

ON A FAMILY OF INHOMOGENEOUS TORSIONAL CREEP PROBLEMS

MARIAN BOCEA AND MIHAI MIHĂILESCU

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ABSTRACT. The asymptotic behavior of solutions to a family of Dirichlet boundary value problems involving inhomogeneous PDEs in divergence form is studied in an Orlicz-Sobolev setting. Solutions are shown to converge uniformly to the distance function to the boundary of the domain. This implies that a well-known result in the analysis of problems modeling torsional creep continues to hold under much more general constitutive assumptions on the stress.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary, $\partial\Omega$, and let $p > 1$. The analysis of the limiting behavior, as $p \rightarrow \infty$, of the weak solutions for elliptic problems of the type

$$(1.1) \quad \begin{cases} -\Delta_p u = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Δ_p is the p -Laplace operator, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$, have been studied for more than two decades in connection with the concept of “torsional creep” - the plastic deformation exhibited by certain materials subject to a torsional moment for an extended period of time at high temperature. The reason one ends up with an equation involving the p -Laplacian for this type of physical phenomena is the constitutive assumption that the stress-strain relation is of a power-law type (see, e.g., Kachanov [10], [11]). Knowledge of the unique solution $u_p \in W_0^{1,p}(\Omega)$ of (1.1) is key to recovering the distribution of stresses (and strains) in the material. Based on physical evidence [10], [11] and on explicit solutions [2], the resulting distribution of stresses appears to approach, as $p \rightarrow \infty$, the stress distribution of an ideally plastic homogeneous material, for which the stress tensor has constant modulus. To briefly describe one of the results that validates this analytically, let $\delta : \bar{\Omega} \rightarrow [0, +\infty)$ be the distance function to the boundary of Ω , given by $\delta(x) := \inf_{y \in \partial\Omega} |x - y|$, $x \in \Omega$. In [13, Theorem 1] (see also [2]) Kawohl proved that the sequence $\{u_p\}$ of solutions

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to (1.1) converges uniformly in Ω to δ , which therefore serves as a kind of ideally plastic stress potential.

The main goal of this paper is to show that such a result continues to hold under more general constitutive assumptions. From a mathematical point of view the question is whether Kawohl’s result is still valid if instead of the family of equations (1.1) with constant $p > 1$ one considers problems involving inhomogeneous differential operators. Precisely, we study the family of problems

$$(1.2) \quad \begin{cases} -\operatorname{div} \left(\frac{\varphi_n(|\nabla u|)}{|\nabla u|} \nabla u \right) = \varphi_n(1) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where, for each $n \in \mathbb{N}$, the mappings $\varphi_n : \mathbb{R} \rightarrow \mathbb{R}$ are odd, increasing homeomorphisms of class C^1 satisfying

$$(1.3) \quad 0 < \varphi_n^- - 1 \leq \frac{t\varphi_n'(t)}{\varphi_n(t)} \leq \varphi_n^+ - 1 < \infty, \quad \forall t \geq 0,$$

for some constants φ_n^- and φ_n^+ with $1 < \varphi_n^- \leq \varphi_n^+ < \infty$,

$$(1.4) \quad \varphi_n^- \rightarrow \infty \text{ as } n \rightarrow \infty,$$

and such that

$$(1.5) \quad \text{there exists a real constant } \beta > 1 \text{ such that } \varphi_n^+ \leq \beta\varphi_n^- \text{ for all } n \in \mathbb{N}.$$

Note that in the particular case where $\varphi_n(t) = |t|^{n-2}t$ the problem (1.2) reduces to (1.1) with $p = n$. However, besides encompassing this basic situation, our framework allows a great deal of additional flexibility in terms of the operators appearing in the family of problems (1.2). We indicate below several examples of functions $\varphi_n : \mathbb{R} \rightarrow \mathbb{R}$ for which our assumptions (1.3), (1.4), and (1.5) are valid. For more details, the reader is referred to [5, Examples 1-3, p. 243] (see also [16]).

- 1) $\varphi_n(t) = |t|^{n-2}t + |t|^{2n-2}t$, with $n > 1$. Then $\varphi_n^- = n$ and $\varphi_n^+ = 2n$.
- 2) $\varphi_n(t) = \log(1 + |t|^p)|t|^{n-2}t$, with $n, p > 1$. In this case $\varphi_n^- = n$, and $\varphi_n^+ = n + p$.
- 3) $\varphi_n(t) = \frac{|t|^{n-2}t}{\log(1+|t|)}$ if $t \neq 0$, $\varphi_n(0) = 0$, with $n > 2$. In this case it turns out that $\varphi_n^- = n - 2$ and $\varphi_n^+ = n - 1$.

Due to the inhomogeneous character of the problem, the classical Lebesgue and Sobolev spaces are not the appropriate functional spaces in which to seek solutions for our problem. Instead, one needs to work in the more general framework of Orlicz and Orlicz-Sobolev spaces. With φ_n as above, define $\Phi_n(t) := \int_0^t \varphi_n(s) ds, \quad \forall t > 0$. The Orlicz space $L^{\Phi_n}(\Omega)$ is defined by

$$L^{\Phi_n}(\Omega) := \left\{ u : \Omega \rightarrow \mathbb{R}; \ u \text{ is measurable and } \int_{\Omega} \Phi_n(|u(x)|) dx < \infty \right\}.$$

Endowed with the Luxemburg norm, given by

$$(1.6) \quad \|u\|_{\Phi_n} := \inf \left\{ \mu > 0 : \int_{\Omega} \Phi_n \left(\frac{u(x)}{\mu} \right) dx \leq 1 \right\},$$

$L^{\Phi_n}(\Omega)$ is a Banach space. The Orlicz-Sobolev space $W_0^{1,\Phi_n}(\Omega)$ is defined as the

closure of $C_0^\infty(\Omega)$ under the norm $\|u\| := \|\nabla u\|_{\Phi_n}$. Under our assumption (1.3), $L^{\Phi_n}(\Omega)$ and $W_0^{1,\Phi_n}(\Omega)$ are reflexive Banach spaces. For more details about Orlicz and Orlicz-Sobolev spaces we refer to the book by Adams [1], and to the papers by Clément et al. [5], Lieberman [14] and Martínez & Wolanski [16].

It is well known that for each positive integer $n \in \mathbb{N}$, the unique weak solution $u_n \in W_0^{1,\Phi_n}(\Omega)$ of problem (1.2) is a minimizer of the Euler-Lagrange functional associated to problem (1.2), that is, $J_n : W_0^{1,\Phi_n}(\Omega) \rightarrow \mathbb{R}$ given by

$$J_n(v) := \int_\Omega \frac{\Phi_n(|\nabla v(x)|)}{\varphi_n(1)} dx - \int_\Omega v(x) dx.$$

In particular, since $J_n(v) \geq J_n(|v|)$ for all $v \in W_0^{1,\Phi_n}(\Omega)$ and u_n is a minimizer of J_n , it is clear that $u_n(x) \geq 0$ for a.e. $x \in \Omega$. Moreover, it follows by standard arguments that $J_n \in C^1(W_0^{1,\Phi_n}(\Omega); \mathbb{R})$ and $\langle J_n'(u_n), u_n \rangle = 0$, i.e., $\frac{1}{\varphi_n(1)} \int_\Omega \varphi_n(|\nabla u_n(x)|) |\nabla u_n(x)| dx - \int_\Omega u_n(x) dx = 0$. By [3, Theorem 3.2], the sequence $\{I_n\}$ of functionals $I_n : L^1(\Omega) \rightarrow [0, \infty]$ defined by

$$I_n(u) := \begin{cases} \int_\Omega \frac{\Phi_n(|\nabla u(x)|)}{\varphi_n(1)} dx - \int_\Omega u(x) dx & \text{if } u \in W_0^{1,\Phi_n}(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

$\Gamma(L^1(\Omega))$ -converges in the strong topology of $L^1(\Omega)$ to the functional $I_\infty : L^1(\Omega) \rightarrow [0, \infty]$, given by

$$I_\infty(u) = \begin{cases} - \int_\Omega u(x) dx & \text{if } |\nabla u(x)| \leq 1 \text{ a.e. } x \in \Omega, \\ +\infty & \text{otherwise.} \end{cases}$$

Since for each positive integer n , u_n is a minimizer of J_n , it also minimizes I_n . Assuming that there exists u_∞ such that $u_\infty = \lim_{n \rightarrow \infty} u_n$ in $L^1(\Omega)$ we deduce by [8, Corollary 6.1.1] that u_∞ must be a minimizer for I_∞ and, in particular, $\|\nabla u_\infty\|_{L^\infty(\Omega; \mathbb{R}^N)} \leq 1$. Note that $|\nabla \delta(x)| = 1$ for a.e. $x \in \Omega$ and thus, the distance function δ is, in fact, a candidate for the minimization of I_∞ . This intuition indicates that an extension of Kawohl’s result might be reasonable to expect in this more general setting. This is indeed the case, as shown in our main theorem.

Theorem 1. *Assume that the hypotheses (1.3), (1.4), and (1.5) hold. Then the sequence $\{u_n\} \subset W_0^{1,\Phi_n}(\Omega)$ of weak solutions of problem (1.2) converges uniformly in Ω to $\delta = \text{dist}(\cdot, \partial\Omega)$.*

Before presenting the proof of Theorem 1 in the next section of the paper it is instructive to check that the conclusion is indeed valid in the particular case where Ω is a ball, namely $\Omega = B_R(0) \subset \mathbb{R}^N$. This simplification has the advantage that the solutions of (1.2) can now be explicitly computed. We follow the strategy described in [13, Section 3]. Since for each integer $n \in \mathbb{N}$ the solution u_n of problem (1.2) is unique, it is radial on $B_R(0)$. For the rest of this section we will make the

identification $u_n(x) = u_n(r)$, $r = |x|$. Using standard symmetrization results it can be checked that u_n is radially decreasing and thus, $|\nabla u_n(x)| = -u'_n(r)$. For each positive integer $n \in \mathbb{N}$, (1.2) can be rewritten as

$$(1.7) \quad \begin{cases} -\left(r^{N-1}a_n(-u'_n(r))u'_n(r)\right)' = \varphi_n(1)r^{N-1}, & r \in (0, R), \\ u'_n(0) = 0, \quad u_n(R) = 0, \end{cases}$$

where $a_n(t) = \frac{\varphi_n(t)}{t}$ for $t > 0$. Letting $v_n(r) := r^{N-1}a_n(-u'_n(r))u'_n(r)$ and integrating in (1.7), we deduce that $v_n(r) - v_n(0) = -\frac{\varphi_n(1)}{N}r^N$, which, in view of the fact that $u'_n(0) = 0$, yields $\varphi_n(-u'_n(r)) = \frac{\varphi_n(1)}{N}r$. Further, note that since φ_n is strictly increasing we can define φ_n^{-1} , and the above relation shows that $-u'_n(r) = \varphi_n^{-1}\left(\frac{r\varphi_n(1)}{N}\right)$. Integrating again we get $u_n(0) - u_n(r) = \int_0^r \varphi_n^{-1}\left(\frac{s\varphi_n(1)}{N}\right) ds$. Taking into account the fact that $u_n(R) = 0$ we deduce that

$$u_n(0) = \int_0^R \varphi_n^{-1}\left(\frac{s\varphi_n(1)}{N}\right) ds.$$

Hence, $u_n(r) = \int_r^R \varphi_n^{-1}\left(\frac{s\varphi_n(1)}{N}\right) ds$, $\forall r \in (0, R)$. Next, we claim that the sequence $\{u_n\}$ converges uniformly to $R - r = \text{dist}(x, \partial B_R(0))$. To this aim, recall first that by [16, Lemma 2.2] we have

$$\min\{s^{1/\varphi_n^-}, s^{1/\varphi_n^+}\}\varphi_n^{-1}(t) \leq \varphi_n^{-1}(st) \leq \max\{s^{1/\varphi_n^-}, s^{1/\varphi_n^+}\}\varphi_n^{-1}(t), \quad \forall s, t \geq 0.$$

Thus, for each positive integer $n \in \mathbb{N}$,

$$\begin{aligned} \min\left\{\left(\frac{s}{N}\right)^{1/\varphi_n^-}, \left(\frac{s}{N}\right)^{1/\varphi_n^+}\right\} &\leq \varphi_n^{-1}\left(\frac{s\varphi_n(1)}{N}\right) \\ &\leq \max\left\{\left(\frac{s}{N}\right)^{1/\varphi_n^-}, \left(\frac{s}{N}\right)^{1/\varphi_n^+}\right\}, \quad \forall s \geq 0. \end{aligned}$$

In particular,

$$\begin{aligned} \min\left\{\left(\frac{s}{N}\right)^{1/\varphi_n^-}, \left(\frac{s}{N}\right)^{1/\varphi_n^+}\right\} &\leq \varphi_n^{-1}\left(\frac{s\varphi_n(1)}{N}\right) \\ &\leq \max\left\{\left(\frac{R}{N}\right)^{1/\varphi_n^-}, \left(\frac{R}{N}\right)^{1/\varphi_n^+}\right\}, \quad \forall s \in (r, R). \end{aligned}$$

Integrating with respect to s leads to

$$\begin{aligned} \int_r^R \min\left\{\left(\frac{s}{N}\right)^{1/\varphi_n^-}, \left(\frac{s}{N}\right)^{1/\varphi_n^+}\right\} ds &\leq u_n(r) \\ &\leq (R - r) \max\left\{\left(\frac{R}{N}\right)^{1/\varphi_n^-}, \left(\frac{R}{N}\right)^{1/\varphi_n^+}\right\}, \\ &\quad \forall r \in [0, R]. \end{aligned}$$

In view of (1.4), it follows that the sequence $\{u_n(\cdot) - (R - \cdot)\}$ is bounded above and below by sequences which converge uniformly to 0 on $[0, R]$. We conclude that $u_n \rightarrow \text{dist}(\cdot, \partial B_R(0))$ uniformly on $[0, R]$.

2. PROOF OF THEOREM 1

The following abstract result (see, e.g., [8, Corollary 6.1.1]) will play an important role in the proof.

Proposition 1. *Let X be a topological space satisfying the first axiom of countability, and assume that the sequence $\{F_n\}$ of functionals $F_n : X \rightarrow \overline{\mathbb{R}}$ Γ -converge to $F : X \rightarrow \overline{\mathbb{R}}$. Let z_n be a minimizer for F_n . If $z_n \rightarrow z$ in X , then z is a minimizer of F , and*

$$F(z) = \liminf_{n \rightarrow \infty} F_n(z_n).$$

Another key result in our analysis, which for the choice $\varphi_n(t) = |t|^{n-2}t$ was first established by Payne & Philippin in [15], is the following.

Proposition 2. *Assume that the hypotheses (1.3), (1.4), and (1.5) hold, and for $n \in \mathbb{N}$ let $u_n \in W_0^{1,\varphi_n}(\Omega)$ be the weak solution of the problem (1.2). Then*

$$\lim_{n \rightarrow \infty} \int_{\Omega} u_n(x) \, dx = \int_{\Omega} \delta(x) \, dx.$$

The proof of Proposition 2 will require two auxiliary results, stated in Lemma 1 and Lemma 2 below.

Lemma 1. *The sequence $\left\{ \int_{\Omega} u_n(x) \, dx \right\}$ is bounded.*

Proof. For every $n \in \mathbb{N}$, Hölder’s inequality gives

$$\int_{\Omega} u_n \, dx \leq \left(\int_{\Omega} u_n^{\varphi_n^-} \, dx \right)^{1/\varphi_n^-} |\Omega|^{(\varphi_n^- - 1)/\varphi_n^-}.$$

Denoting by $\lambda_1(\varphi_n^-)$ the first eigenvalue of the p -Laplacian with $p = \varphi_n^-$, given by

$$\lambda_1(\varphi_n^-) := \inf_{v \in C_0^\infty(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla v|^{\varphi_n^-} \, dx}{\int_{\Omega} |v|^{\varphi_n^-} \, dx},$$

we deduce that

$$(2.1) \quad \left(\int_{\Omega} u_n \, dx \right)^{\varphi_n^-} \leq |\Omega|^{\varphi_n^- - 1} \frac{\int_{\Omega} |\nabla u_n|^{\varphi_n^-} \, dx}{\lambda_1(\varphi_n^-)}.$$

Using (1.3), we deduce by [14, Lemma 1.1] (see also, [16, Lemma 2.1]) that

$$(2.2) \quad 1 < \varphi_n^- \leq \frac{t\varphi_n(t)}{\Phi_n(t)} \leq \varphi_n^+ < \infty, \quad \forall t \geq 0.$$

Further, in view of [6, Lemma A.2] we have

$$\begin{aligned} \Phi_n(t) \begin{cases} \rho^{\varphi_n^+} & \text{if } \rho \in (0, 1] \\ \rho^{\varphi_n^-} & \text{if } \rho \in (1, \infty) \end{cases} &\leq \Phi_n(\rho t) \\ &\leq \Phi_n(t) \begin{cases} \rho^{\varphi_n^-} & \text{if } \rho \in (0, 1], \\ \rho^{\varphi_n^+} & \text{if } \rho \in (1, \infty), \end{cases} \quad \forall \rho, t > 0. \end{aligned}$$

Thus, $\Phi_n(t) \geq \Phi_n(1)t^{\varphi_n^-} \forall t > 1$, giving $t^{\varphi_n^-} \leq 1 + \frac{\Phi_n(t)}{\Phi_n(1)} \forall t > 0$. We deduce that

$$(2.3) \quad |\nabla u_n(x)|^{\varphi_n^-} \leq 1 + \frac{\Phi_n(|\nabla u_n(x)|)}{\Phi_n(1)}, \quad \forall x \in \Omega, \forall n \geq 1.$$

Combining this inequality with (2.1) and using (2.2), (1.5), and the fact that $\langle J'_n(u_n), u_n \rangle = 0$ we obtain

$$(2.4) \quad \begin{aligned} \left(\int_{\Omega} u_n \, dx \right)^{\varphi_n^-} &\leq \frac{|\Omega|^{\varphi_n^- - 1}}{\lambda_1(\varphi_n^-)} \left[|\Omega| + \int_{\Omega} \frac{\Phi_n(|\nabla u_n(x)|)}{\Phi_n(1)} \, dx \right] \\ &\leq \frac{|\Omega|^{\varphi_n^- - 1}}{\lambda_1(\varphi_n^-)} \left[|\Omega| + \frac{1}{\Phi_n(1)\varphi_n^-} \int_{\Omega} \varphi_n(|\nabla u_n(x)|) |\nabla u_n(x)| \, dx \right] \\ &\leq \frac{|\Omega|^{\varphi_n^- - 1}}{\lambda_1(\varphi_n^-)} \left[|\Omega| + \frac{\varphi_n(1)}{\Phi_n(1)\varphi_n^-} \int_{\Omega} u_n(x) \, dx \right] \\ &\leq \frac{|\Omega|^{\varphi_n^- - 1}}{\lambda_1(\varphi_n^-)} \left[|\Omega| + \frac{\varphi_n^+}{\varphi_n^-} \int_{\Omega} u_n(x) \, dx \right] \\ &\leq \frac{|\Omega|^{\varphi_n^- - 1}}{\lambda_1(\varphi_n^-)} \left[|\Omega| + \beta \int_{\Omega} u_n(x) \, dx \right]. \end{aligned}$$

The conclusion of the lemma now follows by contradiction. Indeed, if we assume that the sequence $\left\{ \int_{\Omega} u_n(x) \, dx \right\}$ is unbounded, then there exists a subsequence, not relabelled, such that $\lim_{n \rightarrow \infty} \int_{\Omega} u_n(x) \, dx = +\infty$. Hence, for $n \in \mathbb{N}$ sufficiently large, we have $|\Omega| \leq \int_{\Omega} u_n(x) \, dx$. Dividing both sides of (2.4) by $\int_{\Omega} u_n(x) \, dx$ we find

$$\left(\int_{\Omega} u_n(x) \, dx \right)^{\varphi_n^- - 1} \leq \frac{|\Omega|^{\varphi_n^- - 1}}{\lambda_1(\varphi_n^-)} \left[\frac{|\Omega|}{\int_{\Omega} u_n \, dx} + \beta \right] \leq (1 + \beta) \frac{|\Omega|^{\varphi_n^- - 1}}{\lambda_1(\varphi_n^-)},$$

or, equivalently, $\int_{\Omega} u_n(x) \, dx \leq (1 + \beta)^{1/(\varphi_n^- - 1)} \frac{|\Omega|}{\{[\lambda_1(\varphi_n^-)]^{1/\varphi_n^-}\}^{\varphi_n^-/(\varphi_n^- - 1)}}$. On the other hand, by [9, Lemma 1.5] we know that $\lim_{n \rightarrow \infty} [\lambda_1(\varphi_n^-)]^{1/\varphi_n^-} = \|\delta\|_{L^\infty(\Omega)}^{-1}$, which implies that the right-hand side in the last inequality above is bounded.

This contradicts the fact that $\lim_{n \rightarrow \infty} \int_{\Omega} u_n(x) \, dx = \infty$. □

Lemma 2. *There exists $u_\infty \in W_0^{1,\infty}(\Omega)$ with $\|\nabla u_\infty\|_{L^\infty(\Omega; \mathbb{R}^N)} \leq 1$ such that $u_n \rightarrow u_\infty$ uniformly in Ω .*

Proof. Let $q \geq 1$ be an arbitrary real number. By (1.4), $q < \varphi_n^-$ for sufficiently large $n \in \mathbb{N}$. Using Hölder’s inequality, (2.3), recalling the fact that $\langle J'_n(u_n), u_n \rangle = 0$, and taking into account (2.2) and (1.5), we deduce that

$$\begin{aligned} \int_{\Omega} |\nabla u_n(x)|^q dx &\leq \left[\int_{\Omega} |\nabla u_n(x)|^{\varphi_n^-} dx \right]^{\frac{q}{\varphi_n^-}} |\Omega|^{\frac{\varphi_n^- - q}{\varphi_n^-}} \\ &\leq \left[|\Omega| + \int_{\Omega} \frac{\Phi_n(|\nabla u_n(x)|)}{\Phi_n(1)} dx \right]^{\frac{q}{\varphi_n^-}} |\Omega|^{\frac{\varphi_n^- - q}{\varphi_n^-}} \\ &\leq \left[|\Omega| + \frac{1}{\Phi_n(1)\varphi_n^-} \int_{\Omega} \varphi_n(|\nabla u_n(x)|) |\nabla u_n(x)| dx \right]^{\frac{q}{\varphi_n^-}} |\Omega|^{\frac{\varphi_n^- - q}{\varphi_n^-}} \\ &\leq \left[|\Omega| + \frac{\varphi_n(1)}{\Phi_n(1)\varphi_n^-} \int_{\Omega} u_n(x) dx \right]^{\frac{q}{\varphi_n^-}} |\Omega|^{\frac{\varphi_n^- - q}{\varphi_n^-}} \\ &\leq \left[|\Omega| + \frac{\varphi_n^+}{\varphi_n^-} \int_{\Omega} u_n(x) dx \right]^{\frac{q}{\varphi_n^-}} |\Omega|^{\frac{\varphi_n^- - q}{\varphi_n^-}} \\ &\leq \left[|\Omega| + \beta \int_{\Omega} u_n(x) dx \right]^{\frac{q}{\varphi_n^-}} |\Omega|^{\frac{\varphi_n^- - q}{\varphi_n^-}}. \end{aligned}$$

By Lemma 1, there exists a positive constant M such that $\int_{\Omega} u_n(x) dx \leq M$ for all $n \in \mathbb{N}$ sufficiently large. Thus, for all such $n \in \mathbb{N}$ we must have $\|\nabla u_n\|_{L^q(\Omega; \mathbb{R}^N)} \leq (|\Omega| + \beta M)^{1/\varphi_n^-} |\Omega|^{(\varphi_n^- - q)/(q\varphi_n^-)}$. We deduce that the sequence $\{\nabla u_n\}$ is bounded in $L^q(\Omega; \mathbb{R}^N)$ for any $q \geq 1$. Since by Lemma 1 the sequence $\{u_n\}$ is also bounded in $L^1(\Omega)$ we obtain, in view of the Poincaré-Wirtinger inequality, that $\{u_n\}$ is bounded in $L^q(\Omega)$. Hence, there exists a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ and $u_{\infty} \in W_0^{1,q}(\Omega)$, such that $u_{n_i} \rightharpoonup u_{\infty}$ weakly in $W_0^{1,q}(\Omega)$ and $u_{n_i} \rightarrow u_{\infty}$ strongly in $L^q(\Omega)$. However, since $q \geq 1$ was arbitrary, the compactness of the embedding of $W_0^{1,q}(\Omega)$ into $C^{0,\alpha}(\Omega)$ ($0 < \alpha < 1$) for $q > N$ (one can choose, e.g., $\alpha = 1 - N/q$), allows us to conclude that $u_{n_i} \rightarrow u_{\infty}$ uniformly in Ω . Since, as shown below, $u_{\infty} = \delta$, it follows that all possible subsequences of $\{u_n\}$, and hence the full sequence, must converge to u_{∞} uniformly in Ω . Finally, in view of Proposition 1 (with $X = L^1(\Omega)$, $F_n = I_n$, $F_{\infty} = I_{\infty}$, $z_n = u_n$) and [3, Theorem 1] we conclude that u_{∞} is a minimizer for I_{∞} which, in particular, means that $\|\nabla u_{\infty}\|_{L^{\infty}(\Omega; \mathbb{R}^N)} \leq 1$. Consequently, $u_{\infty} \in W_0^{1,\infty}(\Omega)$. This concludes the proof of Lemma 2. \square

We now prove Proposition 2. Since $\delta \in W_0^{1,\infty}(\Omega) \subset W_0^{1,\Phi_n}(\Omega)$, $|\nabla \delta(x)| = 1$ for a.e. $x \in \Omega$, and u_n is a minimizer of J_n in $W_0^{1,\Phi_n}(\Omega)$, we deduce that for each positive integer $n \in \mathbb{N}$ we have $J_n(u_n) \leq J_n(\delta) = \int_{\Omega} \frac{\Phi_n(1)}{\varphi_n(1)} dx - \int_{\Omega} \delta(x) dx$. Thus, by (2.2), $-\int_{\Omega} u_n(x) dx \leq \frac{|\Omega|}{\varphi_n^-} - \int_{\Omega} \delta(x) dx$ for all $n \geq 1$. Consequently, taking into account (1.4), we find

$$(2.5) \quad \int_{\Omega} \delta(x) dx \leq \liminf_{n \rightarrow \infty} \int_{\Omega} u_n(x) dx.$$

Further, let

$$A_\infty := \max_{\psi \in W_0^{1,\infty}(\Omega) \setminus \{0\}} \frac{\int_\Omega \psi(x) \, dx}{\|\nabla \psi\|_{L^\infty(\Omega; \mathbb{R}^N)}} = \max_{\psi \in W_0^{1,\infty}(\Omega); \|\nabla \psi\|_{L^\infty(\Omega; \mathbb{R}^N)}=1} \int_\Omega \psi(x) \, dx,$$

and note that for any $\psi \in C_0^\infty(\Omega)$ with $\|\nabla \psi\|_{L^\infty(\Omega; \mathbb{R}^N)} = 1$ we have that $\psi(x) = \psi(x) - \psi(y) \leq |x - y| \sup_{z \in [x,y]} |\nabla \psi(z)| \leq \delta(x)$, for every $x \in \Omega$ and $y \in \partial\Omega$ such that $|x - y| = \delta(x)$. Thus, $\int_\Omega \psi(x) \, dx \leq \int_\Omega \delta(x) \, dx, \forall \psi \in C_0^\infty(\Omega)$ with $\|\nabla \psi\|_{L^\infty(\Omega; \mathbb{R}^N)} = 1$. Recalling that $\|\nabla \delta\|_{L^\infty(\Omega; \mathbb{R}^N)} = 1$ we obtain

$$(2.6) \quad A_\infty \leq \int_\Omega \delta(x) \, dx.$$

On the other hand, testing in the definition of A_∞ with u_∞ , given by Lemma 2, we deduce that

$$A_\infty \geq \frac{\int_\Omega u_\infty(x) \, dx}{\|\nabla u_\infty\|_{L^\infty(\Omega; \mathbb{R}^N)}} \geq \int_\Omega u_\infty(x) \, dx = \lim_{n \rightarrow \infty} \int_\Omega u_n(x) \, dx.$$

Combining this with (2.6) it follows that $\int_\Omega \delta(x) \, dx \geq \lim_{n \rightarrow \infty} \int_\Omega u_n(x) \, dx$. Recalling (2.5), we deduce that $\lim_{n \rightarrow \infty} \int_\Omega u_n(x) \, dx = \int_\Omega \delta(x) \, dx$, which concludes the proof of Proposition 2.

We are now ready to give the proof of Theorem 1. By Lemma 2, it is enough to show that $u_\infty = \delta$. To this aim, we first note that for each $x \in \Omega$ and $y \in \partial\Omega$ such that $|x - y| = \delta(x)$, we have $u_\infty(x) = u_\infty(x) - u_\infty(y) \leq |x - y| \sup_{z \in [x,y]} |\nabla u_\infty(z)| \leq \delta(x)$. Next, since $u_n(x) \geq 0$ for a.e. $x \in \Omega$ and for every $n \in \mathbb{N}$, we must also have $u_\infty(x) \geq 0$ for a.e. $x \in \Omega$. Finally, Proposition 2 and the fact that $u_n \rightarrow u_\infty$ uniformly in Ω imply that $\int_\Omega \delta(x) \, dx = \lim_{n \rightarrow \infty} \int_\Omega u_n(x) \, dx = \int_\Omega u_\infty(x) \, dx$. In view of the continuity of δ and u_∞ we conclude that we must have $u_\infty = \delta$.

3. ANOTHER LOOK AT THE PROOF OF THEOREM 1

In this section we present an alternative proof of the fact that u_∞ , the uniform limit of the sequence u_n given by Lemma 2, coincides with δ , the distance function to the boundary of Ω . The key idea is to show that u_∞ is a viscosity solution of the problem

$$(3.1) \quad \begin{cases} \min \{ |\nabla u(x)| - 1, -\Delta_\infty u(x) \} = 0 & \text{if } x \in \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

which can be seen as the limiting problem, as $n \rightarrow \infty$, of the family (1.2) in a sense that will be made precise shortly. Once this is proven, since δ is also a viscosity solution of problem (3.1) (this is an easy consequence of the properties of the distance function to the boundary; see, e.g., [12, Lemma 4.1]), the fact that $u_\infty = \delta$ follows from the uniqueness of viscosity solutions to (3.1), owing to the well-known maximum principle in [7, Theorem 2.1].

Before we recall the definition of viscosity solutions for problems of the type (1.2) let us note that if we assume for a moment that the solutions u_n of (1.2) are

sufficiently smooth so that we can perform the differentiation in the PDE, we get

$$(3.2) \quad -\frac{\varphi_n(|\nabla u_n|)}{|\nabla u_n|} \Delta u_n - \frac{|\nabla u_n| \varphi_n'(|\nabla u_n|) - \varphi_n(|\nabla u_n|)}{|\nabla u_n|^3} \Delta_\infty u_n = \varphi_n(1) \quad \text{in } \Omega,$$

where Δ stands for the Laplace operator, $\Delta u := \text{Trace}(D^2 u) = \sum_{i=1}^N \frac{\partial^2 u}{\partial x_i^2}$, and Δ_∞ stands for the ∞ -Laplace operator, $\Delta_\infty u := \langle D^2 u \nabla u, \nabla u \rangle = \sum_{i,j=1}^N \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j}$. Note that (3.2) can be rewritten as $H_n(\nabla u_n, D^2 u_n) = 0$ in Ω , with

$$H_n(z, S) := -\frac{\varphi_n(|z|)}{z} \text{Trace}(S) - \frac{|z| \varphi_n'(|z|) - \varphi_n(|z|)}{|z|^3} \langle Sz, z \rangle - \varphi_n(1).$$

Here, $z \in \mathbb{R}^N$, and S stands for a real symmetric matrix in $\mathbb{M}^{N \times N}$. We are now ready to give the definition of viscosity solutions for the homogeneous Dirichlet boundary value problem

$$(3.3) \quad \begin{cases} H_n(\nabla u, D^2 u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Definition 1. (i) An upper semicontinuous function $u : \Omega \rightarrow \mathbb{R}$ is called a viscosity subsolution of (3.3) if $u|_{\partial\Omega} \leq 0$ and, whenever $x_0 \in \Omega$ and $\Psi \in C^2(\Omega)$ are such that $u(x_0) = \Psi(x_0)$ and $u(x) < \Psi(x)$ if $x \in B(x_0, r) \setminus \{x_0\}$ for some $r > 0$, then $H_n(\nabla \Psi(x_0), D^2 \Psi(x_0)) \leq 0$.

(ii) A lower semicontinuous function $u : \Omega \rightarrow \mathbb{R}$ is called a viscosity supersolution of (3.3) if $u|_{\partial\Omega} \geq 0$ and, whenever $x_0 \in \Omega$ and $\Psi \in C^2(\Omega)$ are such that $u(x_0) = \Psi(x_0)$ and $u(x) > \Psi(x)$ if $x \in B(x_0, r) \setminus \{x_0\}$ for some $r > 0$, then $H_n(\nabla \Psi(x_0), D^2 \Psi(x_0)) \geq 0$.

(iii) A continuous function $u : \Omega \rightarrow \mathbb{R}$ is called a viscosity solution of (3.3) if it is both a viscosity subsolution and a viscosity supersolution of (3.3).

Note that in both (i) and (ii) above the strict inequalities can be relaxed.

For $z \in \mathbb{R}^N$ and S a real symmetric matrix, define

$$H_\infty(z, S) := \min\{|z| - 1, -\langle Sz, z \rangle\}.$$

As promised, we are now ready to prove

Proposition 3. u_∞ is a viscosity solution of

$$(3.4) \quad \begin{cases} H_\infty(\nabla u, D^2 u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Proof. First, we check that u_∞ is a viscosity supersolution of (3.4). Let $x_0 \in \Omega$ and $\Psi \in C^2(\overline{\Omega})$ be such that $u_\infty(x_0) = \Psi(x_0)$ and $u_\infty - \Psi$ has a minimum at x_0 . The uniform convergence of u_n to u_∞ as $n \rightarrow \infty$ implies that there exists a sequence $\{x_n\} \subset \Omega$ such that $x_n \rightarrow x_0$, $u_n(x_n) = \Psi(x_n)$, and $u_n - \Psi$ has a minimum at x_n . Indeed, we have $u_\infty(x_0) = \Psi(x_0)$ and $u_\infty(x) > \Psi(x)$ for all

$x \in B(x_0, R)$, $x \neq x_0$, with $R > 0$ fixed and such that $B(x_0, 2R) \subset \Omega$. For $0 < r < R$, $\inf_{B(x_0, R) \setminus B(x_0, r)} (u_\infty - \Psi) > 0$. Since $u_n \rightarrow u_\infty$ uniformly in $\overline{B}(x_0, R)$ for $n \in \mathbb{N}$ sufficiently large, the function $u_n - \Psi$ attains its minimum value in $B(x_0, r)$. Let us denote the minimum point of $u_n - \Psi$ by $x_n \in B(x_0, r)$. If we consider a sequence $r_k \rightarrow 0^+$ as $k \rightarrow \infty$, we can construct a subsequence $\{n_k\}$ such that $x_{n_k} \rightarrow x_0$ as $k \rightarrow \infty$. The claim now holds after an appropriate relabeling of the indices. Since u_n is a continuous viscosity solution of (1.2) (this follows via standard arguments; see, e.g., the proof of [4, Lemma 3.3]), we have

$$(3.5) \quad \begin{aligned} & -\frac{\varphi_n(|\nabla\Psi(x_n)|)}{|\nabla\Psi(x_n)|} \Delta\Psi(x_n) - \frac{|\nabla\Psi(x_n)|\varphi'_n(|\nabla\Psi(x_n)|) - \varphi_n(|\nabla\Psi(x_n)|)}{|\nabla\Psi(x_n)|^3} \Delta_\infty\Psi(x_n) \\ & \geq \varphi_n(1), \end{aligned}$$

and note that $\varphi_n(1) \geq \varphi_n^- > 0$. Thus, we must have $|\nabla\Psi(x_n)| > 0$ for each $n \in \mathbb{N}$. Recall that by [16, Lemma 2.1] we have

$$(3.6) \quad \min\{s^{\varphi_n^-}, s^{\varphi_n^+}\} \varphi_n(t) \leq \varphi_n(st) \leq \max\{s^{\varphi_n^-}, s^{\varphi_n^+}\} \varphi_n(t), \quad \forall s, t \geq 0.$$

Combining this with (1.4) we deduce that for each positive integer n sufficiently large the function $b_n : [0, \infty) \rightarrow \mathbb{R}$ defined by $b_n(t) := \frac{t\varphi'_n(t) - \varphi_n(t)}{t^3}$ for $t > 0$, $b_n(0) := 0$ is continuous and the function $a_n : [0, \infty) \rightarrow \mathbb{R}$ defined by $a_n(t) := \frac{\varphi_n(t)}{t}$ for $t > 0$, $a_n(0) := 0$ is continuous and of class C^1 on $(0, \infty)$. Taking into account again hypothesis (1.4), we obtain

$$(3.7) \quad \frac{|\nabla\Psi(x_n)|^3}{|\nabla\Psi(x_n)|\varphi'_n(|\nabla\Psi(x_n)|) - \varphi_n(|\nabla\Psi(x_n)|)} > 0$$

and hence, after multiplying both sides of (3.5) by the expression on the left-hand side of (3.7), we arrive at

$$(3.8) \quad \begin{aligned} & \frac{|\nabla\Psi(x_n)|^2 \varphi_n(|\nabla\Psi(x_n)|)}{|\nabla\Psi(x_n)|\varphi'_n(|\nabla\Psi(x_n)|) - \varphi_n(|\nabla\Psi(x_n)|)} \Delta\Psi(x_n) - \Delta_\infty\Psi(x_n) \\ & \geq \frac{|\nabla\Psi(x_n)|^3 \varphi_n(1)}{|\nabla\Psi(x_n)|\varphi'_n(|\nabla\Psi(x_n)|) - \varphi_n(|\nabla\Psi(x_n)|)}. \end{aligned}$$

In view of (1.3) we deduce that

$$\frac{|\nabla\Psi(x_n)|^2 \varphi_n(|\nabla\Psi(x_n)|)}{|\nabla\Psi(x_n)|\varphi'_n(|\nabla\Psi(x_n)|) - \varphi_n(|\nabla\Psi(x_n)|)} = \frac{|\nabla\Psi(x_n)|^2}{\frac{|\nabla\Psi(x_n)|\varphi'_n(|\nabla\Psi(x_n)|)}{\varphi_n(|\nabla\Psi(x_n)|)} - 1} \leq \frac{|\nabla\Psi(x_n)|^2}{\varphi_n^- - 2}.$$

Letting $n \rightarrow \infty$ in (3.8) and observing (1.4) leads to

$$(3.9) \quad -\Delta_\infty \Psi(x_0) \geq \limsup_{n \rightarrow \infty} \frac{|\nabla \Psi(x_n)|^3 \varphi_n(1)}{|\nabla \Psi(x_n)| \varphi'_n(|\nabla \Psi(x_n)|) - \varphi_n(|\nabla \Psi(x_n)|)}.$$

In particular, we have

$$(3.10) \quad -\Delta_\infty \Psi(x_0) \geq 0.$$

We claim that we also have

$$(3.11) \quad |\nabla \Psi(x_0)| - 1 \geq 0.$$

Indeed, if we had $|\nabla \Psi(x_0)| - 1 < 0$, then $|\nabla \Psi(x_n)| < 1$ for $n \in \mathbb{N}$ sufficiently large. In this case, using (1.3) and (3.6) we deduce that

$$\begin{aligned} & \frac{|\nabla \Psi(x_n)|^3 \varphi_n(1)}{|\nabla \Psi(x_n)| \varphi'_n(|\nabla \Psi(x_n)|) - \varphi_n(|\nabla \Psi(x_n)|)} \\ &= \frac{\varphi_n(1)}{\varphi_n(|\nabla \Psi(x_n)|)} \frac{|\nabla \Psi(x_n)|^3}{\frac{|\nabla \Psi(x_n)| \varphi'_n(|\nabla \Psi(x_n)|)}{\varphi_n(|\nabla \Psi(x_n)|)} - 1} \\ &\geq \frac{\varphi_n(1)}{\varphi_n(|\nabla \Psi(x_n)|)} \frac{|\nabla \Psi(x_n)|^3}{\varphi_n^+ - 2} \\ &\geq \frac{\varphi_n(1)}{\varphi_n(1) |\nabla \Psi(x_n)|^{\varphi_n^-}} \frac{|\nabla \Psi(x_n)|^3}{\varphi_n^+ - 2} \\ &= \left[\frac{1}{(\varphi_n^+ - 2)^{1/(\varphi_n^- - 3)} |\nabla \Psi(x_n)|} \right]^{\varphi_n^- - 3}. \end{aligned}$$

Taking into account the fact that $\lim_{n \rightarrow \infty} \frac{1}{(\varphi_n^+ - 2)^{1/(\varphi_n^- - 3)} |\nabla \Psi(x_n)|} = \frac{1}{|\nabla \Psi(x_0)|} > 1$ we obtain that there exists $\epsilon_0 > 0$ such that $\frac{1}{(\varphi_n^+ - 2)^{1/(\varphi_n^- - 3)} |\nabla \Psi(x_n)|} \geq 1 + \epsilon_0$ for all $n \in \mathbb{N}$ sufficiently large. The above estimates lead to

$$\limsup_{n \rightarrow \infty} \frac{|\nabla \Psi(x_n)|^3 \varphi_n(1)}{|\nabla \Psi(x_n)| \varphi'_n(|\nabla \Psi(x_n)|) - \varphi_n(|\nabla \Psi(x_n)|)} \geq \lim_{n \rightarrow \infty} (1 + \epsilon_0)^{\varphi_n^- - 3} = +\infty,$$

which is a contradiction to (3.9). Hence, (3.11) holds, as claimed. From (3.10) and (3.11) we now get $\min\{-\Delta_\infty \Psi(x_0), |\nabla \Psi(x_0)| - 1\} \geq 0$.

It remains to check that u_∞ is a viscosity subsolution of (3.4). Let us consider $x_0 \in \Omega$ and a test function $\Psi \in C^2(\bar{\Omega})$ such that $u_\infty(x_0) = \Psi(x_0)$ and $u_\infty(x) < \Psi(x)$ for every x in a punctured neighborhood of x_0 . We need to show that $H_\infty(\nabla \Psi(x_0), D^2 \Psi(x_0)) \leq 0$. To this aim, first note that if $\nabla \Psi(x_0) = 0$, then the inequality trivially holds. Hence, we may assume that $\nabla \Psi(x_0) \neq 0$. Next, we argue as follows: assuming that

$$(3.12) \quad |\nabla \Psi(x_0)| - 1 > 0,$$

we will show that $-\Delta_\infty \Psi(x_0) \leq 0$. Indeed, we can again construct, as above, a sequence $x_n \rightarrow x_0$ as $n \rightarrow \infty$ such that

$$\begin{aligned} & \frac{|\nabla \Psi(x_n)|^2 \varphi_n(|\nabla \Psi(x_n)|)}{|\nabla \Psi(x_n)| \varphi_n'(|\nabla \Psi(x_n)|) - \varphi_n(|\nabla \Psi(x_n)|)} \Delta \Psi(x_n) - \Delta_\infty \Psi(x_n) \\ & \leq \frac{|\nabla \Psi(x_n)|^3 \varphi_n(1)}{|\nabla \Psi(x_n)| \varphi_n'(|\nabla \Psi(x_n)|) - \varphi_n(|\nabla \Psi(x_n)|)}. \end{aligned}$$

Letting $n \rightarrow \infty$ and using (3.12), (3.6) and (1.3), we obtain

$$-\Delta_\infty \Psi(x_0) \leq \liminf_{n \rightarrow \infty} \left[\frac{1}{(\varphi_n^- - 2)^{1/(\varphi_n^- - 3)} |\nabla \Psi(x_n)|} \right]^{\varphi_n^- - 3} = 0.$$

This completes the proof. \square

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DEPARTMENT OF MATHEMATICS AND STATISTICS, LOYOLA UNIVERSITY CHICAGO, 1032 W. SHERIDAN ROAD, CHICAGO, ILLINOIS 60660

E-mail address: mbocea@luc.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CRAIOVA, 200585 CRAIOVA, ROMANIA — AND — “SIMION STOILOW” INSTITUTE OF MATHEMATICS OF THE ROMANIAN ACADEMY, 010702 BUCHAREST, ROMANIA

E-mail address: mmihaiiles@yahoo.com