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STABILITY OF TALAGRAND'S INEQUALITY UNDER CONCENTRATION TOPOLOGY

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ABSTRACT. In this paper, we study the compatibility between Talagrand's inequality and the concentration topology; i.e., if a sequence of mm-spaces satisfying Talagrand's inequality converges with respect to the observable distance, then the limit space satisfies Talagrand's inequality.

1. Introduction

Gromov [4, Chapter $3.\frac{1}{2}{}_+$] introduced the observable distance function d_{conc} on the set $\mathcal{X}$ of isomorphism classes of mm-spaces (metric measure spaces). This comes from the idea of measure concentration phenomenon which is stated as any 1-Lipschitz function on an mm-space is close to a constant function on a Borel set with almost full measure. The observable distance function is defined by the difference between the sets of 1-Lipschitz functions on two mm-spaces. The topology generated by the observable distance function admits a convergence sequence of Riemannian manifolds of unbounded dimension. For example, the sequence $\{S^n\}_{n=1}^{\infty}$ of n-dimensional unit spheres d_{conc}-converges to one-point mm-space.

Talagrand's inequality is one of the functional approaches to the concentration phenomenon. An mm-space (X, d_X, μ_X) satisfies Talagrand's inequality $(T_p(K))$ if we have

$$W_p(\nu, \mu_X)^2 \leq \frac{2}{K}\mathrm{Ent}(\nu|\mu_X)$$

for any $\nu \in \mathcal{P}_p(X)$. Here, W_p is the L^p-Wasserstein distance function, $\mathrm{Ent}(\nu|\mu_X)$ is the relative entropy of ν with respect to μ_X, and $\mathcal{P}_p(X)$ is the set of Borel probability measures with finite p^{th} moment. The case $p = 2$ was first proved by Talagrand [8]. He proved that n-dimensional Gaussian space satisfies Talagrand's inequality $(T_2(1))$ for any $n \in \mathbb{N}$. After that, Sturm [7] and Lott-Villani [5] introduced the curvature-dimension condition $\mathrm{CD}(K, \infty)$ for mm-spaces. This is a generalized notion of Ricci curvature bound from below by $K \in \mathbb{R}$. Lott-Villani [5] proved that the curvature-dimension condition $\mathrm{CD}(K, \infty)$ implies Talagrand's inequality $(T_2(K))$.

In this paper, we study the compatibility between d_{conc}-convergence and Talagrand's inequality. Our main theorem is stated as follows.

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Theorem 1.1. *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of mm-spaces satisfying Talagrand's inequality $(T_p(K))$ for $K > 0$ and p with $1 \le p < \infty$. If X_n concentrates to an mm-space Y as $n \to \infty$, then Y also satisfies Talagrand's inequality $(T_p(K))$.*

2. Preliminaries

In this section, we give the definitions and properties stated in [4, Chapter $3\frac{1}{2}{}_+$], [6], and [9, 10].

2.1. **Observable distance function.**

Definition 2.1 (mm-Space)**.** A triple $X = (X, d_X, \mu_X)$ is called an *mm-space* (*metric measure space*) if (X, d_X) is a complete separable metric space and if μ_X is a Borel probability measure on X.

Definition 2.2 (mm-Isomorphism)**.** Two mm-spaces X and Y are said to be *mm-isomorphic* to each other if there exists an isometry $f : \mathrm{supp}\mu_X \to \mathrm{supp}\mu_Y$ such that $f_*\mu_X = \mu_Y$, where $f_*\mu_X$ is the push-forward measure of μ_X by f. Such an f is called an *mm-isomorphism.*

Note that X is mm-isomorphic to $(\mathrm{supp}(\mu_X), d_X, \mu_X)$. Denote by $\mathcal{X}$ the set of mm-isomorphism classes of mm-spaces.

Let $I := [\,0, 1\,]$ and X be an mm-space. A Borel measurable map $\varphi : I \to X$ is called a *parameter of* X if φ satisfies $\varphi_*\mathcal{L} = \mu_X$, where $\mathcal{L}$ is the Lebesgue measure. Any mm-space has a parameter (see [6, Proposition 4.1]). For two μ_X-measurable functions $f, g : X \to \mathbb{R}$, we define the *Ky Fan distance between f and g* by

$$d_{\mathrm{KF}}(f, g) := \inf\{\,\varepsilon > 0 \mid \mu_X(\{\,x \in X \mid |\,f(x) - g(x)\,| > \varepsilon\,\}) \le \varepsilon\,\}.$$

The distance function d_{KF} is called the *Ky Fan metric* on the set of μ_X-measurable functions on X. Note that the Ky Fan metric is a metrization of convergence in measure of μ_X-measurable functions.

Definition 2.3 (Observable distance)**.** Denote by $\mathcal{L}ip_1(X)$ the set of 1-Lipschitz continuous functions on an mm-space X. For any parameter φ of X, we set $\varphi^*\mathcal{L}ip_1(X) := \{\,f \circ \varphi \mid f \in \mathcal{L}ip_1(X)\,\}$. We define the *observable distance* $d_{\mathrm{conc}}(X, Y)$ *between two mm-spaces X and Y* by

$$d_{\mathrm{conc}}(X, Y) := \inf_{\varphi, \psi} d_{\mathrm{H}}(\varphi^*\mathcal{L}ip_1(X), \psi^*\mathcal{L}ip_1(Y)),$$

where $\varphi : I \to X$ and $\psi : I \to Y$ run over all parameters of X and Y, respectively, and where d_{H} is the Hausdorff distance function with respect to the Ky Fan metric d_{KF}. We say that a sequence of mm-spaces X_n, $n = 1, 2, \ldots$, *concentrates* to an mm-space Y if X_n d_{conc}-converges to Y as $n \to \infty$.

The observable distance d_{conc} is a metric on $\mathcal{X}$ (see [4, Section $3\frac{1}{2}$.45] and [6, Theorem 5.16]). We call the topology on $\mathcal{X}$ induced by d_{conc} the *concentration topology.*

Proposition 2.4 ([3, Proposition 3.5, Proposition 3.11, Lemma 5.4], [6, Lemma 5.27, Corollary 5.35, Proposition 9.31])**.** *Let X_n and Y be mm-spaces, $n = 1, 2, \ldots$. If X_n concentrates to Y as $n \to \infty$, then there exist Borel measurable maps $p_n : X_n \to Y$, positive real numbers ε_n with $\varepsilon_n \to 0$ as $n \to \infty$ and Borel subsets $\tilde{X}_n \subset X_n$ with $\mu_{X_n}(\tilde{X}_n) \ge 1 - \varepsilon_n$ such that*

(1) $d_{\mathrm{H}}(\mathcal{L}ip_1(X_n), p_n^*\mathcal{L}ip_1(Y)) \le \varepsilon_n$,

(2) $(p_n)_*\mu_{X_n}$ *converges weakly to* μ_Y *as* $n \to \infty$,
(3) $d_Y(p_n(x_n), p_n(x'_n)) \leq d_{X_n}(x_n, x'_n) + \varepsilon_n$ *for any* $x_n, x'_n \in \tilde{X}_n$,
(4) $\limsup_{n\to\infty} \sup_{x_n \in X_n \setminus \tilde{X}_n} d_Y(p_n(x_n), y_0) < +\infty$ *for any* $y_0 \in Y$.

We call $\tilde{X}_n$ *the non-exceptional domain of* p_n *for an additive error* ε_n.

Remark 2.5.

(1) By the inner regularity of μ_{X_n}, we may assume $\tilde{X}_n$ is a compact set.
(2) The conditions (1) and (2) of Proposition 2.4 imply the d_{conc}-convergence (see [3, Proposition 3.5], [6, Corollary 5.36]).

2.2. **Talagrand's inequality.** Let X be a complete separable metric space. A Borel probability measure π on X^2 is a *coupling* of two Borel probability measures ν_0 and ν_1 on X if π satisfies $(\mathrm{proj}_0)_*\pi = \nu_0$ and $(\mathrm{proj}_1)_*\pi = \nu_1$, where $\mathrm{proj}_i : X \times X \to X$, $i = 0, 1$, are the projections defined by $\mathrm{proj}_0(x_0, x_1) = x_0$, $\mathrm{proj}_1(x_0, x_1) = x_1$.

Definition 2.6 (Wasserstein distance)**.** Let (X, d_X) be a complete separable metric space and $p \in [\,1, \infty\,)$. For two Borel probability measures μ and ν on X, we define the *L^p-Wasserstein distance between μ and ν* by

$$W_p(\mu, \nu) := \inf_\pi \left(\int_{X\times X} d_X(x, x')^p \, d\pi(x, x') \right)^{1/p}, \tag{2.1}$$

where π runs over all couplings of μ and ν.

Denote by $\mathcal{P}_p(X)$ the set of Borel probability measures μ satisfying

$$W_p(\mu, \delta_{x_0})^p = \int_X d_X(x, x_0)^p \, d\mu(x) < \infty$$

for the Dirac measure δ_{x_0} of some point $x_0 \in X$. The L_p-Wasserstein distance W_p is a metric on $\mathcal{P}_p(X)$ (see [9, Theorem 7.3] and [10, Chapter 6]).

Remark 2.7.

(1) There exists a minimizer for the infimum in (2.1). We will call it optimal coupling of ν_0 and ν_1 (see [10, Theorem 4.1]).
(2) The topology generated by the Wasserstein distance is stronger than the weak topology. If a complete separable metric space X is bounded, then the topology generated by the Wasserstein distance and the weak topology coincide to each other (see [9, Theorem 7.12] and [10, Theorem 6.9]).

Definition 2.8 (Relative entropy)**.** Let X be an mm-space and ν a Borel probability measure on X. The *relative entropy* $\mathrm{Ent}(\nu|\mu_X)$ of ν with respect to μ_X is defined as follows. If ν is absolutely continuous with respect to μ_X, then

$$\mathrm{Ent}(\nu|\mu_X) := \int_X \frac{d\nu}{d\mu_X} \log\left(\frac{d\nu}{d\mu_X}\right) d\mu_X;$$

otherwise $\mathrm{Ent}(\nu|\mu_X) := \infty$.

For an mm-space X, we denote by $\mathcal{P}^{cb}(X)$ the set of Borel probability measures on X with compact support which is absolutely continuous with respect to μ_X and the Radon-Nikodym derivative is essentially bounded on X. Note that $\mathcal{P}^{cb}(X)$ is a dense subset in $(\mathcal{P}_p(X), W_p)$.

Lemma 2.9 ([6, Lemma 9.20]). *Let X be an mm-space and $\nu \in \mathcal{P}_p(X)$ with $\mathrm{Ent}(\nu|\mu_X) < \infty$. Then, for any $\varepsilon > 0$, there exists $\tilde{\nu} \in \mathcal{P}^{cb}(X)$ such that*

$$W_p(\tilde{\nu}, \nu) < \varepsilon \quad \textit{and} \quad |\,\mathrm{Ent}(\tilde{\nu}|\nu_X) - \mathrm{Ent}(\nu|\mu_X)\,| < \varepsilon.$$

Definition 2.10 (Talagrand's inequality). Let X be an mm-space. X satisfies *Talagrand's inequality* ($T_p(K)$) for positive real numbers K and p with $1 \leq p < \infty$ if we have

$$W_p(\nu, \mu_X)^2 \leq \frac{2}{K}\mathrm{Ent}(\nu|\mu_X) \qquad (T_p(K))$$

for any $\nu \in \mathcal{P}_p(X)$.

Sturm [7] and Lott-Villani [5] introduced the curvature-dimension condition $\mathrm{CD}(K, \infty)$. This is a generalized notion of Ricci curvature bound from below by $K \in \mathbb{R}$ (see [7, Theorem 4.9] and [5, Theorem 7.3]). Lott-Villani proved the following.

Example 2.11 ([5, Theorem 6.1]). Let $K > 0$ and X be an mm-space satisfying $\mathrm{CD}(K, \infty)$. Then X satisfies Talagrand's inequality ($T_2(K)$). In particular, if M is a complete Riemannian manifold with $\mathrm{Ric}_M \geq K$, then M satisfies Talagrand's inequality ($T_2(K)$).

Remark 2.12. Consider the n-dimensional standard Gaussian measure γ^n on $(\mathbb{R}^n, \|\cdot\|_2)$. Since $(\mathbb{R}^n, \|\cdot\|_2, \gamma^n)$ satisfies $\mathrm{CD}(1, \infty)$ (see [7, Example 4.10]), this space satisfies Talagrand's inequality ($T_2(1)$). This coincides with Talagrand's result (see [8, Theorem 1.1]).

Combining Csiszár-Kullback-Pinsker's inequality (see e.g. [1, Theorem 8.2.7]) and [10, Theorem 6.15], we obtain the following example.

Example 2.13. Let K_n be the complete graph with n vertices, unit distance and uniform probability distribution. Then K_n satisfies Talagrand's inequality ($T_1(1/4)$).

3. Proof of Theorem 1.1

For a Borel subset B of an mm-space X with positive measure, we define a Borel probability measure μ_B by

$$\mu_B := \frac{\mu_X|_B}{\mu_X(B)}.$$

For two Borel measures μ and ν on a metric space X, we write $\mu \leq \nu$ if $\mu(B) \leq \nu(B)$ for any Borel set B of X.

Lemma 3.1. *Let X be an mm-space. If we assume that every $\nu \in \mathcal{P}^{cb}(X)$ satisfies the condition of the definition of Talagrand's inequality ($T_p(K)$), then we have the following:*

1. $\mu_X \in \mathcal{P}_p(X)$.
2. *X satisfies Talagrand's inequality ($T_p(K)$).*

Proof. We prove (1). Let $C \subset X$ be a compact set with $\mu_X(C) > 0$ and $x_0 \in X$. Then, we obtain

$$\begin{aligned} W_p(\mu_X, \delta_{x_0}) &\le W_p(\mu_X, \mu_C) + W_p(\mu_C, \delta_{x_0}) \\ &\le \sqrt{\frac{2}{K}\mathrm{Ent}(\mu_C|\mu_X)} + \sup_{x \in C} d_X(x, x_0) \\ &= \sqrt{\frac{2}{K}\log\frac{1}{\mu_X(C)}} + \sup_{x \in C} d_X(x, x_0) \\ &< \infty. \end{aligned}$$

(2) follows from Lemma 2.9. □

Lemma 3.2 ([3, Lemma 3.13], [6, Lemma 9.33])**.** *Let X_n and Y be mm-spaces, $n = 1, 2, \dots$. Assume that a sequence of Borel measurable maps $p_n : X_n \to Y$ and a sequence $\{\varepsilon'_n\}_{n=1}^\infty$ of positive real numbers with $\varepsilon'_n \to 0$ satisfy* (1)–(3) *of Proposition* 2.4*. For a real number $\delta > 0$, we give two Borel subsets $B_0, B_1 \subset Y$ such that*

$$\mathrm{diam} B_i \le \delta, \quad \mu_Y(B_i) > 0, \quad \textit{and} \quad \mu_Y(\partial B_i) = 0$$

for $i = 0, 1$, and set

$$\tilde{B}_i := p_n^{-1}(B_i) \cap \tilde{X}_n \subset X_n,$$

where $\tilde{X}_n$ is a non-exceptional domain of p_n. Then, there exist a sequence $\{\varepsilon_n\}_{n=1}^\infty$ of positive real numbers with $\varepsilon_n \to 0$, Borel probability measures $\tilde{\xi}_0^n$, $\tilde{\xi}_1^n$ on X_n and couplings $\tilde{\pi}_n$ between $\tilde{\xi}_0^n$ and $\tilde{\xi}_1^n$, $n = 1, 2, \dots$, such that, for every sufficiently large natural number n,

1. $\tilde{\xi}_i^n \le (1 + O(\delta^{1/2}))\mu_{\tilde{B}_i}$ $(i = 0, 1)$*, where $O(\cdot)$ is a Landau symbol,*
2. $d_{X_n}(x_0, x_1) \ge d_Y(B_0, B_1) - \varepsilon_n$ *for any* $x_i \in \tilde{B}_i$, $i = 0, 1$,
3. $\mathrm{supp}\tilde{\pi}^n \subset \{\, (x_n, x'_n) \in X_n^2 \mid d_{X_n}(x_n, x'_n) \le d_Y(B_0, B_1) + \delta^{1/2} \,\}$,
4. $-\varepsilon_n \le W_p(\tilde{\xi}_0^n, \tilde{\xi}_1^n) - d_Y(B_0, B_1) \le \delta^{1/2}$ *for any* $p \ge 1$.

Let $\theta(\cdot) : \mathbb{R} \to \mathbb{R}$ be a function such that $\theta(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Proof of Theorem 1.1. By Lemma 3.1, it suffices to prove that

$$W_p(\nu, \mu_Y)^2 \le \frac{2}{K}\mathrm{Ent}(\nu|\mu_Y) \tag{3.1}$$

for any $\nu \in \mathcal{P}^{cb}(Y)$. Let $p_n : X_n \to Y$, $n = 1, 2, \dots$, be Borel measurable maps as in Proposition 2.4. To prove the theorem, we first prove the inequality

$$W_p(\mu, \nu)^2 \le \frac{2}{K}\left(\sqrt{\mathrm{Ent}(\mu|\mu_Y)} + \sqrt{\mathrm{Ent}(\nu|\mu_Y)}\right)^2 \tag{3.2}$$

for any $\mu, \nu \in \mathcal{P}^{cb}(Y)$.

We take any $\mu, \nu \in \mathcal{P}^{cb}(Y)$ and fix them. For any natural number m, there are finite disjoint Borel subsets $B_j \subset Y$, $j = 1, 2, \dots, J$, such that $\bigcup_{j=1}^J \overline{B_j} = \mathrm{supp}\mu \cup \mathrm{supp}\nu$, $\mathrm{diam} B_j \le m^{-1}$, $\mu_Y(B_j) > 0$, and $\mu_Y(\partial B_j) = 0$ for any j. For each $(j, k) \in \{\, 1, \dots, J \,\}^2$, we apply Lemma 3.2 for B_j and B_k to obtain Borel probability measures $\tilde{\xi}_{jk}^{mn} \in \mathcal{P}^{cb}(X_n)$, $n = 1, 2, \dots$, such that

$$\tilde{\xi}_{jk}^{mn} \le (1 + \theta(m^{-1}))\mu_{\tilde{B}_j}, \tag{3.3}$$

for any sufficiently large natural number n. By the diagonal argument, we may assume that $(p_n)_*\tilde{\xi}_{jk}^{mn}$ converges weakly to a Borel probability measure $\tilde{\xi}_{jk}^{m} \in \mathcal{P}^{cb}(Y)$ as $n \to \infty$ for each $(j,k,m) \in \{1,\dots,J\}^2 \times \mathbb{N}$. Let π be an optimal coupling for $W_p(\mu,\nu)$. We define

$$\begin{aligned}
w_{jk} &:= \pi(B_j \times B_k),\\
\tilde{\mu}^{mn} &:= \sum_{j,k=1}^{J} w_{jk}\tilde{\xi}_{jk}^{mn}, \qquad & \tilde{\nu}^{mn} &:= \sum_{j,k=1}^{J} w_{jk}\tilde{\xi}_{kj}^{mn} \quad \in \mathcal{P}^{cb}(X_n),\\
\tilde{\mu}^{m} &:= \sum_{j,k=1}^{J} w_{jk}\tilde{\xi}_{jk}^{m}, \qquad & \tilde{\nu}^{m} &:= \sum_{j,k=1}^{J} w_{jk}\tilde{\xi}_{kj}^{m} \quad \in \mathcal{P}^{cb}(Y).
\end{aligned}$$

Then, $(p_n)_*\tilde{\mu}^{mn}$ and $(p_n)_*\tilde{\nu}^{mn}$ converge weakly to $\tilde{\mu}^{m}$ and $\tilde{\nu}^{m}$, respectively, as $n \to \infty$. $\tilde{\mu}^{m}$ and $\tilde{\nu}^{m}$ converge weakly to μ and ν, respectively, as $m \to \infty$. Moreover, $W_p((p_n)_*\tilde{\mu}^{mn},\mu)$, $W_p((p_n)_*\tilde{\nu}^{mn},\nu) \to 0$ as $n \to \infty$ and then $m \to \infty$.

Let $\tilde{\pi}$ be an optimal coupling for $W_p(\tilde{\mu}^{mn},\tilde{\nu}^{mn})$. By $\mathrm{supp}\tilde{\mu}^{mn}$, $\mathrm{supp}\tilde{\nu}^{mn} \subset \tilde{X}_n$, and Proposition 2.4 (3), we have

$$\begin{aligned}
W_p((p_n)_*\tilde{\mu}^{mn},(p_n)_*\tilde{\nu}^{mn})^p &\le \int_{X_n\times X_n} d_Y(p_n(x_n),p_n(x_n'))^p\, d\tilde{\pi}(x_n,x_n')\\
&\le \int_{X_n\times X_n} (d_{X_n}(x_n,x_n')+\varepsilon_n)^p\, d\tilde{\pi}(x_n,x_n')\\
&\le (W_p(\tilde{\mu}^{mn},\tilde{\nu}^{mn})+\varepsilon_n)^p.
\end{aligned}$$

Then, we have

$$\begin{aligned}
W_p(\mu,\nu) &= \lim_{m\to\infty}\lim_{n\to\infty} W_p((p_n)_*\tilde{\mu}^{mn},(p_n)_*\tilde{\nu}^{mn})\\
&\le \liminf_{m\to\infty}\liminf_{n\to\infty} W_p(\tilde{\mu}^{mn},\tilde{\nu}^{mn}).
\end{aligned} \tag{3.4}$$

By (3.3), we have

$$\begin{aligned}
\frac{d\tilde{\mu}^{mn}}{d\mu_{X_n}} &= \sum_{j,k=1}^{J} w_{jk}\frac{d\tilde{\xi}_{jk}^{mn}}{d\mu_{X_n}}\\
&\le (1+\theta(m^{-1}))\sum_{j,k=1}^{J}\frac{w_{jk}}{\mu_{X_n}(\tilde{B}_j)}\chi_{\tilde{B}_j}\\
&= (1+\theta(m^{-1}))\sum_{j=1}^{J}\frac{\mu(B_j)}{\mu_{X_n}(\tilde{B}_j)}\chi_{\tilde{B}_j}.
\end{aligned}$$

In particular, we have $\tilde{\mu}^{mn}(\tilde{B}_j) \leq (1+\theta(m^{-1}))\mu(B_j)$. The monotonicity of $f(x) = \log x$ and the previous inequality imply that

$$\begin{aligned}
&\mathrm{Ent}(\tilde{\mu}^{mn}|\mu_{X_n}) \\
&= \int_{X_n} \log\left(\frac{d\tilde{\mu}^{mn}}{d\mu_{X_n}}(x_n)\right) d\tilde{\mu}^{mn}(x_n) \\
&\leq \int_{X_n} \log\left((1+\theta(m^{-1}))\sum_{j=1}^{J} \frac{\mu(B_j)}{\mu_{X_n}(\tilde{B}_j)}\chi_{\tilde{B}_j}(x_n)\right) d\tilde{\mu}^{mn}(x_n) \\
&= \sum_{j=1}^{J} \tilde{\mu}^{mn}(\tilde{B}_j)\log\left((1+\theta(m^{-1}))\frac{\mu(B_j)}{\mu_{X_n}(\tilde{B}_j)}\right) \\
&\leq (1+\theta(m^{-1}))\sum_{j=1}^{J} \mu(B_j)\log\frac{\mu(B_j)}{\mu_{X_n}(\tilde{B}_j)} + \theta(m^{-1}).
\end{aligned}$$

Since B_j satisfies $\mu_Y(\partial B_j) = 0$, Proposition 2.4 (2) and the portmanteau theorem (see [2, Corollary 8.2.10]) imply that

$$\lim_{n\to\infty} \mu_{X_n}(\tilde{B}_j) = \lim_{n\to\infty} \mu_{X_n}(p_n^{-1}(B_j)\cap \tilde{X}_n) = \mu_Y(B_j),$$

and then we obtain

$$\begin{aligned}
&\limsup_{n\to\infty} \mathrm{Ent}(\tilde{\mu}^{mn}|\mu_{X_n}) \\
&\leq (1+\theta(m^{-1}))\sum_{j=1}^{J} \mu(B_j)\log\frac{\mu(B_j)}{\mu_Y(B_j)} + \theta(m^{-1}).
\end{aligned} \tag{3.5}$$

Define a probability measure $\overline{\mu}^m$ by

$$\overline{\mu}^m := \sum_{j=1}^{J} \frac{\mu(B_j)}{\mu_Y(B_j)}\mu_Y|_{B_j}.$$

Jensen's inequality implies that

$$\begin{aligned}
&\mathrm{Ent}(\mu|\mu_Y) \\
&= \int_Y \frac{d\mu}{d\mu_Y}(y)\log\frac{d\mu}{d\mu_Y}(y)\, d\mu_Y(y) \\
&= \sum_{j=1}^{J}\int_{B_j} \frac{d\mu}{d\mu_Y}(y)\log\frac{d\mu}{d\mu_Y}(y)\, d\mu_Y(y) \\
&\geq \sum_{j=1}^{J}\left(\int_{B_j}\frac{d\mu}{d\mu_Y}(y)\, d\mu_Y(y)\right)\log\left(\frac{1}{\mu_Y(B_j)}\int_{B_j}\frac{d\mu}{d\mu_Y}(y)\, d\mu_Y(y)\right) \\
&= \sum_{j=1}^{J} \mu(B_j)\log\frac{\mu(B_j)}{\mu_Y(B_j)} \\
&= \mathrm{Ent}(\overline{\mu}^m|\mu_Y).
\end{aligned}$$

Combining this inequality and (3.5) and taking the limit as $n\to\infty$, we obtain

$$\limsup_{m\to\infty}\limsup_{n\to\infty} \mathrm{Ent}(\tilde{\mu}^{mn}|\mu_{X_n}) \leq \mathrm{Ent}(\mu|\mu_Y). \tag{3.6}$$

In the same way, we also obtain

$$\limsup_{m\to\infty}\limsup_{n\to\infty}\mathrm{Ent}(\tilde{\nu}^{mn}|\mu_{X_n})\le\mathrm{Ent}(\nu|\mu_Y). \tag{3.7}$$

The triangle inequality and Talagrand's inequality on X_n imply that

$$\begin{aligned}W_p(\tilde{\mu}^{mn},\tilde{\nu}^{mn})&\le W_p(\tilde{\mu}^{mn},\mu_{X_n})+W_p(\mu_{X_n},\tilde{\nu}^{mn})\\&\le\sqrt{\frac{2}{K}}\left(\sqrt{\mathrm{Ent}(\tilde{\mu}^{mn}|\mu_{X_n})}+\sqrt{\mathrm{Ent}(\tilde{\nu}^{mn}|\mu_{X_n})}\right),\end{aligned}$$

which together with (3.4), (3.6), and (3.7) imply (3.2).

Let us next prove that μ_Y belongs to $\mathcal{P}_p(Y)$. We take an optimal coupling $\overline{\pi}$ for $W_p(\mu_{X_n},\tilde{\mu}^{mn})$. By Proposition 2.4 (4) and $\tilde{\mu}^{mn}(X_n\setminus\tilde{X}_n)=0$, there exists a constant $D>0$ such that

$$d_Y(p_n(x_n),p_n(x_n'))\le D$$

for $\overline{\pi}|_{(X_n\setminus\tilde{X}_n)\times X_n}$-a.e. $(x_n,x_n')\in X_n^2$. This together with Proposition 2.4 (3) and Talagrand's inequality on X_n implies that

$$\begin{aligned}&W_p((p_n)_*\mu_{X_n},(p_n)_*\tilde{\mu}^{mn})^p\\&\le\int_{\tilde{X}_n\times\tilde{X}_n}(d_{X_n}(x_n,x_n')+\varepsilon_n)^p\,d\overline{\pi}(x_n,x_n')\\&\quad+\int_{(X_n\setminus\tilde{X}_n)\times\tilde{X}_n}d_Y(p_n(x_n),p_n(x_n'))^p\,d\overline{\pi}(x_n,x_n')\\&\le(W_p(\mu_{X_n},\tilde{\mu}^{mn})+\varepsilon_n)^p+D^p\varepsilon_n\\&\le\left(\sqrt{\frac{2}{K}\mathrm{Ent}(\tilde{\mu}^{mn}|\mu_{X_n})}+\varepsilon_n\right)^p+D^p\varepsilon_n.\end{aligned}$$

By the inequality just before and (3.6), we have

$$\limsup_{m\to\infty}\limsup_{n\to\infty}W_p((p_n)_*\mu_{X_n},(p_n)_*\tilde{\mu}^{mn})\le\sqrt{\frac{2}{K}\mathrm{Ent}(\mu|\mu_Y)}. \tag{3.8}$$

We take any point $y_0\in Y$ and fix this. Fatou's lemma, Proposition 2.4 (2), and $W_p((p_n)_*\tilde{\mu}^{mn},\mu)\to 0$ as $n,m\to\infty$ together imply that

$$\begin{aligned}\int_Y d_Y(y,y_0)^p\,d\mu_Y(y)&\le\liminf_{R\to\infty}\int_Y(d_Y(y,y_0)\wedge R)^p\,d\mu_Y(y)\\&=\liminf_{R\to\infty}\lim_{n\to\infty}\int_Y(d_Y(y,y_0)\wedge R)^p\,d(p_n)_*\mu_{X_n}(y)\\&\le\liminf_{n\to\infty}\int_Y d_Y(y,y_0)^p\,d(p_n)_*\mu_{X_n}(y)\\&=\liminf_{n\to\infty}W_p((p_n)_*\mu_{X_n},\delta_{y_0})^p\\&\le\left(\sqrt{\frac{2}{K}\mathrm{Ent}(\mu|\mu_Y)}+W_p(\mu,\delta_{y_0})\right)^p\\&<\infty.\end{aligned}$$

This means μ_Y belongs to $\mathcal{P}_p(Y)$. We apply Lemma 2.9 for μ_Y and then obtain the inequality (3.1). This completes the proof. □

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