lim_{k \rightarrow \infty} \delta(N_k) = \frac{1}{2} \delta(F_n) \quad \text{and} \quad \lim_{k \rightarrow \infty} \text{growth}(\Gamma_{N_k}) = \text{growth}(T_n).$$

We see that a particular sequence of normal subgroups satisfies the above condition. For cogrowth this was proved by Grigorchuk [Gri80], and for growth this follows from a result by Shukhov [Shu99]. For  $g_1, \dots, g_s \in F_n$  we denote by  $\langle\langle g_1, \dots, g_s \rangle\rangle$  the normal closure of  $\{g_1, \dots, g_s\}$  in  $F_n$ .

**Corollary 1.4.** *Let  $F_n = \langle g_1, \dots, g_n \rangle$  and let  $N_k := \langle\langle g_1^{k_1}, g_2^{k_2}, \dots, g_s^{k_s} \rangle\rangle \triangleleft F_n$ ,  $s \in \mathbb{N}$ , and put  $k = \min\{k_i \mid 1 \leq i \leq s\}$ . Then we have*

$$\lim_{k \rightarrow \infty} \delta(N_k) = \frac{1}{2} \delta(F_n) \quad \text{and} \quad \lim_{k \rightarrow \infty} \text{growth}(\Gamma_{N_k}) = \text{growth}(T_n).$$

*Proof.* We have to verify that  $\Gamma_N$  is planar for  $N = N_k$ . Then the corollary follows from Theorem 1.3. To prove this consider the graph  $\Gamma_{F_n} = T_n/F_n$ , which consists of one vertex and  $n$  edges that are the loops based at the vertex. We embed this graph into  $(n + 1)$ -punctured sphere  $S$  so that each loop of  $\Gamma_{F_n}$  wraps around a different puncture, and hence the inclusion map induces the isomorphism  $\theta : \pi_1(\Gamma_{F_n}) = F_n \rightarrow \pi_1(S)$  between their fundamental groups. By [Mas88, Proposition X.A.3], if  $w_1, \dots, w_n \in \pi_1(S)$  correspond to mutually disjoint non-trivial simple closed curves in  $S$ , then the normal closure  $H = \langle\langle w_1^{k_1}, \dots, w_n^{k_n} \rangle\rangle$  for any  $k_1, \dots, k_n \in \mathbb{N}$  defines the normal covering surface  $\tilde{S}$  that is planar; namely, all simple closed curves are dividing. Note that any planar surface can be embedded into the sphere. Set  $N = \theta^{-1}(H) < F_n$ . By the covering projection  $p : \tilde{S} \rightarrow S$ , we lift the embedded graph  $\Gamma_{F_n} \subset S$  to  $\tilde{S}$ . Then the lifted graph is  $\Gamma_N$ . Since  $\tilde{S}$  is planar, we see that  $\Gamma_N$  is a planar graph.  $\square$

For the more general case of  $G < F_n$  we have the following immediate consequence of our results.

**Corollary 1.5.** *Let  $G = \langle g_i : i \in I \rangle$  be a subgroup of  $F_n$  such that  $\delta(G) = \delta(F_n)$ . Then there exists a sequence of normal subgroups  $N_k \triangleleft G$ ,  $k \in \mathbb{N}$ , such that*

$$\lim_{k \rightarrow \infty} \delta(N_k) = \frac{1}{2} \delta(G).$$

*Proof.* Put  $\tilde{N}_k := \langle\langle g_1^k \rangle\rangle \triangleleft F_n$ . It is easy to see that  $\sup_k d(\text{id}, g_1^k) = \infty$ . Hence, by passing to a subsequence, we may assume that  $\ell_k \rightarrow \infty$ , as  $k \rightarrow \infty$ . By Theorem 1.3 we have  $\lim_{k \rightarrow \infty} \delta(\tilde{N}_k) = \delta(F_n)/2$ . Put  $N_k := \tilde{N}_k \cap G$ . Since  $N_k \triangleleft G$  we have  $\delta(G)/2 \leq \delta(N_k) \leq \delta(\tilde{N}_k)$  by Proposition 1.1. The corollary follows because  $\lim_{k \rightarrow \infty} \delta(\tilde{N}_k) = \delta(F_n)/2 = \delta(G)/2$ .  $\square$

To prove our main result, we make use of the concept of isoperimetric inequalities to estimate the bottom of the spectrum of the discrete Laplacian on graphs. This concept also allows us to give new proofs for known results on the growth and cogrowth of quotient graphs of the tree. In Section 2 we introduce the necessary preliminaries on the discrete Laplacian and isoperimetric inequalities. The proof of Theorem 1.3 is given in Section 3.

Finally, in Section 4 we derive the following relation between the growth and the cogrowth.

**Theorem 1.6.** *For every non-trivial  $N \triangleleft F_n$  we have*

$$\delta(N) + \frac{1}{2} \log(\text{growth}(T_n/N)) \geq \delta(F_n).$$

## 2. PRELIMINARIES

**2.1. Discrete Laplacian.** Let  $n \in \mathbb{N}$  and let  $\Gamma$  be a  $(2n)$ -regular graph with vertex set  $\mathcal{V}(\Gamma)$ . The transition operator of the simple random walk on  $\Gamma$  is for  $f : \mathcal{V}(\Gamma) \rightarrow \mathbb{R}$  given by

$$Pf(x) := \frac{1}{2n} \sum_{y \sim x} f(y), \quad x \in \mathcal{V}(\Gamma),$$

where the sum is taken over all edges connecting  $x$  and  $y$ . The discrete Laplacian  $\Delta$  on  $\Gamma$  is given by  $\Delta f := f - Pf$ . Denote by  $\lambda_0(\Gamma)$  the bottom of the spectrum of  $\Delta$  given by

$$\lambda_0(\Gamma) := \inf \{ \lambda \in \mathbb{R} \mid \Delta - \lambda : \ell^2(\mathcal{V}(\Gamma)) \rightarrow \ell^2(\mathcal{V}(\Gamma)) \text{ has no bounded inverse} \}.$$

The following two facts are well known.

**Fact 2.1.** *The bottom of the spectrum is given by*

$$\lambda_0(\Gamma) = \inf_{f: \mathcal{V}(\Gamma) \rightarrow \mathbb{R}, \text{card}(\text{supp}(f)) < \infty} \frac{1}{2n} \frac{\sum_{x \sim y} |f(x) - f(y)|^2}{\sum_x |f(x)|^2}.$$

**Fact 2.2.** *Let  $\lambda \in \mathbb{R}$ . Then we have  $\lambda \leq \lambda_0(\Gamma)$  if and only if there exists  $f : \mathcal{V}(\Gamma) \rightarrow \mathbb{R}_{>0}$  such that  $\Delta f = \lambda f$ .*

In order to explain the relation between the bottom of the spectrum and the Poincaré exponent, we will first state Grigorchuk’s cogrowth formula. Denote by  $\rho(F_n/N)$  the spectral radius of the transition operator  $P : \ell^2(\mathcal{V}(\Gamma_N)) \rightarrow \ell^2(\mathcal{V}(\Gamma_N))$  of the simple random walk on the quotient graph  $\Gamma_N := T_n/N$ .

**Theorem 2.3** (Grigorchuk’s cogrowth formula). *For every  $\{1\} \neq N \triangleleft F_n$  we have*

$$(2.1) \quad \rho(F_n/N) = \frac{\sqrt{2n-1}}{2n} \left( \frac{\sqrt{2n-1}}{e^{\delta(N)}} + \frac{e^{\delta(N)}}{\sqrt{2n-1}} \right).$$

*Remark.* In [GdlH97] the cogrowth formula is stated for arbitrary subgroups  $H < F_n$ . A proof of this formula can be given by using the Patterson-Sullivan theory.

The relation between the bottom of the spectrum of the Laplacian and the Poincaré exponent is stated in the following proposition. The analogue result for Kleinian groups is known as the theorem of Elstrodt, Patterson and Sullivan [Sul87].

**Proposition 2.4.** *For every  $\{1\} \neq N \triangleleft F_n$  we have*

$$\lambda_0(\Gamma_N) = \frac{1}{2n} (2n - 1 - e^{\delta(N)})(1 - e^{-\delta(N)}).$$

*Proof.* First observe that  $\lambda_0(\Gamma_N) = 1 - \rho(F_n/N)$ , where we used the fact that  $\rho(F_n/N)$  is contained in the spectrum of the transition operator of the simple random walk on  $F_n/N$  ([Moh88]; see also [MW89, Theorem 4.4]). The proposition now follows from (2.1). More precisely, we have that

$$\begin{aligned} \lambda_0(\Gamma_N) &= 1 - \frac{\sqrt{2n-1}}{2n} \left( \frac{\sqrt{2n-1}}{e^{\delta(N)}} + \frac{e^{\delta(N)}}{\sqrt{2n-1}} \right) = 1 - \left( \frac{2n-1}{2n} e^{-\delta(N)} + \frac{1}{2n} e^{\delta(N)} \right) \\ &= \frac{1}{2n} \left( 2n - (2n-1)e^{-\delta(N)} - e^{\delta(N)} \right) = \frac{1}{2n} \left( 2n - 1 - e^{\delta(N)} \right) \left( 1 - e^{-\delta(N)} \right). \end{aligned}$$

□

**2.2. Isoperimetric constant.** The isoperimetric constant of a  $(2n)$ -regular graph  $\Gamma$  is given by

$$i(\Gamma) := \inf_{A \subset \mathcal{V}(\Gamma), \text{card}(A) < \infty} \frac{1}{2n} \frac{\text{card}(\partial A)}{\text{card}(A)},$$

where  $\partial A$  denotes the set of edges  $e$  such that  $e$  connects  $x, y$  with  $x \in A$  and  $y \in \mathcal{V}(\Gamma) \setminus A$ . It is well known that

$$i(T_n) = (n-1)/n \quad \text{and} \quad \lambda_0(T_n) = 1 - (\sqrt{2n-1})/n.$$

The following analogue of the well-known Cheeger inequality was proved by Mohar. Note that our isoperimetric constant is normalized by the factor  $(2n)^{-1}$ .

**Proposition 2.5** ([Moh88, Theorem 2.1]). *We have*

$$i(\Gamma) \leq \sqrt{1 - (1 - \lambda_0(\Gamma))^2}.$$

The following relation between the isoperimetric constant and the growth is due to Mohar [Moh88]. As we mentioned above, our definition of the isoperimetric constant is different from Mohar's.

**Lemma 2.6** ([Moh88, Theorem 4.1]). *We have*

$$\text{growth}(\Gamma) \geq \frac{1 + i(\Gamma)}{1 - i(\Gamma)}.$$

*That is, we have*

$$i(\Gamma) \leq \frac{\text{growth}(\Gamma) - 1}{\text{growth}(\Gamma) + 1}.$$

### 3. PROOF OF THE MAIN RESULT

In order to obtain estimates on the isoperimetric constant we show that, for a subgroup  $H < F_n$ , it suffices to consider all the finite core subgraphs of  $\Gamma_H$  in the definition of  $i(\Gamma_H)$ .

**Definition 3.1.** Let  $\Gamma \subset \Gamma_H$  be a finite subgraph. The minimal subgraph  $C \subset \Gamma$  such that the inclusion  $\iota : C \rightarrow \Gamma$  is a homotopy equivalence is called the core of  $\Gamma$ .

**Lemma 3.2.** *Suppose that  $H \neq \{1\}$ . Then we have*

$$i(\Gamma_H) = \inf \frac{1}{2n} \frac{\text{card}(\partial C_\Gamma)}{\text{card}(C_\Gamma)},$$

where the infimum is taken over all finite connected subgraphs  $\Gamma \subset \Gamma_H$  and  $C_\Gamma$  denotes the core of  $\Gamma$ .

*Proof.* To prove that we can restrict to connected subgraphs, we use the fact that if  $(a/b) \leq (c/d)$ , then  $(a/b) \leq (a+c)/(b+d)$ . To prove the reduction to finite core subgraphs, let  $\Gamma \subset \Gamma_H$  be a finite connected subgraph and consider the core of  $\Gamma$ . Note that the core is obtained from  $\Gamma$  by successively removing vertices  $v \in \Gamma$  for which there exists only one  $v' \in \Gamma$  such that  $v \sim v'$ . Hence, it suffices to show that

$$(3.1) \quad \frac{1}{2n} \frac{\text{card}(\partial \Gamma) - 2(n-1)}{\text{card}(\Gamma) - 1} \leq \frac{1}{2n} \frac{\text{card}(\partial \Gamma)}{\text{card}(\Gamma)}.$$

We may assume that  $\text{card}(\Gamma) \geq 2$ . If

$$(3.2) \quad \frac{1}{2n} \frac{\text{card}(\partial \Gamma)}{\text{card}(\Gamma)} \leq \frac{n-1}{n},$$

then (3.1) follows from the fact that  $(a-c)/(b-d) \leq a/b$  whenever  $a/b \leq c/d$  and  $b > d$ . If  $i(\Gamma_H) < (n-1)/n$ , then we may without loss of generality assume that (3.2) holds. If  $i(\Gamma_H) = (n-1)/n$ , then the lemma holds, because the infimum is attained if we consider a single cycle, which defines a core subgraph.  $\square$

We denote by  $\chi$  the Euler characteristic of a topological space.

**Lemma 3.3.** *If  $C$  is a connected core subgraph, then*

$$\text{card}(\partial C) = (2n - 2) \text{card}(C) + 2\chi(C).$$

*Proof.* First observe that the formula holds when  $C$  is a single loop, that is,  $\chi(C) = 0$ . The general case follows by induction on the Euler characteristic, because if we remove a cycle of edges or an edge path, 2 boundary elements appear and the Euler characteristic increases by 1. □

**Definition 3.4.** The injectivity radius of a connected graph  $\Gamma$  is given by

$$\ell(\Gamma) := \inf_{x \in \mathcal{V}(\Gamma)} \{\ell_x(\Gamma)\},$$

where we have set

$$\ell_x(\Gamma) := \frac{1}{2} \min \{\text{length}(\gamma) \mid \gamma \text{ is a non-backtracking edge path from } x \text{ to } x\}.$$

Note that if  $\Gamma = F_n$ , then  $\ell(\Gamma) = \infty$ , and if a graph  $C$  consists of a single vertex and no edge, then  $\ell(C) = \infty$ . Moreover, if  $C$  is a subgraph of  $\Gamma$ , then  $\ell(C) \geq \ell(\Gamma)$ .

**Proposition 3.5.** *Let  $C \subset \Gamma_H$  be a core subgraph. Suppose that  $\ell(C) < \infty$  and that  $C$  is planar. Then we have*

$$\text{card}(C) \geq (-\chi(C) + 2) \cdot (\ell(C) - 1).$$

*Proof.* Since  $C$  is planar, we can consider  $C$  as a subspace of the sphere  $S^2$ . Hence,

$$\chi(C) + \text{card}\{\text{faces of } C\} = \chi(S^2) = 2,$$

giving that

$$(3.3) \quad \text{card}\{\text{faces of } C\} = -\chi(C) + 2.$$

Since every edge of  $C$  is between two faces and each face is bounded by at least  $2\ell(C)$  edges, we obtain

$$2 \text{card}\{\text{edges of } C\} \geq 2\ell(C) \text{card}\{\text{faces of } C\}.$$

Combining with (3.3) yields

$$\text{card}\{\text{edges of } C\} \geq \ell(C) (-\chi(C) + 2).$$

Finally, we deduce for the number of vertices that

$$\text{card}\{\text{vertices of } C\} = \text{card}\{\text{edges of } C\} + \chi(C) \geq (\ell(C) - 1) (-\chi(C) + 2).$$

□

**Proposition 3.6.** *Let  $(H_k)$  be a sequence of non-trivial subgroups of  $F_n$ .*

- (1) *Suppose that  $\Gamma_{H_k}$  is planar for each  $k \in \mathbb{N}$ . If  $\ell(\Gamma_{H_k}) \rightarrow \infty$ , as  $k \rightarrow \infty$ , then  $\lim_{k \rightarrow \infty} i(\Gamma_{H_k}) = i(T_n)$ .*
- (2) *If  $\lim_{k \rightarrow \infty} i(\Gamma_{H_k}) = i(T_n)$  and  $H_k \triangleleft F_n$ , then*

$$\lim_{k \rightarrow \infty} \delta(H_k) = \frac{1}{2} \log(2n - 1) = \delta(F_n)/2.$$

- (3) *If  $\lim_{k \rightarrow \infty} i(\Gamma_{H_k}) = i(T_n)$ , then  $\lim_{k \rightarrow \infty} \text{growth}(\Gamma_{H_k}) = 2n - 1 = \text{growth}(T_n)$ .*

*Proof.* We first prove (1). By Lemma 3.2, Lemma 3.3 and Proposition 3.5 we have

$$\begin{aligned} i(\Gamma_H) &= \inf \frac{1}{2n} \frac{\text{card}(\partial C_\Gamma)}{\text{card}(C_\Gamma)} = \inf \frac{1}{2n} \frac{(2n-2) \text{card}(C_\Gamma) + 2\chi(C_\Gamma)}{\text{card}(C_\Gamma)} \\ &= \frac{n-1}{n} + \inf \frac{1}{n} \frac{\chi(C_\Gamma)}{\text{card}(C_\Gamma)} \geq \frac{n-1}{n} + \inf \frac{1}{n} \frac{\chi(C_\Gamma)}{(-\chi(C_\Gamma) + 2) \cdot (\ell(C_\Gamma) - 1)} \\ &> \frac{n-1}{n} - \frac{1}{n(\ell(\Gamma_H) - 1)}, \end{aligned}$$

where the infimum is taken over all finite connected subgraphs  $\Gamma \subset \Gamma_H$  and  $C_\Gamma$  denotes the core of  $\Gamma$ . Since  $i(\Gamma_H) \leq i(T_n) = (n-1)/n$ , the assertion in (1) follows.

To prove (2) observe that by Proposition 2.5 we have  $\lambda_0(\Gamma_{H_k}) \geq 1 - \sqrt{1 - i(\Gamma_{H_k})^2}$ . Consequently, if  $\lim_{k \rightarrow \infty} i(\Gamma_{H_k}) = i(T_n) = (n-1)/n$ , then  $\liminf_{k \rightarrow \infty} \lambda_0(\Gamma_{H_k}) \geq 1 - \sqrt{2n-1}/n$ . By Proposition 2.4 we conclude that  $\limsup_{k \rightarrow \infty} \delta(H_k) \leq \log \sqrt{2n-1}$ . Since  $H_k \triangleleft F_n$  we have  $\delta(H_k) \geq \log \sqrt{2n-1}$  by Proposition 1.1, which finishes the proof of (2).

Finally we turn to the proof of (3). By Proposition 2.6 we have as  $k \rightarrow \infty$ ,

$$\text{growth}(\Gamma_{H_k}) \geq \frac{1 + i(\Gamma_{H_k})}{1 - i(\Gamma_{H_k})} \rightarrow \frac{1 + (n-1)/n}{1 - (n-1)/n} = 2n - 1 = \text{growth}(T_n).$$

□

*Proof of Theorem 1.3.* The first assertion follows from Proposition 3.6 (1) and (2). The second assertion follows from Proposition 3.6 (1) and (3). □

#### 4. A RELATION BETWEEN GROWTH AND COGROWTH

We prove a relation between growth and cogrowth for  $N \triangleleft F_n$ . In the proof of Theorem 1.3, we obtained both the convergences of growth and cogrowth from that of the isoperimetric constant. However, it follows from Theorem 1.6 that  $\lim_{k \rightarrow \infty} \delta(N_k) = \delta(F_n)/2$  always implies  $\lim_{k \rightarrow \infty} \text{growth}(T_n/N_k) = \text{growth}(T_n)$ .

*Proof of Theorem 1.6.* For ease of notation we write  $\delta = \delta(N)$ ,  $\lambda_0 = \lambda_0(T_n/N)$  and  $\kappa = \log(\text{growth}(T_n/N))$ . It follows from Proposition 2.4 that

$$(4.1) \quad e^\delta = n(1 - \lambda_0) + \sqrt{n^2(1 - \lambda_0)^2 - (2n - 1)}.$$

By [Fuj96, Theorem 1] we have

$$\lambda_0 \leq 1 - \frac{2e^{\kappa/2}}{1 + e^\kappa}.$$

By using this estimate to replace  $1 - \lambda_0$  in (4.1) and multiplying by  $e^{\kappa/2}$  we have

$$(4.2) \quad e^{\delta + \kappa/2} \geq \frac{2ne^\kappa}{1 + e^\kappa} + e^{\kappa/2} \sqrt{n^2 \frac{4e^\kappa}{(1 + e^\kappa)^2} - (2n - 1)}.$$

We will show that the right side of (4.2) is greater than or equal to  $e^{\delta(F_n)} = 2n - 1$ . For this, we verify that

$$e^{\kappa/2} \sqrt{n^2 \frac{4e^\kappa}{(1 + e^\kappa)^2} - (2n - 1)} \geq (2n - 1) - \frac{2ne^\kappa}{1 + e^\kappa}$$

holds true. Taking the square of both sides, we see that this is equivalent to the inequality

$$\frac{4n^2 e^{2\kappa} - (2n - 1)(1 + e^\kappa)^2 e^\kappa}{(1 + e^\kappa)^2} \geq \frac{((2n - 1) - e^\kappa)^2}{(1 + e^\kappa)^2}$$

and is satisfied for  $0 \leq \kappa \leq \log(2n - 1)$ . This finishes the proof.  $\square$

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