POSITIVE RADIAL SOLUTIONS OF A MEAN CURVATURE EQUATION IN MINKOWSKI SPACE WITH STRONG SINGULARITY

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Abstract. The existence of positive radial solution is obtained for a mean curvature equation in Minkowski space of the form
\[ \begin{cases} \text{div}\left( \frac{\nabla v}{\sqrt{1-|\nabla v|^2}} \right) + f(|x|, v) = 0 & \text{in } \Omega; \\ v = 0 & \text{on } \partial \Omega, \end{cases} \]
where \( \Omega \) is a unit ball in \( \mathbb{R}^N \), \( f(r, u) \) has singularities at \( u = 0 \), \( r = 0 \) and/or \( r = 1 \). The main tool is the perturbation technique and nonlinear alternative of Leray-Schauder type. The interesting point is that the nonlinear term \( f(r, u) \) at \( u = 0 \) may be strongly singular.

1. Introduction

In this paper, we will consider a strongly singular Dirichlet problem involving the mean curvature operator in Minkowski space of the form
\[ \begin{cases} \text{div}\left( \frac{\nabla v}{\sqrt{1-|\nabla v|^2}} \right) + f(|x|, v) = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial \Omega, \end{cases} \]
where \( f(r, u) \) is nonnegative and continuous on \( (0, 1) \times (0, +\infty) \) and may be singular at \( r = 0 \) and/or \( r = 1 \) and strongly singular at \( u = 0 \) and \( \Omega \) is a unit ball in \( \mathbb{R}^N \). The model example is
\[ f(r, u) = r^{-\gamma}(au^{-\alpha} + bu^\beta + 1); \]
here \( \beta, \gamma, a, b \) are positive constants with \( \gamma < 1 \) and \( \alpha > 1 \).

In recent years, the Dirichlet problem involving the mean curvature operator in Minkowski space has been discussed by many authors and many excellent results have been obtained, for instance, see [1]–[8], [11] and the references therein. However, most of the results in the above mentioned references are concerned with nonsingular problems, there are only a few works on weakly singular problems; see [5], [11]. To the best knowledge of the authors, no work has been done for the strongly singular Dirichlet problem (1.1).
Motivated by the results mentioned above and [9], the purpose of this paper is to establish the existence results of positive solutions of problem (1.1) by applying the perturbation technique and nonlinear alternative of Leray-Schauder type.

Throughout this paper, we make the following assumptions:

$(C_1)$ function $f \in C((0, 1) \times (0, +\infty), [0, +\infty))$ has the decomposition

$$f(r, u) = q(r)(g(u) + h(u)),$$

where $q \in C((0, 1), (0, +\infty))$ with $\int_0^1 q(r)dr < +\infty$, $g \in C((0, +\infty), (0, +\infty))$ is nonincreasing on $(0, +\infty)$, $h \in C([0, +\infty), [0, +\infty))$, and $h/g$ is nondecreasing on $(0, +\infty)$;

$(C_2)$ there exists a constant $r^* > 0$ such that

$$\frac{1}{1 + h(r^*)/g(r^*)} \int_0^{r^*} \frac{dr}{g(r)} > \int_0^1 \frac{1}{r^{N-1}} \int_0^t \tau^{N-1}q(\tau)d\tau dt.$$

2. Main Results

In order to obtain the existence results of positive radial solutions for problem (1.1), setting $r = |x|$ and $v(x) = u(r)$, we reduce the problem (1.1) to

\[
\begin{aligned}
(r^{N-1} \phi(u'))' + r^{N-1} f(r, u) &= 0, \quad r \in (0, 1), \\
\phi(r(0)) &= 0, \quad \phi(r(1)) = 0,
\end{aligned}
\]

where $\phi(s) = s/\sqrt{1 - s^2}$.

In this paper, we say that a function $u(r)$ is a positive solution of problem (2.1) if $u \in C[0, 1] \cap C^1[0, 1]$ with $u(r) > 0$ on $[0, 1]$ be such that $(r^{N-1} \phi(u'(r)))' + r^{N-1} f(r, u(r)) = 0$ holds for all $r \in (0, 1)$ and $u'(0) = 0$, $u(1) = 0$.

**Lemma 2.1.** Let $w \in C((0, 1), (0, +\infty))$ with $\int_0^1 w(r)dr < +\infty$. Then, the following problem

\[
\begin{aligned}
(r^{N-1} \phi(w'))' + r^{N-1} w(r) &= 0, \quad r \in (0, 1), \\
w'(0) &= 0, \quad w(1) = 0
\end{aligned}
\]

has a unique solution $u \in C[0, 1] \cap C^1[0, 1]$.

**Proof.** From the assumptions on $w$,

$$\frac{1}{r^{N-1}} \int_0^r \tau^{N-1}w(\tau)d\tau \in C(0, 1)$$

and

$$\frac{1}{r^{N-1}} \int_0^r \tau^{N-1}w(\tau)d\tau \to 0 \text{ as } r \to 0^+.$$

Then integrating both sides of equation of (2.2) on $[0, r] \subset [0, 1)$, we get

$$u'(r) = -\phi^{-1}\left(\frac{1}{r^{N-1}} \int_0^r \tau^{N-1}w(\tau)d\tau\right), \quad r \in [0, 1).$$

Integrating both sides of the above equality from $r$ to 1, the unique solution of problem (2.2)

$$u(r) = \int_r^1 \phi^{-1}\left(\frac{1}{r^{N-1}} \int_0^r \tau^{N-1}w(\tau)d\tau\right)dt, \quad r \in [0, 1)$$

is obtained. Also, since $\int_r^1 \phi^{-1}\left(\frac{1}{r^{N-1}} \int_0^r \tau^{N-1}w(\tau)d\tau\right)dt \to 0$ as $r \to 1^-$, it follows that $u \in C[0, 1] \cap C^1[0, 1)$. This completes the proof of the lemma. \qed
Let $P = \{ u \in C[0,1] : u(r) \text{ is nonnegative and nonincreasing on } [0,1] \}$, then $P$ is a convex set in $C[0,1]$.

Let $n \in \mathbb{N} = \{1, 2, \ldots \}$ be a fixed natural number. Now, we consider the modified problem

\[(2.3)\]
\[
\begin{align*}
(r^{N-1}\phi'(w))' + r^{N-1}f^*(r, w(r)) &= 0, & r \in (0,1), \\
u'(0) &= 0, & u(1) = \frac{1}{n},
\end{align*}
\]

where $w \in P$ and $f^*(r, u) = q(r)(g(u) + h(u))$ with

\[g^*(u) = \begin{cases} 
  g(u), & u \geq \frac{1}{n}, \\
  g\left(\frac{1}{n}\right), & n \leq \frac{1}{n}.
\end{cases}\]

It is easy to see that $g^* \in C[0, +\infty)$ is nonincreasing on $[0, +\infty)$ and $g^*(u) \leq g(u), \forall u \in (0, +\infty)$ if condition $(C_1)$ holds.

By Lemma 2.1, we have the following.

**Lemma 2.2.** Assume that condition $(C_1)$ is satisfied. Then, for each fixed $w \in P$, problem (2.3) has a unique solution $u \in P$ with

\[u(r) = (Tw)(r),\]

where

\[(2.4)\]
\[(Tw)(r) := \frac{1}{n} + \int_0^1 \phi^{-1}\left(\frac{1}{t^{N-1}} \int_0^t \tau^{N-1}f^*(\tau, w(\tau))d\tau\right)dt, \quad w \in P.\]

**Lemma 2.3.** Assume $w \in P$ and $f_i(r, u) = q(r)(g_i(u) + h_i(u))$ ($i = 1, 2$) satisfies condition $(C_1)$. Let $u_i(r)$ be a solution of problem (2.3) with $f^*(r, u) = f_i^*(r, u)$, $i = 1, 2$, respectively. If $f_1^*(r, w(r)) \leq f_2^*(r, w(r))$ on $(0,1)$, then $u_1(r) \leq u_2(r)$ on $[0,1]$.

**Proof.** It is a direct consequence of formula (2.4) and from the fact that integrals respect order and that $\phi^{-1}$ is nondecreasing. This completes the proof of the lemma.

Let $v_M(r)$ be a positive solution to problem (2.2) with $w(r) = Mq(r)$ ($M > 0$) and $v_m(r)$ be a positive solution to problem (2.2) with $w(r) = mq(r)$ ($m > 0$).

By Lemmas 2.1–2.3, we have the following remarks.

**Remark 2.1.** Let $w \in P$ and $u(r)$ be a solution to problem (2.3) with $f^*(r, w(r)) \leq Mq(r)$. Then $u(r) \leq 1/n + v_M(r)$ on $[0,1]$, that is, $(Tw)(r) \leq 1/n + v_M(r)$ on $[0,1]$.

**Remark 2.2.** Let $w \in P$ and $u(r)$ be a solution to problem (2.3) with $f^*(r, w(r)) \geq mq(r)$. Then $u(r) \geq 1/n + v_m(r)$ on $[0,1]$, that is, $(Tw)(r) \geq 1/n + v_m(r)$ on $[0,1]$.

**Lemma 2.4.** Assume that condition $(C_1)$ is satisfied. Then, for any bounded and closed set $K \subset P$, the set $T(K)$ is equicontinuous on $[0,1]$.

**Proof.** We note that the set $\{(Tw)'(r) : w \in K\}$ is uniformly bounded on $(0,1)$, and hence the lemma follows from the mean value theorem for differentiable functions. This completes the proof of the lemma.

**Lemma 2.5.** Assume that condition $(C_1)$ is satisfied. Then, for any bounded and closed set $K \subset P$, the mapping $T : K \to P$ is continuous.
Proof. Assume that \( \{w_k\}_{k=0}^{\infty} \subset K \) and \( w_k(r) \) converges to \( w_0 \in K \) uniformly on \([0,1]\). Then there exists an \( M > 0 \) such that

\[
0 \leq f^*(r, w_k(r)) \leq M q(r) \quad \text{on} \quad [0,1],
\]

and hence from Remark 2.1, it follows that

\[
0 \leq (Tw_k)(r) \leq 1/n + v_M(r) \quad \text{on} \quad [0,1],
\]

i.e., \( \{(Tw_k)(r)\} \) is uniformly bounded on \([0,1]\). This together with Lemma 2.4 implies that \( \{(Tw_k)(r)\} \) is uniformly bounded and equicontinuous on \([0,1]\). So from the Arzela-Ascoli Theorem, there exist uniformly convergent subsequences in \( \{(Tw_k)(r)\} \). Let \( \{(Tw_{k_j})(r)\} \) be a any subsequence of \( \{(Tw_k)(r)\} \) which converges to \( v(r) \) uniformly on \([0,1]\). Notice that (2.5) and

\[
(Tw_{k_j})(r) = \frac{1}{n} + \int_0^1 \frac{1}{N-1} \int_0^t \tau^{N-1} f^*(\tau, w_{k_j}(\tau)) d\tau dt,
\]

by Lebesgue dominated convergence theorem, we have

\[
v(r) = \frac{1}{n} + \int_0^1 \frac{1}{N-1} \int_0^t \tau^{N-1} f^*(\tau, w_0(\tau)) d\tau dt,
\]

i.e., \( v(r) \equiv (Tw_0)(r) \) on \([0,1]\). This shows that each convergent subsequence of \( \{(Tw_k)(r)\} \) uniformly converges to \( (Tw_0)(r) \) on \([0,1]\). Therefore, the sequence \( \{(Tw_k)(r)\} \) itself uniformly converges to \( (Tw_0)(r) \) on \([0,1]\). Therefore, \( T \) is continuous on \( K \). This completes the proof of the lemma. \( \square \)

Combining Lemmas 2.4 and 2.5 and Remark 2.1, we have the following lemma.

**Lemma 2.6.** Assume that condition \((C_1)\) is satisfied. Then, the mapping \( T : P \to P \) is completely continuous.

Our existence principles will be proved by using the following fixed point result.

**Lemma 2.7** \([10]\). Assume that \( U \) is a relatively open subset of a convex set \( C \) in a normal space \( E \). Let \( T : \bar{U} \to C \) be a compact map with \( 0 \in U \). Then either

(A1) \( T \) has a fixed point in \( \bar{U} \); or
(A2) there is an \( x \in \partial U \) and a \( \lambda \in (0,1) \) such that \( x = \lambda Tx \).

**Theorem 2.1.** Assume that conditions \((C_1)\) and \((C_2)\) are satisfied. Then, the problem \((1.1)\) has at least one positive radial solution \( v = u(r) \) with \( \|u\| < r^* \).

**Proof.** It is sufficient to show that problem (2.1) has at least one positive solution, \( u = u(r) \) with \( \|u\| < r^* \).

We choose \( \varepsilon \in (0,r^*) \) such that

\[
\frac{1}{1 + h(r^*)/g(r^*)} \int_\varepsilon^{r^*} \frac{dr}{g(r)} > \int_0^1 \frac{1}{N-1} \int_0^t \tau^{N-1} q(\tau) d\tau dt.
\]

Let \( n_0 \in \mathbb{N} \) be chosen such that \( 1/n_0 < \varepsilon \) and let \( \mathbb{N}_{n_0} = \{n_0, n_0 + 1, \ldots\} \).

We first show that the following problem

\[
(2.7)_n \quad \left\{ \begin{array}{ll}
(r^{N-1}\phi(u'))' + r^{N-1} f(r, u) = 0, \quad r \in (0,1), \\
u'(0) = 0, \quad u(1) = \frac{1}{n}, \quad n \in \mathbb{N}_{n_0},
\end{array} \right.
\]

has a solution \( u_n \) with \( u_n(r) > 1/n \) on \([0,1]\) and \( \|u_n\| < r^* \) for \( n \in \mathbb{N}_{n_0} \).


To do this, we deal with the modified problem
\[(2.8)_n \quad \left\{ \begin{array}{l}
(r^{N-1}\phi(u'))' + r^{N-1}f^*(r, u) = 0, \quad r \in (0, 1), \\
u'(0) = 0, \quad u(1) = \frac{1}{n}, \quad n \in \mathbb{N}_{n_0},
\end{array} \right.
\]
where \(f^*\) is defined by (2.3).

Fix \(n \in \mathbb{N}_{n_0}\). Let \(T : \bar{\Omega}_{r^*} \to P\) be defined by (2.4), i.e.,
\[(Tu)(r) := \frac{1}{n} + \int_0^1 \phi^{-1}(\frac{1}{t^{N-1}} \int_0^t \tau^{N-1} f^*(\tau, u(\tau))d\tau)dt, \quad u \in \bar{\Omega}_{r^*},
\]
where \(\Omega_{r^*} = P \cap \{u \in C[0, 1] : \|u\| < r^*\}\). Then, from Lemma 2.6, \(T : \bar{\Omega}_{r^*} \to P\) is completely continuous.

We now show that
\[(2.9) \quad u \neq \lambda Tu, \quad \text{for } \lambda \in (0, 1), \quad u \in \partial \Omega_{r^*}.
\]
Assume by contradiction, there exist a \(\lambda_0 \in (0, 1)\) and \(u_0 \in \partial \Omega_{r^*}\) such that \(u_0 = \lambda_0 Tu_0 \in P\). Then, \(u_0(0) = r^*\) and
\[\left\{ \begin{array}{l}
(r^{N-1}\phi(u_0'/(r/\lambda_0)))' + r^{N-1}f^*(r, u_0) = 0, \quad r \in (0, 1), \\
u_0'(0) = 0, \quad u_0(1) = \frac{\lambda_0}{n}, \quad n \in \mathbb{N}_{n_0}.
\end{array} \right.
\]
Notice that
\[f^*(r, u_0(r)) \leq q(r)(g(u_0(r)) + h(u_0(r))), \quad r \in (0, 1),\]
we have
\[(2.10) \quad -(r^{N-1}\phi(u_0'/(r/\lambda_0)))' \leq r^{N-1}q(r)g(u_0(r)) \left(1 + \frac{h(u_0(r))}{g(u_0(r))}\right), \quad r \in (0, 1).
\]
Integrate both sides of (2.10) from 0 to \(t (0 \leq t < 1)\), then from condition (C_1) and the fact \(u_0 \in P\), it follows that
\[-t^{N-1}\phi(u_0'(t)/\lambda_0) \leq \left(1 + \frac{h(r^*)}{g(r^*)}\right) g(u_0(t)) \int_0^t \tau^{N-1}q(\tau)d\tau,
\]
and hence we obtain
\[-u_0'(t) \leq \frac{\lambda_0}{t^{N-1}} \left(1 + \frac{h(r^*)}{g(r^*)}\right) g(u_0(t)) \int_0^t \tau^{N-1}q(\tau)d\tau \]
Consequently,
\[(2.11) \quad -\frac{u_0'(t)}{g(u_0(t))} \leq \frac{1}{t^{N-1}} \left(1 + \frac{h(r^*)}{g(r^*)}\right) \int_0^t \tau^{N-1}q(\tau)d\tau.
\]
Also, integrating both sides of (2.11) from 0 to 1, one has
\[
\int_{r_0/n}^{r^*} \frac{dr}{g(r)} \leq \left(1 + \frac{h(r^*)}{g(r^*)}\right) \int_0^1 \left(\frac{1}{t^{N-1}} \int_0^t \tau^{N-1}q(\tau)d\tau\right)ds,
\]
and so
\[(2.12) \quad \int_{r_0/n}^{r^*} \frac{dr}{g(r)} \leq \left(1 + \frac{h(r^*)}{g(r^*)}\right) \int_0^1 \left(\frac{1}{t^{N-1}} \int_0^t \tau^{N-1}q(\tau)d\tau\right)ds.
\]
This contradicts (2.6), and thus (2.9) is true.

Now Lemma 2.7 implies that \(T\) has a fixed point \(u_n \in \bar{\Omega}_{r^*}\) which is a solution of \((2.8)_n\) with \(1/n \leq \|u_n\| < r^*\)(note if \(\|u_n\| = r^*\), then following essentially the
same argument from (2.10)–(2.12) will yield a contradiction. Since \( u_n(r) \geq 1/n \) on \([0, 1]\), we know that \( u_n(r) \) is a solution of (2.7)\(_n\) also.

Since \( u_n(r) \leq r^* \) on \([0, 1]\), then from condition \( (C_1) \) it follows that \( f^*(r, u_n(r)) = q(r)(g(u_n(r)) + h(u_n(r))) \geq q(r)g(r^*) \) on \([0, 1]\). Hence, by Remark 2.2, we have

\[
(2.13) \quad u_n(r) \geq \frac{1}{n} + v_{g(r^*)}(r), \quad \forall r \in [0, 1],
\]

where \( v_{g(r^*)}(r) > 0 \) on \([0, 1]\).

Next, we claim that

\[
(2.14) \quad \{u_n(r)\}_{n \in \mathbb{N}_{n_0}} \text{ is a bounded and equicontinuous family on } [0, 1].
\]

In fact, note that \( 0 \leq \|u_n\| < r^* \) for each \( n \in \mathbb{N}_{n_0} \), then \( \{u_n(r)\}_{n \in \mathbb{N}_{n_0}} \) is bounded on \([0, 1]\). Also, since for each \( n \in \mathbb{N}_{n_0} \), \( |u_n'(r)| = |(Tu_n)''(r)| < 1 \) on \((0, 1)\), and from the mean value theorem for differentiable functions, \( \{u_n(r)\}_{n \in \mathbb{N}_{n_0}} \) is equicontinuous on \([0, 1]\). Thus (2.14) is true.

The Arzela-Ascoli theorem guarantees the existence of a subsequence \( \{u_{n_j}(r)\}_{j=1}^\infty \subset \{u_n(r)\}_{n \in \mathbb{N}_{n_0}} \) with \( u_{n_j} \) converging to \( u \) in \( C[0, 1] \) as \( j \to \infty \). Clearly, \( u(1) = 0 \) and by (2.13), \( u(r) \geq v_{g(r^*)}(r) \) on \([0, 1]\). In particular, \( u(r) > 0 \) on \([0, 1]\). Fixing \( r \in (0, 1) \), then for any \( \eta \in (0, 1-r) \) we have

\[
u_{n_j}(r) = u_{n_j}(1-\eta) + \int_{r}^{1-\eta} \phi^{-1}(\frac{1}{tN-1}) \int_{0}^{t} \tau^{N-1} f(\tau, u_{n_j}(\tau))d\tau dt.
\]

Notice that \( g(\cdot) + h(\cdot) \) is uniformly continuous on a compact subset of \((0, r^*)\), let \( j \to \infty \) in the above equality, and it follows that

\[
u(r) = u(1-\eta) + \int_{r}^{1-\eta} \phi^{-1}(\frac{1}{tN-1}) \int_{0}^{t} \tau^{N-1} f(\tau, u(\tau))d\tau dt.
\]

Also, let \( \eta \to 0^+ \) in the above equality and one has

\[
u(r) = \int_{r}^{1} \phi^{-1}(\frac{1}{tN-1}) \int_{0}^{t} \tau^{N-1} f(\tau, u(\tau))d\tau dt, \quad r \in [0, 1),
\]

and so

\[
u'(r) = -\phi^{-1}(\frac{1}{rN-1}) \int_{0}^{r} \tau^{N-1} f(\tau, u(\tau))d\tau, \quad r \in [0, 1),
\]

thus \( \nu'(0) = 0 \). Then we have

\[
\left\{ \begin{array}{l}
(rN-1\phi(u'(r)))' + rN-1f(r, u(r)) = 0, \quad r \in (0, 1),
\nu'(0) = 0, \quad u(1) = 0.
\end{array} \right.
\]

Finally, it is easy to see that \( \|u\| < r^* \) (note, if \( \|u\| = r^* \), then following essentially the same argument from (2.10)–(2.12) will yield a contradiction). This completes the proof of the theorem.

Finally, we give two examples to illustrate our result.

**Example 2.1.** Consider the strongly singular problem

\[
(2.15) \quad \left\{ \begin{array}{l}
\text{div} \left( \frac{\nabla v}{\sqrt{1-|\nabla v|^2}} \right) + \lambda (av^{-\alpha} + bv^{\beta} + 1) = 0 \quad \text{in } \Omega,
\vspace{0.1cm}
\vspace{0.1cm}
\vspace{0.1cm}
v = 0 \quad \text{on } \partial \Omega,
\end{array} \right.
\]

where \( a, b, \alpha, \beta \) are positive constants, \( \lambda > 0 \) is a parameter, and \( \Omega \) is a unit ball in \( \mathbb{R}^N \).
Let 
\[ f(r, u) = \lambda (au^{-\alpha} + bu^\beta + 1) =: q(r)(g(u) + h(u)), \]
where 
\[ q(r) = \lambda, \quad g(u) = au^{-\alpha}, \quad h(u) = bu^\beta + 1. \]
It is easy to see that the function \( f \) satisfies condition \((C_1)\). We now choose \( r^* = 1 \).

Then
\[ \frac{1}{1 + h(r^*)/g(r^*)} \int_0^{r^*} \frac{dr}{g(r)} = \frac{1}{(a + b + 1)(\alpha + 1)}. \]

On the other hand,
\[ \int_0^1 \frac{1}{t^{N-1}} \int_0^t \tau^{N-1} q(\tau) d\tau dt = \frac{\lambda}{2N}. \]

Hence, when \( \lambda < \frac{2N}{(a + b + 1)(\alpha + 1)} \), the function \( f \) satisfies condition \((C_2)\). So from Theorem 2.1, the strongly singular problem \((2.15)\) has at least one positive radial solution \( v(x) = u(r) \) with \( \|u\| < 1 \) provided the positive parameter \( \lambda < \frac{2N}{(a + b + 1)(\alpha + 1)} \).

**Example 2.2.** Consider the strongly singular problem
\[
\begin{align*}
\text{div} \left( \frac{\nabla v}{\sqrt{1 - |\nabla v|^2}} \right) + \lambda |x|^{-\gamma} (au^{-\alpha} + u^\beta) & = 0 \quad \text{in} \quad \Omega, \\
v & = 0 \quad \text{on} \quad \partial \Omega,
\end{align*}
\]
where \( \alpha, \beta, \gamma, a \) are positive constants with \( \gamma \in (0, 1) \), \( \lambda > 0 \) is a parameter, \( \Omega \) is a unit ball in \( \mathbb{R}^N \).

Let 
\[ f(r, u) = \lambda r^{-\gamma} (au^{-\alpha} + u^\beta) =: q(r)(g(u) + h(u)), \]
where 
\[ q(r) = \lambda r^{-\gamma}, \quad g(u) = au^{-\alpha}, \quad h(u) = u^\beta. \]
It is easy to see that function \( f \) satisfies condition \((C_1)\). Choosing \( r^* = 1 \), then
\[ \frac{1}{1 + h(r^*)/g(r^*)} \int_0^{r^*} \frac{dr}{g(r)} = \frac{1}{(\alpha + 1)(\alpha + 1)}. \]

On the other hand,
\[ \int_0^1 \frac{1}{t^{N-1}} \int_0^t \tau^{N-1} q(\tau) d\tau dt = \frac{\lambda}{(N - \gamma)(2 - \gamma)}. \]

Hence, if \( \lambda < \frac{(N - \gamma)(2 - \gamma)}{(\alpha + 1)(\alpha + 1)} \), then function \( f \) satisfies condition \((C_2)\). So from Theorem 2.1, the strongly singular problem \((2.16)\) has at least one positive radial solution \( v(x) = u(r) \) with \( \|u\| < 1 \) provided the positive parameter \( \lambda < \frac{(N - \gamma)(2 - \gamma)}{(\alpha + 1)(\alpha + 1)} \).

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